THEORY AND PRACTICE OF ALTERNATING CURRENTS

GENERAL PRINCIPLES, CIRCUITS
INSTRUMENTS, MEASUREMENTS
TRANSFORMERS, MACHINES
SYMMETRICAL COMPONENTS

BY

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PREFACE

In the present (third) edition of this book new chapters have been added dealing with single-phase transformers, three-phase transformers, single-phase and polyphase alternators, three-phase induction motors, transmission circuits, symmetrical components.

The original text has been revised and the collection of examples (with answers) has been extended. These examples, which are additional to the worked examples in the text, now comprise 250 numerical exercises, and have been carefully selected and graded. They have been taken mainly from the examination papers of the University of London [B.Sc. (Eng.)], Institution of Electrical Engineers, and City and Guilds of London Institute, but exercises suitable for National Certificate courses have been included.

The thanks of the author are due to his colleagues and numerous friends at home and abroad who have made suggestions, criticisms, and given help during the preparation of the present edition. Special thanks are due to Messrs. H. C. Mann, B.Sc. (Eng.), J. G. Fleming, M.Sc. (Eng.), H. Buckingham, M.Sc. (Eng.), H. A. Hayden, D.Sc., C. G. Paradine, M.A., Harry Silk (U.S.A.).

The thanks of the author are also due to the Senate of the University of London, the Council of the Institution of Electrical Engineers, and the Examinations Board of the City and Guilds of London Institute for the use of examples from examination papers.

A. T. DOVER

PREFACE

TO THE FIRST EDITION

THE original intention of the author, when undertaking this work, was to write a single volume general textbook dealing with the theory and practice of alternating current electrical engineering. As the work progressed, however, it was realized that the scope of the subject was too great for adequate treatment in a single volume. Accordingly, the present volume is devoted to general principles, circuits, polyphase systems, non-sinusoidal wave-forms, the magnetization of iron, instruments, measurements, and an elementary treatment of the initial conditions in the simpler electric circuits.

The author is of opinion that a thorough grounding in the principles of alternating currents is essential before proceeding to the study of alternating current machines and apparatus. This portion of the subject has, therefore, been treated on a broader basis than is the case in some textbooks.

Among the special features of the present volume are an extended application of the circle (locus) diagram to series, parallel series-parallel circuits; the reduction of the general circuit to its equivalent series-parallel form and the development of a locus diagram therefor; the calculation of currents in unbalanced polyphase circuits and the determination of the neutral point potential in the case of unbalanced star-connected three-phase circuits; the theory of the principal types of measuring instruments employed in alternating current engineering; and a large number of worked examples in the text.

The deduction of the circle, or current locus, diagram for a general circuit involves the application of geometrical inversion, which principle has not hitherto received much consideration in textbooks for the English-speaking students, although it has received considerable development in Arnold and La Cour's *Wechselstromtechnik*, a portion of which has been translated into English by Professor Stanley Parker Smith.[1] The author has given considerable attention to the application of this (inversion) principle to circle diagrams for the simple, or fundamental, circuit, as well as its

application to the general circuit. As an extension of the methods here developed the student is referred to the Advanced Course of Lectures in Engineering given, in 1923, by Professor Miles Walker under the auspices of the University of London. Some interesting applications of the principle of inversion to the predetermination of the performance of special types of induction motors were included in the lectures.[1]

In connection with the worked examples in the text, analytical, graphical and complex algebraic methods of solution have been employed. In some cases alternative methods of solution are given in order to show the student the steps involved in the application of the alternative methods.

The thanks of the author are due to his colleagues and numerous friends who have made suggestions, criticisms, and given help during the preparation of the work. Special thanks are due to Messrs. H. C. Mann, B.Sc. (Eng.), A.M.I.E.E., F. W. Harvey, B.A., B.Sc., and B. Hague, M.Sc. (Eng.), D.I.C., A.M.I.E.E.

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The thanks of the author are also due to the Senate of the University of London for permission to use questions from the Final Engineering (B.Sc.) examination papers, and to the City and Guilds of London Institute for permission to use questions from the Electrical Engineering examination papers.

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ABBREVIATIONS

NOTE. The abbreviations and symbols employed in this volume are generally in accordance with the recommendations of the International Electro-technical Commission.

Ampere	amp., A.
Centimetre	cm.
Electro-motive force	E.M.F.
Centigrade	C.
Cubic centimetre	cm.^{3}
Cycles per second	c.p.s.
Foot, feet	ft.
Inch	in.
Henry	H.
Farad	F.
Kilo-cycles per second	kc.p.s.
Kilo-volt-amperes	kVA.
Kilowatt	kW.
Kilogramme	kg.
Gramme	gm.
Gramme-centimetre	gm-cm.
Magneto-motive force	M.M.F.
Micro-farad	μF.
Micro-henry	μH.
Micro-micro-farad	$\mu\mu$F.
Microhm	$\mu\Omega$.
Milli-ampere	mA.
Milli-henry	mH.
Millimetre	mm.
Milli-volt-amperes	mVA.
Milli-volts	mV.
Pound	lb.
Potential difference	P.D.
Ohm	Ω.
Reactive volt-amperes	VAr
Revolutions per second	revs. per sec.
Revolutions per minute	r.p.m.
Root-mean-square	R.M.S.
Second	sec.
Square centimetre	cm.^{2}
Square millimetre	mm.^{2}
Square inch	in.^{2} or sq. in.
Volts	V.
Volt-amperes	VA.
Watts	W.
Yard	yd.

CHIEF SYMBOLS

A = Coefficient of sine term in Fourier series. Constant.
a = Area of cross section.
B = Flux density.
= Coefficient of cosine term in Fourier series. Constant.
= Susceptance.
C = Capacitance.
$\underset{\bullet}{C}$ = Complex constant.
D = Value of determinant.
= Distance between centres.
d = Diameter.
E = E.M.F. (root-mean-square value).
E_m = E.M.F. (maximum value of quantity varying with respect to time).
e = E.M.F. (instantaneous value).
F = Force.
= Ampere turns.
= Frequency.
G = Conductance.
I = Current (root-mean-square value).
I_m = Current (maximum value of quantity varying with respect to time).
i = Current (instantaneous value).
J, j = Symbolic operators for vectors.
K = Ratio of transformation.
K_b = Winding distribution factor.
K_{bn} = Winding distribution factor for nth harmonic.
K, k = Constants.
K_f = Form factor.
L = Inductance.
L, l = Length.
M = Mutual inductance.
N = Number of turns.
n = Angular speed in revolutions per second.

n = Number of phases in polyphase systems.
P = Power.
p = Power (instantaneous value).
$\mathfrak{P}$ = Permeance.
Q = Electrostatic charge.
q = Number of slots per pole per phase.
R = Resistance.
r = Radius.
S = Reluctance.
s = Slip.
T = Period.
$\mathfrak{T}$ = Torque.
t = Time (instantaneous value).
V = Voltage, potential difference.
v = Voltage, potential difference (instantaneous value).
= Linear velocity.
W = Energy.
X = Reactance.
x = Abscissa of point with respect to co-ordinate axes.
Y = Admittance.
y = Ordinate of point with respect to co-ordinate axes.
Z = Impedance.

GREEK SYMBOLS

α = Coefficient of linear expansion.
= Current density.
. $\alpha, \beta, \gamma, \theta, \psi$ = Angles.
δ = Small quantity. Thickness.
Δ = Small quantity. Delta connection of three-phase circuit.
Σ = Sign denoting summation of series.
ε = Base of natural logarithms = 2·7318. . .
η = Hysteretic coefficient.
= Efficiency.
Φ = Flux.
φ = Phase difference.
Θ = Temperature.

κ = Dielectric constant

μ = Permeability.

ξ = Eddy-current loss coefficient.

= Young's modulus of elasticity.

λ = 120° turning operator for vectors.

π = Ratio of circumference to diameter of circle (= 3·14 . . .).

ρ = Specific resistance.

σ = Density.

τ = Time constant (= L/R).

ω = Angular velocity.

ALGEBRAIC SYMBOLS

$\neq$ = Not equal to.

$\cong$ = Approximately equal to.

$>$ = Greater than

$<$ = Less than.

$\angle$ = Angle.

! = Factorial (e.g. 5 ! = 5 × 4 ×3 × 2 × 1).

∞ = Infinity.

NOTES. **Maximum values** of quantities varying with respect to time are denoted by the subscript m attached to the symbol of the quantity.

Vector diagrams. All vector diagrams have been drawn for counter-clockwise rotation.

E.M.F. vectors are represented by an ordinary arrow-head. ——→
Flux vectors are represented by a double arrow-head. ——↠
Ampere-turn vectors are represented by a solid arrow-head ——▶
Current vectors are represented by a closed arrow-head. ——▷

Vector quantities are denoted by dotted italic upper-case symbols, thus $\dot{E}$, $\dot{I}$, $\dot{Y}$, $\dot{Z}$: their *rectangular components* are denoted by heavy-faced italic lower-case symbols and numerals, vertical components being compounded with the symbolic operator j, thus $\dot{E}$ = ***110*** + j ***191***.

Phase rotation. In the analytical treatment, and vector diagrams, of polyphase systems the clockwise direction of phase rotation, or phase sequence, is adopted, except where statements are made to the contrary.

Phase and line quantities in polyphase systems. Symbols denoting "phase" E.M.Fs. and currents usually have Roman numeral subscripts, thus e_{I}, e_{II} ; those denoting "line" E.M.Fs., and currents have Arabic numeral subscripts, thus $\boldsymbol{v}_{1-2}$, $\boldsymbol{v}_{2-3}$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$.

THEORY AND PRACTICE OF ALTERNATING CURRENTS

CHAPTER I

GENERAL CONSIDERATIONS AND DEFINITIONS

Definitions. An alternating current, or E.M.F., is one which changes periodically in magnitude and direction.

The *graphical representation of an alternating current* is shown in Fig. 1, in which abscissæ denote time and ordinates current. During the time interval AB the current increases from zero to its positive maximum value; it decreases to zero during the interval BC, attains its maximum negative value at the instant D, and returns to zero at E. The negative half-wave is usually an exact reproduction of the positive half-wave with the sign reversed, and the succeeding waves are identical with the initial wave.

The complete set of changes through which the current, or E.M.F., passes is called a *cycle*, and the time interval during which these changes occur is called a *period*. For example, the full-line curve in Fig. 1 represents a cycle, and the time interval AE represents a period.

The number of cycles per second is called the *frequency*. Hence if T denote the time, in seconds, of a period, the frequency (f) is equal to $1/T$.

The graphical representation of Fig. 1 refers to steady conditions in an alternating-current circuit. In special cases Fig. 1 is also representative of the initial conditions occurring when a circuit is connected to a source of alternating E.M.F., but more generally the initial conditions give rise to transient phenomena and the waves are unsymmetrical.*

Application of Alternating Currents in Practice. When electric power is required in large quantities it is always produced in the alternating-current form, as the alternating-current generator can be built in much larger sizes than the direct-current generator and gives satisfactory operation at the high speeds at which large steam turbines have a high efficiency. Moreover, when electrical

* A few special cases of transient phenomena are considered in Chap. XXIII.

energy is generated in the alternating-current form the transmission, distribution, and utilization pressures are not limited to that of the generator, as by means of stationary transformers the energy may be transformed, at high efficiency, to pressures either higher or lower than that of the generator. Thus the pressures adopted for transmission and distribution may be chosen entirely from economic considerations without reference to the generator pressure, which is governed by a number of technical considerations connected with design.

Production of Alternating E.M.Fs. Alternating E.M.Fs. may be produced either dynamically, by the relative motion of an electric circuit and a magnetic circuit, or statically, by the variation of

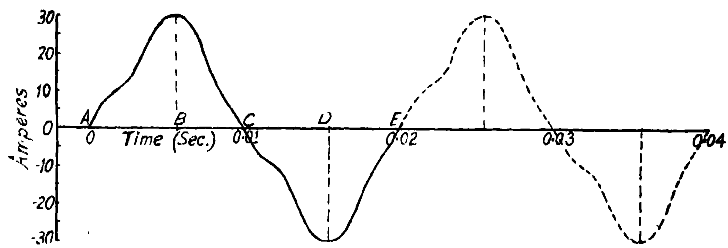


FIG. 1. REPRESENTATION, IN RECTANGULAR CO-ORDINATES, OF AN ALTERNATING CURRENT

flux in a stationary magnetic circuit which is interlinked with a stationary electric circuit. In both cases the magnitude of the E.M.F. at any instant is proportional to the rate of change of the linkage of flux with the electric circuit, i.e. $e \propto N.d\Phi/dt$, where e is the instantaneous value of the E.M.F., N the number of turns in the electric circuit, and $d\Phi/dt$ the rate of change of the flux. An E.M.F. of 1 volt is produced when the rate of change of linkages per second is equal to 10^8. Thus

$$e = 10^{-8} N \, d\Phi/dt$$

In alternating-current engineering we are concerned with E.M.Fs. produced both dynamically and statically. For instance, in the majority of circuits carrying alternating currents and in electromagnetic apparatus and transformers, E.M.Fs. are produced statically; but in all alternating-current machines operating on load both dynamically- and statically-produced E.M.Fs. occur simultaneously. In these cases the dynamically-produced E.M.F. is usually called the *E.M.F. of rotation* or the *generated E.M.F.*, and the statically-produced E.M.F. is called the *E.M.F of pulsation* or the *induced E.M.F.*

Producion of E.M.F. in a Simple Alternating-current Generator. Let a single conductor be rotated at constant angular velocity in a uniform magnetic field, the axis of revolution being perpendicular to the magnetic lines as indicated in Fig. 2. Also let

B = flux density (in lines per cm^2.) of the magnetic field,
l = length (in cm.) of the conductor in the field,
ω = angular velocity (in radians per second) of the conductor,
r = radius (in cm.) of the path in which the conductor moves,
$v = r\omega$ = linear velocity (in cm. per second) of the conductor.

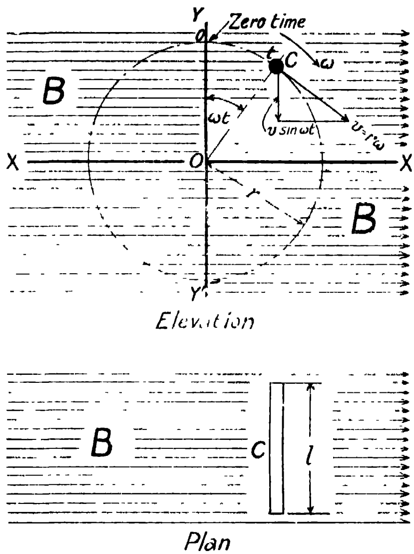


FIG. 2. PERTAINING TO THE PRODUCTION OF E.M.F. IN A SIMPLE ALTERNATOR

Further, let time, in seconds, be measured from a plane of reference (YOY', Fig. 2) which contains the axis of revolution and is perpendicular to the magnetic lines.

Then at any instant t, when the conductor occupies the position C, Fig. 2, the component of the linear velocity perpendicular to the flux is

$$r\omega \sin \angle\, YOC = r\omega \sin \omega t.$$

If the conductor were to continue its motion in this direction the flux cut in one second would be

$$B\, l\, r\, \omega \sin \omega t.$$

Hence at this particular instant, t, the E.M.F. generated in the conductor is

$$e = B\,l\,r\,\omega \sin \omega t \times 10^{-8} \text{ volts}$$
$$= \tfrac{1}{2}\,\Phi\,\omega \sin \omega t \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad (1)$$

where Φ ($= 2Blr$) is the flux cut by the conductor during one half of a revolution.

Thus the E.M.F. varies as a sine function of the time-angle ωt, and if E.M.F. and time be plotted in rectangular co-ordinates we obtain the curve shown in Fig 3, which is called a sine curve. An E.M.F. which varies in this manner is called a *sinusoidal E.M.F.* (sometimes the term *simple harmonic E.M.F.* is used) and the generator in which it is produced is called a *sine-wave generator.*

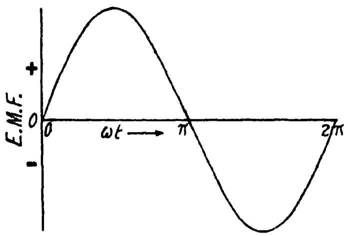


FIG. 3. REPRESENTATION OF SINUSOIDAL E.M.F.

The maximum, or peak, value of the E.M.F. occurs when the direction of motion of the conductor is perpendicular to the flux—i.e., when $\omega t = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, etc.—and its value (E_m) is

$$E_m = \tfrac{1}{2}\,\Phi\,\omega \times 10^{-8}.$$

Hence, equation (1) may be written

$$e = E_m \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If instead of a single conductor we have a coil consisting of a single turn, and the plane of the coil passes through the axis of rotation, the E.M.F generated in the coil at any instant will equal twice that generated in one conductor, or coil-side, so that in this case*

$$e = \Phi\,\omega \sin \omega t \times 10^{-8} \quad . \quad . \quad . \quad (1a)$$

* This equation may be obtained directly as follows—

Let Φ = flux passing through the coil when the latter is perpendicular to the magnetic lines (i.e. when the plane containing the coil coincides with the plane of reference). Then the flux (φ) passing through the coil at the instant t is

$$\varphi = \Phi \cos \omega t,$$

and the instantaneous E.M.F. generated in the coil is

$$e = -10^{-8} \times d\varphi/dt = -10^{-8} \times \frac{d}{dt}(\Phi \cos \omega t)$$
$$= \Phi\,\omega \sin \omega t \times 10^{-8}$$

The sine wave of E.M.F. is therefore produced naturally in the simplest type of alternator. The type of construction represented in Fig. 2 would, however, not be commercially practicable on account of the poor utilization of the magnetic and electric materials. Commercial alternators are constructed with iron magnetic circuits and slotted armatures for the purpose of reducing the magnetic reluctance and the ampere-turns required for excitation. The slotted construction results in deviations from the sine wave, even when the flux is distributed sinusoidally in the air gap, and accordingly features have to be introduced into the construction to correct or neutralize the defects due to the slotted construction.

Forms of E.M.F. Equation. In each of the above cases the E.M.F. passes through a complete set of changes during each revolution of the conductor, or coil. Hence, the frequency (f) is equal to the number of revolutions per second (n), and the period (T) is equal to $1/n$.

Now the angular velocity $(\omega) = 2\pi n = 2\pi f$, so that $f = \omega/2\pi$. Equation (2) may therefore be expressed in the following forms—

$$e = E_m \sin \omega t \qquad (2)$$

$$= E_m \sin 2\pi f t \qquad (2a)$$

$$= E_m \sin (2\pi/T)t \qquad (2b)$$

In these equations we observe that—

(1) *the maximum value* (also called the peak value or amplitude) *of the E.M.F. is given by the coefficient of the sine of the time-angle* ;

(2) *the frequency is given by* : (coefficient of time $\div$ 2π).

For example, if the equation to an alternating E.M.F. is

$$e = 100 \sin 314\, t$$

the maximum value of the E.M.F. is 100, and the frequency is $(314/2\pi) = 50$.

Similarly if the equation takes the form

$$e = I_m\sqrt{(R^2 + 9\omega^2 L^2)} \sin 3\omega t,$$

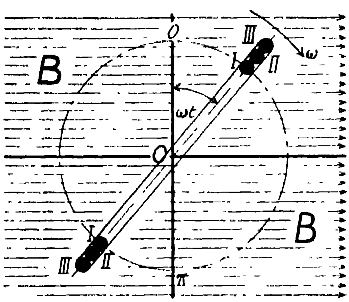
the maximum E.M.F. is given by $E_m = I_m\sqrt{(R^2 + 9\omega^2 L^2)}$ and the frequency by $3\omega/2\pi$.

Phase. Consider now the effect of adding additional turns to the above coil. These turns may be arranged either radially in the same plane, as in Fig. 4, or side by side on the surface of a cylindrical core, as in Fig. 6. In the former case the time-angle (ωt) is the same for each turn, but the instantaneous E.M.Fs. generated in the several turns differ in magnitude, since these turns do not cut the flux at the same speed. If the turns are connected in series

the total E.M.F. of the coil at any instant is equal to the sum of the instantaneous E.M.Fs. in the several turns and is given by

$$\begin{aligned} e &= e_1 + e_2 + e_3 + \dots \\ &= (\Phi_1\omega \sin \omega t + \Phi_2\omega \sin \omega t + \Phi_3\omega \sin \omega t + \dots) \times 10^{-8} \\ &= (E_{1m} + E_{2m} + E_{3m} + \dots) \sin \omega t \qquad (3) \end{aligned}$$

where Φ_1, Φ_2, Φ_3, ... denote the maximum fluxes enclosed by the several turns, and E_{1m}, E_{2m}, E_{3m}, ... denote the corresponding maximum E.M.Fs. A graphical representation of the E.M.Fs. for the case of a three-turn coil is shown in Fig. 5, from which it will



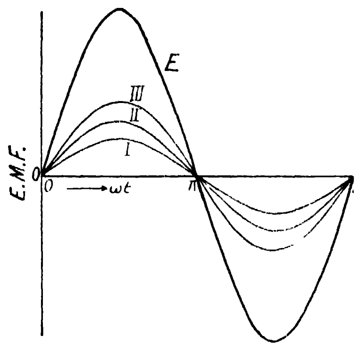


Fig. 4 Fig. 5

Pertaining to the Production of E.M.F. in a Simple Alternator with Co-planar Coils

be observed that the instantaneous value of the resultant E.M.F., as well as that of each of the individual E.M.Fs., is proportional to $\sin \omega t$. Thus all the E.M.Fs. become zero at the same instant and reach their maxima together, i.e. *the E.M.Fs. are in phase with one another*. Therefore, alternating quantities of the same frequency have the same phase, or are in phase, when their zero values—or their maximum or other corresponding values—occur at the same instant.

With the conditions shown in Fig. 6 each turn cuts the same maximum flux (Φ) during a revolution, but the time-angle is different for each turn. If the turns are connected in series the total E.M.F. generated in the coil is given by

$$\begin{aligned} e &= e_1 + e_2 + e_3 + \dots \\ &= \{\Phi\omega \sin \omega t + \Phi\omega \sin(\omega t - \alpha) + \Phi\omega \sin[\omega t - (\alpha + \beta)] + \dots\} \times 10^{-8} \\ &= \Phi\omega \times 10^{-8}\{\sin \omega t + \sin(\omega t - \alpha) + \sin[\omega t - (\alpha + \beta)] + \dots\} \\ &= E_m \{\sin \omega t + \sin(\omega t - \alpha) + \sin(\omega t - (\alpha + \beta)) + \dots\} \qquad (4) \end{aligned}$$

where ωt is the time-angle referred to the first turn, and α, β, ... are the angles by which the planes containing the several turns are displaced from one another (see Fig. 6).

A graphical representation of these conditions is given in Fig. 7, which refers to the case of a coil having three turns. In this case the maxima of the several E.M.Fs. occur at different instants, i.e. *the E.M.Fs. are out of phase with one another.*

In simple harmonic motion the term "phase" is used to denote the point or stage in the period, considered in relation to a standard position or to the instant of starting, to which the oscillation has

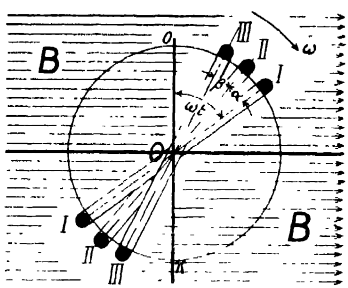


FIG. 6

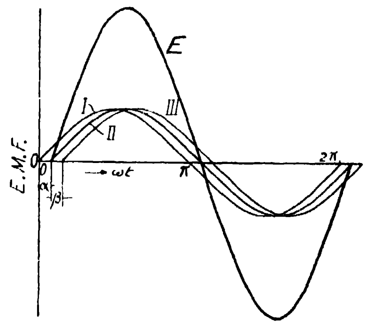


FIG. 7

PERTAINING TO THE PRODUCTION OF E.M.F. IN A SIMPLE ALTERNATOR WITH DISTRIBUTED COILS

advanced. For example, in Fig. 2 the phase of the conductor at time t is given by the angle ωt. Hence this angle is sometimes called the "phase angle." In electrical engineering, however, we are concerned more with the *relative* phases of alternating quantities, with respect to a quantity of reference, rather than their *absolute* phases, and hence the term "phase difference" is more appropriate. Moreover the term "phase" has other meanings than that given above.

Phase Difference. The phase difference between two alternating quantities is measured by the time-angle between their zero values. When the quantities have the same frequency* this angle is constant for corresponding values (e.g. zero, maximum, etc.) of the quantities. For the case represented in Fig. 7 the phase difference between the E.M.Fs. in turns I and II is α, and is equal to the angle by which these turns are displaced from each other. Similarly, the phase

* The meaning of "phase difference" when applied to quantities of different frequencies is considered in Chap. XIV.

difference between the E.M.Fs. in turns II and III is β, while that between the E.M.Fs. in turns I and III is $(\alpha + \beta)$. These expressions, however, need qualification, as they give no indication of which of the E.M.Fs. first reaches its maximum value. To supply this deficiency the terms " lead " and " lag " are employed.

Lead denotes that the maximum value of an alternating quantity occurs earlier than the corresponding maximum value of a second quantity.

Lag denotes that the maximum value occurs later than the corresponding maximum value of a second quantity.

Phase difference accordingly is usually expressed as either the " angle of lead " or the " angle of lag." For example, in Fig. 7 the E.M.F. wave I is leading with respect to II, the " angle of lead " being α; conversely, III is lagging with respect to II, the " angle of lag " being β.

If in Fig. 6 the time-angle is referred to turn II instead of to turn I, as above, and this angle is now denoted by $\omega t'$, then on substituting $(\omega t' + \alpha)$ for ωt in equation (4) we obtain

$$e = E_m \{\sin(\omega t' + \alpha) + \sin \omega t' + \sin(\omega t' - \beta) + \ldots\} \quad (5)$$

The E.M.F. in turn I is now given by

$$e'_1 = E_m \sin(\omega t' + \alpha),$$

that in turn II is given by

$$e'_2 = E_m \sin \omega t';$$

while that in turn III is given by

$$e'_3 = E_m \sin(\omega t' - \beta).$$

Now the E.M.F. in turn I is leading, and that in turn III is lagging, with respect to the E.M.F. in turn II. Hence *a plus* (+) *sign employed in connection with phase difference denotes " lead " and a minus* (–) *sign denotes " lag."*

Thus in general we can determine from the equation of an alternating quantity: (1) its maximum value, (2) its frequency, (3) its phase difference with respect to another quantity of the same frequency.

Examples. Determine the maximum value and frequency of the following alternating E.M.Fs. and currents. Determine also the phase differences between the respective E.M.Fs. and currents—

(a) $\begin{cases} e = 3250 \sin 157t \\ i = 35 \sin(157t - 15^\circ) \end{cases}$ (b) $\begin{cases} e = E'_m \sin 5\omega t \\ i = 5\omega C E'_m \sin(5\omega t + \frac{1}{2}\pi) \end{cases}$

(c) $\begin{cases} e = \omega L I'_m \sin(\omega t + \frac{1}{2}\pi) \\ i = I'_m \sin \omega t \end{cases}$ (d) $\begin{cases} e = E''_m \sin(\omega t + \alpha) \\ i = E''_m / \sqrt{(R^2 + X^2)} \sin(\omega t - \tan^{-1} X/R) \end{cases}$

Denoting, in each case, the maximum value of the E.M.F. by E_m, the maximum value of the current by I_m, the frequency by f, and the phase difference by φ, we have

(*a*) $E_m = 3250$;	$I_m = 35$;	$f = 157/2\pi = 25$;	$\varphi = -15°$
(*b*) $E_m = E'_m$;	$I_m = 5\omega CE'_m$	$f = 5\omega/2\pi$;	$\varphi = +\frac{1}{2}\pi = +90°$
(*c*) $E_m = \omega LI'_m$;	$I_m = I'_m$:	$= \omega/2\pi$;	$\varphi = -\frac{1}{2}\pi = -90°$
(*d*) $E_m = E''_m$;	$I_m = E''_m/\sqrt{(R^2 + X^2)}$	$= \omega/2\pi$;	$\varphi = -(\alpha + \tan^{-1}X/R)$

Observe that both frequency and phase difference are determined from the time-angle, the former being given by ($1/2\pi \times$ coefficient of time), and the latter by the constant term, if any, of the time-angle. Also observe that in expressing phase difference, the E.M.F. is taken as the quantity of reference. Thus in (*c*) the E.M.F. leads the current by 90°, therefore the current lags behind the E.M.F. by this amount: in (*d*) the E.M.F. is zero when $\omega t = -\alpha$, and the current is zero when $\omega t = +\tan^{-1} X/R$, so that the phase difference between E.M.F. and current is $(\alpha + \tan^{-1} X/R)$, lagging.

Properties of Sine Curves. The study of alternating-current phenomena requires a knowledge of the properties of sine curves. A fundamental property of such curves is that any ordinate is proportional to the sine of the corresponding abscissa. Other properties are (1) that the mean ordinate of a half-wave is equal to $2/\pi$ times the maximum ordinate; (2) that the square root of the mean of the squared ordinates of a half-wave or a complete wave is equal to $1/\sqrt{2}$ times the maximum ordinate. We shall now show how these quantitative relations are obtained.

Mean Ordinate, or Arithmetic Mean Value, of a Sine Curve. Let the curve be represented by the equation

$$e = E_m \sin \omega t = E_m \sin (2\pi t/T)$$

where E_m denotes the maximum ordinate and T the period. Then the mean ordinate is given by

$$E_{av} = \frac{\text{Area included between the curve and abscissa axis for one half-period}}{\text{Half-period}}$$

$$= \frac{1}{\frac{1}{2}T}\int_0^{\frac{1}{2}T} e\,.\,dt = \frac{2}{T}\int_0^{\frac{1}{2}T} E_m \sin\left(\frac{2\pi}{T}\right)t\,.\,dt$$

$$= \frac{2}{T}\cdot\frac{T}{2\pi}\cdot E_m\left[-\cos\left(\frac{2\pi}{T}\right)t\right]_0^{\frac{1}{2}T}$$

$$= \frac{E_m}{\pi}[-\cos\pi + \cos 0]$$

$$= (2/\pi)E_m = 0{\cdot}637\ E_m$$

An approximation to the value of the mean ordinate can be obtained arithmetically by determining the mean value of the sines

of angles between 0° and 180°. Thus taking angles at intervals of 15° we have

Angle	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Sine	0·259	0·5	0·707	0·866	0·966	1·0	0·966	0·866	0·707	0·5	0·259	0

Total = 7 56 ; Mean = 7 56/12 = 0 633

The arithmetic mean value of an alternating current is of little importance in practice, as the heating effect due to a current is proportional, not to the square of the mean value of the current, but to the mean of the squared instantaneous values of the current as explained later. However, a knowledge of the mean value of an alternating E.M.F. is frequently required in calculations connected with alternating-current machines.

Root-mean-square (R.M.S.)* Value of a Sine Curve. The root-mean-square value is obtained by determining the square root of the mean value of the squared ordinates for a cycle or half-cycle. Let the curve be represented by the equation

$$i = I_m \sin \omega t = I_m \sin(2\pi/T)t$$

The square of any ordinate (i) is then

$$i^2 = I^2_m \sin^2 (2\pi/T)t,$$

and if I^2 denote the mean value of the squares of the ordinates during a period, we have

$$I^2 = \frac{1}{T}\int_o^T I_m^2 \sin^2\left(\frac{2\pi}{T}\right)t\,.\,dt$$

$$= \frac{I_m^2}{T}\int_o^T \tfrac{1}{2}(1 - \cos 2(2\pi/T)t)\,dt$$

$$= \frac{I_m^2}{2T}\left[t - \frac{T}{4\pi}\sin\left(\frac{4\pi}{T}\right)t\,\right]_o^T$$

$$= \tfrac{1}{2} I_m^2$$

Whence $\quad I = \sqrt{\tfrac{1}{2}I_m^2} = I_m/\sqrt{2} = 0{\cdot}707\, I_m$

An approximation to the R.M.S. value can be obtained arithmetically by determining the mean value of the squares of the sines of angles between 0° and 180° and taking the square root of this quantity. Thus taking angles at intervals of 15° we have

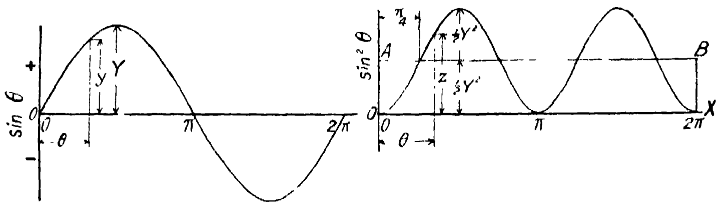
Angle	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Sine	0·259	0·5	0·707	0·866	0·966	1·0	0·966	0·866	0·707	0·5	0·259	0
(Sine)²	0·067	0·25	0·5	0·75	0·933	1·0	0·933	0·75	0·5	0·25	0·067	0

Sum of (sine)² = 6 0, Mean of (sine)² = 6·0/12 = 0·5, $\sqrt{\{\text{Mean of (sine)}^2\}}$ = 0·707

* In this book the square root of the mean value of the squared ordinates of a periodic curve is called the "root-mean-square" (R.M.S.) value. The terms "effective" and "virtual" are also used in practice, the former being widely used in America.

Graphical Representation of R.M.S. Value of a Sine Curve. In Fig. 8 (a) is shown a sine curve and at (b) in the same Fig. is shown a curve the ordinates of which are proportional to the squares of the corresponding ordinates of the sine curve (a). The horizontal line AB is drawn such that its height is equal to the mean ordinate of curve (b), i.e. the area of rectangle $OABX$ = area between curve and the abscissa axis. Hence OA is the mean value of the squared ordinates of the sine curve (a) and $\sqrt{OA}$ is the R.M.S. value of (a).

It can be shown that curve (b) is a sine curve having AB as axis and a frequency twice that of (a). Thus if any ordinate of (a) is given by $y = Y \sin \theta$, then the corresponding ordinate of (b) is $z = y^2 = Y^2 \sin^2 \theta$. Now $\sin^2 \theta$ may be expressed as $(\frac{1}{2} - \frac{1}{2} \cos 2\theta)$,



(a) (b)

FIG. 8. GRAPHICAL DETERMINATION OF R.M.S. VALUE OF A SINE CURVE

which, since $\cos \theta = - \sin (\theta - \frac{1}{2}\pi)$, reduces to $[\frac{1}{2} + \frac{1}{2} \sin (2\theta - \frac{1}{2}\pi)]$. Hence $z = \frac{1}{2} Y^2 + \frac{1}{2} Y^2 \sin (2\theta - \frac{1}{2}\pi)$. Now $\frac{1}{2} Y^2 \sin (2\theta - \frac{1}{2}\pi)$ is a sine curve having a maximum ordinate $\frac{1}{2} Y^2$, a frequency twice that of curve (a), its zero at $2\theta = 90°$, or $\theta = 45°$, and its axis at a distance $\frac{1}{2} Y^2$ above the abscissa axis. Obviously the mean ordinate of this curve, taken over a cycle, is zero. Therefore the mean value of curve (b) is $\frac{1}{2} Y^2$, whence the R.M.S. value of (a) is $(1/\sqrt{2})Y = (1/\sqrt{2}) \times$ maximum ordinate of (a).

Examples. Give the R.M.S. value and frequency of the following—

(a) $\sqrt{(R^2 + X^2)} \sin (\omega t + \frac{1}{2}\pi)$;

(b) $(A + B) \sin (3\omega t - a)$;

(c) $A \sin \omega t + B \cos \omega t$;

(d) $141{\cdot}4 \sin 377t$.

The maximum value of the quantities (a), (b), (d) may be written down directly, but that of quantity (c) must be deduced as follows—

Multiply and divide each term by $\sqrt{(A^2 + B^2)}$ thus

$$\sqrt{(A^2 + B^2)} \frac{A}{\sqrt{(A^2 + B^2)}} \sin \omega t + \sqrt{(A^2 + B^2)} \frac{B}{\sqrt{(A^2 + B^2)}} \cos \omega t$$

Now if $B/A = \tan \varphi$, $A/\sqrt{(A^2 + B^2)} = \cos \varphi$, and $B/\sqrt{(A^2 + B^2)} = \sin \varphi$. Hence expression ($c$) may be written

$$\sqrt{(A^2 + B^2)} \{\sin \omega t \cos \varphi + \cos \omega t \sin \varphi\} = \sqrt{(A^2 + B^2)} \sin(\omega t + \varphi)$$

Therefore the R.M.S. values and frequencies are

Quantity.	R.M.S. Value.	Frequency.
$\sqrt{(R^2 + X^2)} \sin(\omega t + \frac{1}{4}\pi)$	$\sqrt{\frac{1}{2}(R^2 + X^2)} = 0{\cdot}707\sqrt{(R^2+X^2)}$	$\omega/2\pi$
$(A + B)\sin(3\omega t - \alpha)$	$(A + B)/\sqrt{2} = 0{\cdot}707\,(A + B)$	$3\omega/2\pi$
$A \sin \omega t + B \cos \omega t$	$\sqrt{\frac{1}{2}(A^2 + B^2)} = 0{\cdot}707\sqrt{(A^2 + B^2)}$	$\omega/2\pi$
$141{\cdot}4 \sin 377t$	$141{\cdot}4/\sqrt{2} = 100$	$377/2\pi = 60$

Practical Importance of R.M.S. Value of Alternating Quantities. The heat produced in a conductor carrying a current is proportional to the square of the current. Hence, if i denotes the instantaneous value of the current in amperes, R, the resistance of the conductor in ohms, the energy (dW) expended in heat during the time dt is

$$dW = i^2R\,.\,dt \text{ joules}$$

and the mean rate at which energy is expended (i.e. the mean heating effect) during the time T is

$$P = \int_0^T \frac{dW}{T} = \frac{1}{T}\int_0^T i^2R\,.\,dt$$

If the current follows a sine law

$$i = I_m \sin(2\pi/T)t$$

the mean heating effect during a period (T) is

$$P = \frac{1}{T}\int_0^T RI_m^{\,2}\sin^2(2\pi/T)t\,.\,dt$$

$$= \frac{RI_m^{\,2}}{T}\int_0^T \sin^2\left(\frac{2\pi}{T}\right)t\,.\,dt$$

$$= \tfrac{1}{2}RI_m^{\,2} = R(I_m/\sqrt{2})^2 = RI^2$$

where $I(= I_m/\sqrt{2})$ is the R.M.S. value of the current.

Thus the mean heating effect due to a sinusoidal alternating current of maximum value I_m is the same as that due to a continuous current equal to $0{\cdot}707\,I_m$.

The R.M.S. value of an alternating current is therefore of considerable importance in practice. Moreover, as will be shown later in Chapter XVI, the indications of ammeters and voltmeters give R.M.S. values of current and E.M.F. respectively. In commercial electrical engineering when values of current and E.M.F. are specified, these values always refer to the R.M.S. values.

Form-factor of a Sine Curve. The form-factor of a periodic curve

is the ratio (R.M.S. value/arithmetic mean value). Hence for a sine curve the form-factor (K_f) is

$$K_f = \frac{1}{\sqrt{2}} \div \frac{2}{\pi} = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11$$

A knowledge of the form-factor of a curve therefore enables the R.M.S. value to be obtained from the arithmetic mean value, or *vice versâ*.

Example. A conductor rotates at a uniform speed of n r.p.s. in a two-pole magnetic field. The flux per pole is Φ lines and the flux distribution is sinusoidal. Obtain an expression for the R.M.S. value of the E.M.F. generated in the conductor.

Now the mean E.M.F. generated in the conductor is equal to (flux cut per second $\times\ 10^{-8}$), and since the conductor cuts the flux (Φ) twice in one revolution, the mean E.M.F. (E_{av}) is

$$E_{av} = 2\,\Phi\, n \times 10^{-8} \text{ volts.}$$

But since the velocity is constant the E.M.F. is, at any instant, proportional to the flux density of the field in which the conductor is moving. Therefore the E.M.F. follows a sine law and its R.M.S. value (E) is

$$\begin{aligned} E &= 1{\cdot}11\, E_{av} = 2{\cdot}22\,\Phi\, n \times 10^{-8} \text{ volts} \\ &= 2{\cdot}22\,\Phi f \times 10^{-8} \text{ volts,} \end{aligned}$$

since the frequency, f, is equal to n.

Amplitude- or Peak-factor of a Sine Curve. The amplitude-factor of a periodic curve is the ratio (maximum value/R.M.S.value). Hence for a sine curve the amplitude-factor (K_a) is

$$K_a = 1 \div (1/\sqrt{2}) = \sqrt{2} = 1{\cdot}414$$

The amplitude-factor is used only in connection with E.M.F. waves and is of importance in insulation testing and high-voltage apparatus, as the maximum stress to which the insulation is subjected is proportional to the maximum or peak value of the E.M.F. A knowledge of the peak value of the E.M.F. is also necessary when measuring iron losses, as the iron loss depends upon the maximum value of the flux.

If E is the R.M.S. value of the applied E.M.F. as given by the reading of a voltmeter and K_a is the amplitude-factor, the maximum E.M.F. is

$$E_m = K_a E$$

CHAPTER II

FORMS OF REPRESENTATION

THE study of alternating-current phenomena is considerably simplified by assuming the quantities to vary sinusoidally with respect to time; in fact, the elementary principles of alternating-current circuits can only be developed in an intelligible manner by making this assumption, which, in many cases, is justifiable. Moreover, the sine curve is, in practice, considered as the standard wave-form, and deviations therefrom are classed as distorted waves.

In view of the importance of the sine wave in alternating-current engineering we shall devote the present chapter to a consideration of the various methods of representating sinusoidal quantities. These methods may be classified as follows—

I. Graphical methods.	(1) Rectangular co-ordinates.
	(2) Polar co-ordinates.
	(3) Vector diagrams.
II. Analytical methods.	(4) Trigonometric functions.
	(5) Complex algebra.

REPRESENTATION BY RECTANGULAR CO-ORDINATES

In this method time is measured horizontally, as abscissae, and instantaneous values of the alternating quantity are measured vertically, as ordinates. Examples are given in Chapter I, Figs. 1, 3, 5, 7, 8.

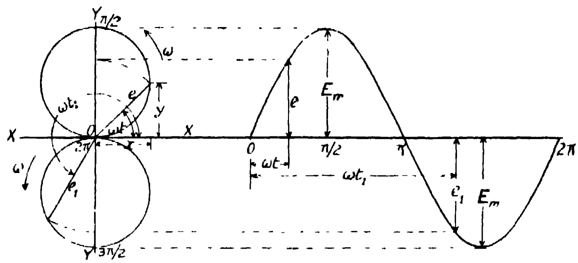
The method can be also employed for representing non-sinusoidal quantities.

REPRESENTATION BY POLAR CO-ORDINATES

In this method the alternating quantity is represented by a rotating line—called a radius-vector—of variable length. Time is measured by the *angle* between this line and a reference axis. The *length* of the line at any position represents the instantaneous value of the quantity at the time corresponding to this position. The fixed point about which the line rotates is called the *pole*. The positive direction of rotation is counter-clockwise, and one revolution corresponds to one cycle of the alternating quantity

When this method of representation is applied to sinusoidal

quantities the locus of the free end of the radius-vector forms two circles (one corresponding to each half-cycle) which touch each other, the line joining the centres and point of contact passing through the pole. The diameter of each circle is equal to the maximum value of the alternating quantity. If time is measured from the horizontal, or (X), axis and the zero value of the quantity occurs at zero time, the centres of the circles lie in the vertical, or (Y), axis as shown in Fig. 9 (*a*), which is a polar representation of the sine curve of Fig. 3, reproduced in Fig. 9 (*b*).



(*a*) (*b*)

FIG. 9. POLAR REPRESENTATION OF A SINE CURVE

Proof. Let the equation to the alternating quantity be

$$e = E_m \sin \omega t.$$

Then the length of the rotating radius-vector at any instant (t) is equal to e, and the angle between it and the axis of reference is equal to ωt. Let x, y, be the co-ordinates of the free end of the vector referred to rectangular axes XOX, YOY, passing through the pole O (see Fig. 9). Then

$$x = e \cos \omega t, \qquad y = e \sin \omega t,$$

$$x^2 + y^2 = e^2 \cos^2 \omega t + e^2 \sin^2 \omega t = e^2 = e(E_m \sin \omega t) = E_m\, y.$$

Hence $x^2 + y^2 - E_m\, y = 0.$

This is the equation to a circle, referred to rectangular coordinates, of radius $\frac{1}{2} E_m$, having its centre in the vertical (Y) axis at a distance $\frac{1}{2} E_m$ from the origin (i.e. the circle is tangential to the horizontal (X) axis). Therefore the locus of the free end of the rotating vector is a circle.

If the zero value of the quantity does not occur at zero time the line of centres is displaced from the vertical by an angle corresponding to the time at which the zero value of the quantity occurs. *Lead* is represented by a displacement in the clockwise direction (see Fig. 10 (*a*)) ; *lag* is represented by a displacement in the counter-clockwise direction (see Fig. 10 (*b*)).

Applications of Polar Co-ordinates to the Representation of Non-sinusoidal Quantities. Non-sinusoidal quantities may also be represented by polar co-ordinates, but in this case the locus of the rotating radius-vector will form two irregular figures, which will be

symmetrical if the two half-waves of the alternating quantity are symmetrical. Fig. 11 shows the non-sinusoidal curve of Fig. 1 plotted in polar co-ordinates.

In the case of symmetrical waves the equivalent sine wave—i.e. the sine wave which has the same R.M.S. value and frequency

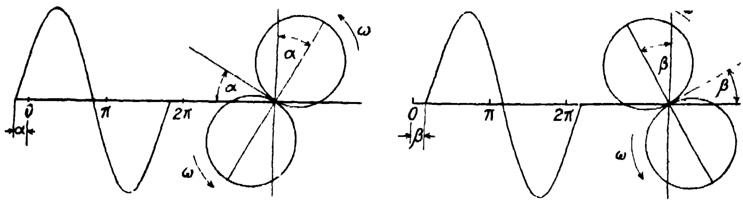


FIG. 10. REPRESENTATION OF LEAD (*a*) AND LAG (*b*) BY POLAR CO-ORDINATES

as the non-sinusoidal wave—is obtained by drawing a circle having an area equal to that of one of the irregular figures. The diameter

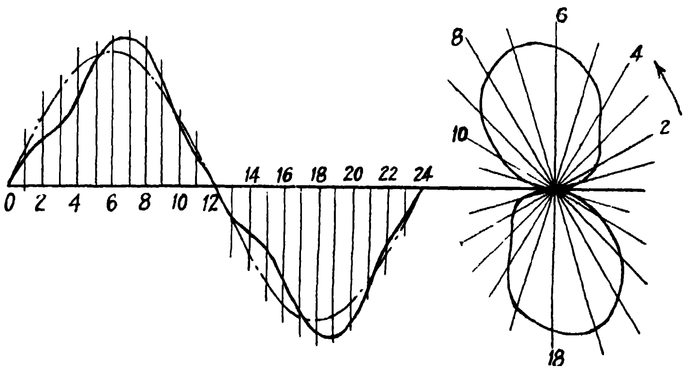


FIG. 11. REPRESENTATION OF NON-SINUSOIDAL CURVE BY POLAR CO-ORDINATES. THE CHAIN-DOTTED CURVE IS THE EQUIVALENT SINE CURVE

of this circle gives the maximum value of the equivalent sine wave and hence the R.M.S. value of the distorted wave is given by

$$(1/\sqrt{2}) \times \text{diameter of circle, or}$$

$$\sqrt{2} \times \text{radius of circle.}$$

Proof. Let the value of the periodic quantity at time t be denoted by e and its phase by ωt (see Fig. 9). Then the increase in polar area for an increment of time dt (i.e. an increment $d\omega t$ in phase) is

$$\tfrac{1}{2}e.e\,d\omega t = \tfrac{1}{2}e^2\,d\omega t.$$

Hence the area of the polar diagram corresponding to half a period of the distorted wave is

$$\int_0^{\pi} \tfrac{1}{2}\, e^2\, d\omega t = \int_0^{\frac{1}{2}T} \tfrac{1}{2}\, e^2 d\left(\frac{2\pi}{T}\right) t = \frac{\pi}{T}\int^{\frac{1}{2}T} e^2\, dt$$

Let r denote the radius of circle which has an area equal to that of the polar diagram. Then

$$\pi r^2 = \frac{\pi}{T}\int^{\frac{1}{2}T} e^2\, dt$$

and $$r\sqrt{2} = \sqrt{\left(\frac{1}{\frac{1}{2}T}\int_0^{\frac{1}{2}T} e^2\, dt\right)} = \text{R.M.S. value of distorted wave.}$$

Representation by Vector Diagrams

A vector diagram is a graphical representation of one or more vector quantities, a vector quantity being a physical quantity which has *direction* as well as magnitude. Examples of vector quantities are force, velocity, magnetic flux, E.M.F., current, ampere-turn.

Such quantities are only completely specified when particulars of their magnitudes, direction, and sense are given.

Vector quantities are represented graphically by straight lines called *vectors*. The *length* of the line represents the *magnitude* of the quantity; the *inclination* of the line with respect to some axis of reference denotes the *direction* of the quantity, and an *arrow-head*, usually placed at one end of the line, indicates the *sense* in which the quantity acts.

In electrical engineering we may require to represent a number of vector quantities in the same vector diagram; it is necessary, therefore, to adhere to a system of nomenclature in order that the several vectors may be readily distinguished from one another. The vector nomenclature in this treatise is as follows*—

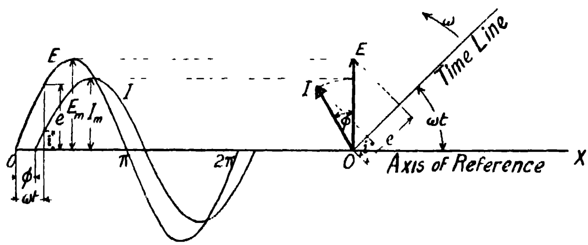
E.M.F. vectors are represented by an ordinary arrow-head— $\longrightarrow$
Flux vectors are represented by a double arrow-head— $\twoheadrightarrow$
Ampere-turn vectors are represented by a solid arrow-head— $\longrightarrow\!\!\blacktriangleright$
Current vectors are represented by a closed arrow-head— $\longrightarrow\!\!\triangleright$

With alternating quantities varying harmonically, two methods of vector representation may be adopted, viz. (1) the polar vector diagram, (2) the crank vector diagram.

Polar Vector Diagram. In this diagram a fixed vector, representing the maximum value of the quantity, occupies a definite

* This nomenclature is the same as that employed in vector diagrams in the author's *Electric Traction* and *Electric Motors and Control Systems*. (Pitman.)

position relative to an axis of reference, one end of the vector being located at the origin. *Instantaneous values* of the quantity are represented by the projections of the vector on a rotating line (called the "time line") which rotates, in a plane containing the vector, with uniform angular velocity. The counter-clockwise direction of rotation is positive, and *time* is measured from the horizontal axis in the first quadrant. Hence if the zero value of the quantity occurs at zero time the fixed vector lies in the vertical axis, but otherwise this vector will be displaced from the vertical axis by an angle equal to the value of the time-angle at which zero value occurs; *lead* being represented by an angle measured in the clockwise direction, and *lag* by an angle in the counter-clockwise



(a) (b)

Fig. 12. Polar Vector Diagram (*b*) for the Sinusoidal E.M.F. and Current Represented by (*a*)

direction. A positive phase-angle must be measured in the clockwise direction because if one quantity is leading relative to another quantity, the revolving time line must meet the vector of the former before that of the latter. Fig. 12 shows the polar vector diagram for a current, I, lagging φ with respect to an E.M.F. E.

Crank Vector Diagram. In this diagram a vector, equal to the maximum value of the quantity, rotates with uniform angular velocity, ω, about a fixed point which is the origin of rectangular axes. *Time* is measured from the horizontal axis in the first quadrant, the counter-clockwise direction of rotation being positive. *Instantaneous values* of the quantity are represented by the projections of the vector on the vertical axis. *Phase* is given by the angle, measured in the counter-clockwise direction, which the rotating vector makes with the time axis. Hence the phase difference between two quantities is represented by the angle between their vectors, an angle measured in the counter-clockwise (+) direction denoting *lead*, and one measured in the opposite

direction denoting *lag*. Fig. 13 shows a crank vector diagram representing the same conditions as the polar diagram of Fig. 12.

Polar and Crank Vector Diagrams Compared. In both polar and crank vector diagrams *time* is measured by the angle, from the horizontal axis, in the counter-clockwise (+) direction, but *phase difference* is measured in the clockwise direction in the polar diagram and in the counter-clockwise direction in the crank diagram. Thus a vector representing a lagging quantity is shown in one diagram on one side of the vector of reference and in the other diagram on the opposite side of the latter. Hence when the complex algebraic, or symbolic, method of representation is employed ambiguity in signs is likely to occur if the polar and crank

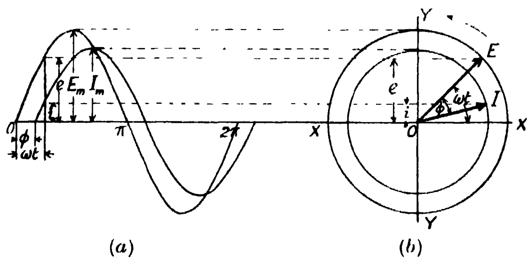


FIG. 13. CRANK VECTOR DIAGRAM (*b*) FOR THE SINUSOIDAL E.M.F. AND CURRENT REPRESENTED BY (*a*)

vector diagrams are used indiscriminately. To remove this ambiguity the crank vector diagram, in which alternating quantities are represented by rotating vectors, has been adopted by the International Electro-technical Commission as the standard form of representation for electrical vector quantities. All vector diagrams and complex algebraic expressions in this book conform to this standard, it being understood that such diagrams and expressions are used solely in connection with quantities varying sinusoidally.

Addition and Subtraction of Vector Quantities. In problems connected with alternating-current circuits we may be concerned with a number of E.M.Fs. or currents of the same frequency but of different phases, and may wish to obtain the resultant E.M.F. or current. The quantities, if sinusoidal, may be represented by a number of rotating vectors having a common axis of rotation and displaced from one another by invariable angles which are equal to the phase differences between the respective quantities. The instantaneous value of the resultant E.M.F., or current, is given by the algebraic sum of the projections of the vectors on the vertical

axis. The maximum value of the resultant is obtained by compounding the several vectors according to the parallelogram and polygon laws.

The *parallelogram law of vectors* states—

If two co-planar vectors, not lying in the same straight line, meet at a point and a parallelogram be constructed having these vectors as adjacent sides, then their resultant is given in magnitude and direction by the diagonal which passes through the point of intersection of the vectors.

The *polygon law of vectors* states—

If a number of co-planar vectors, not lying in the same straight line, meet at a point and an open polygon be constructed having one side formed by one of the vectors and the remaining sides equal and parallel to the remaining vectors, taken in order, then their resultant is given in magnitude and *reversed* direction by the closing side of the polygon.

To prove these laws it is necessary to show that the sum of the vertical projections of the vectors at a given instant is equal to the vertical projection of their resultant at that instant. Thus in the case of the parallelogram of vectors let OA, OB, Fig. 14 (*a*), represent two rotating vectors having a phase difference φ, and let OC—the diagonal of the parallelogram $OACB$—represent their resultant. The vertical projections of OA, OB, OC are given by Oa, Ob, Oc, respectively.

From Fig. 14 we have

$$Oc = Oa + ac.$$

But ac is the vertical projection of AC, and as AC is equal and parallel to OB, therefore $ac = Ob$. Hence

$$Oc = Oa + ac = Oa + Ob$$

and the proposition is proved.

The polygon law of vectors may be proved in a similar manner.

If the *vector difference* of the quantities is required, the vector representing one of the quantities is reversed, and this reversed vector is compounded with the other vector according to the parallelogram law. For example, if in Fig. 14 (*b*), OA and OB are two vectors and their difference $(A - B)$ is required, the vector OB is reversed, i.e. another vector OB' is drawn equal to, but in the reverse direction to OB, and OA and OB' are compounded according to the parallelogram law, thus giving the resultant OD. Then OD represents the difference of the vector quantities A, B. Similarly

if the difference $B - A$ is required the vector OA is reversed and the resultant OE is obtained.

Vector Representation of Root-mean-square Values of Alternating Quantities. In alternating-current circuits we are chiefly concerned with R.M.S. values of current and E.M.F., and their phase differences. Provided that the law of variation of the quantities is

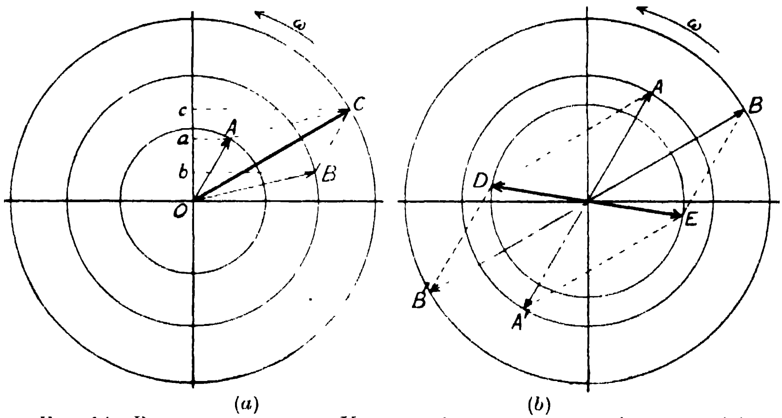


FIG. 14. PARALLELOGRAM OF VECTORS APPLIED TO THE ADDITION (*a*) AND SUBTRACTION (*b*) OF TWO VECTORS *OA*, *OB*

known, a knowledge of their instantaneous values throughout a period is unnecessary and is of little interest in practice. Now with sinusoidal quantities the R.M.S. value bears a fixed relation to the maximum value. Hence R.M.S. values may be determined directly from the vector diagram by a suitable change of scale Therefore, when the resultant of a number of currents, or E.M.Fs., is to be determined we may represent the quantities by *fixed* vectors, the lengths of which represent the R.M.S. values of the quantities and the angular displacements of the vectors with respect to one vector, or quantity of reference, represent their phase differences with respect to this quantity. *Lead* is represented by an angle in the counter-clockwise direction, and *lag* is represented by an angle in the clockwise direction.

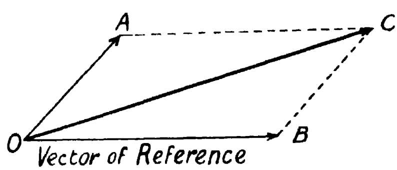


FIG. 15. VECTOR DIAGRAM FOR CONSTANT QUANTITIES

Thus the crank diagram of Fig. 14 (*a*) may be replaced by the stationary vector diagram of Fig. 15, which is virtually a diagram

for constant quantities, as would be employed in connection with applied mechanics.

Example. Represent the following quantities by vectors—

$$5 \sin (2\pi ft - 1)\ ;\ 3 \cos (2\pi ft + 1)\ ;\ 2 \sin (2\pi ft + 2\cdot 5)\ ;\ 4 \sin (2\pi ft - 1).$$

Add these vectors together and express the result in the form—

$$A \sin (2\pi ft + \varphi)\ . \quad \text{[C. and G.]}$$

Observe that all the quantities have the same frequency, f. Hence the vectors representing them may be added together. Before doing so, however, it will be convenient to express all the quantities as sine functions. Thus the second quantity $3 \cos (2\pi ft) + 1)$,

may be written $3 \sin (2\pi ft + 1 + \frac{1}{2}\pi) = 3 \sin (2\pi ft + 1 + 1\cdot 57)$
$= 3 \sin (2\pi ft + 2\cdot 57)$

The maximum value of each quantity and its phase difference with respect to the quantity of reference, viz. $X \sin 2\pi ft$, is given in the following table—

Quantity.	Maximum value.	Phase difference with respect to $X \sin 2\pi ft$.	
		Radians.	Degrees.
(*a*) $5 \sin (2\pi ft - 1)$	5	-1	$-57\cdot 3$
(*b*) $3 \cos (2\pi ft + 1)$	3	$+2\cdot 57$	$+147\cdot 2$
(*c*) $2 \sin (2\pi ft + 2\cdot 5)$	2	$+2\cdot 5$	$+143\cdot 2$
(*d*) $4 \sin (2\pi ft - 1)$	4	-1	$-57\cdot 3$

The vector diagram is shown in Fig. 16, in which OX represents the vector of reference, and OA, OB, OC, OD represent the four quantities a, b, c, d respectively. The resultant, or sum, of these vectors is obtained by compounding them according to the polygon law and is represented by OG, which is 4·81 units long and lags 82°, or $(82/57\cdot 3) = 1\cdot 43$ radians, with respect to the vector of reference. Hence the sum of the quantities is

$$4\cdot 81 \sin (2\pi ft - 1\cdot 43).$$

Representation by Trigonometrical Functions

We have already shown that in the case of quantities varying harmonically the instantaneous values are proportional to the sine, or in some cases to the cosine, of the time-angle. Hence in the analytical treatment of these quantities they are expressed as a trigonometrical function (sine or cosine) of the time-angle, e.g. $e = E_m \sin \omega t\ ;\ \ i = I_m \sin (\omega t - \varphi)\ ;\ \ i = I_m \cos \omega t.$

This method is particularly suitable for dealing with instantaneous values, as it enables the relationship between current and E.M.F.

for the simpler circuits to be deduced analytically and the quantitative relations between these quantities to be calculated accurately as shown in the following chapters. By means of Fourier's series the method can be extended to the calculations of circuits when the wave-form of the supply E.M.F. is non-sinusoidal (see Chapter XIV).

As both maximum and R.M.S. values can, in the case of sinusoidal quantities, readily be obtained from the instantaneous value, the trigonometrical method is largely employed in the calculations of electric circuits and is developed further in the following chapters.

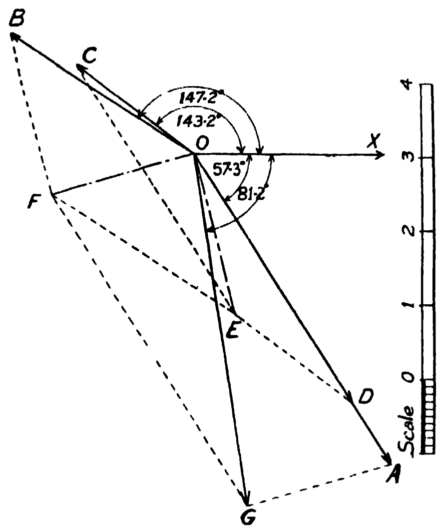


FIG. 16. GRAPHICAL SOLUTION AND VECTOR DIAGRAM FOR EXAMPLE (p. 22)

REPRESENTATION BY COMPLEX ALGEBRA

Symbolic Notation. This form of representation is an algebraic method of representing vector quantities and enables the operations which are carried out graphically in a vector diagram to be performed analytically. It is, moreover, of considerable value in the solution of alternating-current problems, as the results obtained are of the same order of accuracy as those obtained by trigonometrical methods although the calculations are usually simpler and less tedious.

The basis of the method is the representation of vector quantities by their components in the direction of arbitrarily chosen axes of

reference. For example, the vector OE_1, Fig. 17, may be completely specified by stating that its horizontal component is $\boldsymbol{a}_1$ and its vertical component is $\boldsymbol{b}_1$. Instead of stating these conditions verbally we may express them symbolically, thus

$$\underset{\cdot}{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$$

where j indicates that the component $\boldsymbol{b}_1$ is perpendicular to the component $\boldsymbol{a}_1$.

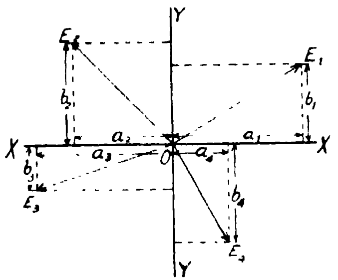


FIG. 17. COMPONENTS OF VECTORS ALONG AXES OF REFERENCE

This method of representation is similar to that adopted in analytical geometry for representing co-planar points in terms of their rectangular co-ordinates but it admits of more extensive application owing to the two components, or co-ordinates, being connected by an algebraic sign, the use of which requires the introduction of the distinguishing symbol j to prevent these components being treated as ordinary algebraic quantities.

Axes of Reference. The horizontal axis, XOX', Fig. 17, is called the *real* or *in-phase axis*. Components in this axis are called *in-phase components*: they are positive when measured to the right of O, and negative when measured in the reverse direction.

The vertical axis, YOY', Fig. 17, is called the *imaginary*, or *quadrature, axis*. Components in this axis are called *quadrature components*: they are positive when measured upwards and negative when measured downwards. These components are distinguished from those in the horizontal axis by being compounded with the symbol j.

For example, in Fig. 17, the vector OE_1 is represented symbolically

by the expression $\underset{\cdot}{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$,
the vector OE_2 by $\underset{\cdot}{E}_2 = -\boldsymbol{a}_2 + j\boldsymbol{b}_2$,
the vector OE_3 by $\underset{\cdot}{E}_3 = -\boldsymbol{a}_3 - j\boldsymbol{b}_3$,
and the vector OE_4 by $\underset{\cdot}{E}_4 = \boldsymbol{a}_4 - j\boldsymbol{b}_4$.

Similarly, vectors having directions along the axes of reference are represented by

$$\underset{\cdot}{I}_1 = \boldsymbol{g}_1 \ : \quad \underset{\cdot}{I}_2 = j\boldsymbol{g}_2 \ : \quad \underset{\cdot}{I}_3 = -\boldsymbol{g}_3 \ : \quad \underset{\cdot}{I}_4 = -j\boldsymbol{g}_4$$

We may therefore consider j as an operator, the application of

which to a vector, or a vector component, causes a rotation through 90° in the positive (counter-clockwise) direction. The double application of the operator rotates a vector through 180° and reverses its sense, e.g. the double application of j to the vector $\dot{I} = \boldsymbol{g}$ gives the vector $\dot{I}' = jj\boldsymbol{g} = -\boldsymbol{g}$. Hence operation by jj or j^2 is equivalent to multiplication by -1, so we may regard the effect of j^2 as being equivalent numerically to -1 and write

$$j^2 (\ ) = -1 (\ )$$

Whence

$$j (\ ) = \sqrt{-1} (\ )$$

i.e. j has numerically the effect of the imaginary root of -1.*

In ordinary algebra quantities having the factor $\sqrt{-1}$ are called imaginary. This term may also be applied to vector components in the vertical axis, since when considering a uni-directional quantity such as a length, measured along a given axis, any measurement taken along a perpendicular axis has no physical meaning and must therefore be regarded as imaginary. But when dealing with electrical quantities the term "quadrature component" is preferable.

Complex Quantities. A quantity which is represented by two components along perpendicular axes is called a complex quantity. In this book complex quantities, i.e. vector quantities and complex numbers, are denoted by dotted italic capitals, thus $\dot{E}$; vector components are denoted by bold italic lower-case characters, thus $\boldsymbol{a}$, $\boldsymbol{b}$; simple magnitudes or scalar quantities are denoted by upper and lower-case italic characters, thus E, I, a, b.

For example, the vector OE_1, Fig. 17, is represented by the complex quantity

$$\dot{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1 ;$$

its *absolute value*, or magnitude, is given by

$$E_1 = \sqrt{(a_1^2 + b_1^2)} ;$$

and its *phase*, or inclination, (φ), to the horizontal axis is given by

$$\varphi = \tan^{-1} b_1/a_1 .$$

φ is also called the *argument* of the complex quantity.

Again, if a vector in the horizontal axis is represented by

$$\dot{I}_1 = \boldsymbol{g}_1 + j0 = \boldsymbol{g}_1$$

its absolute value (I_1) is

$$I_1 = \sqrt{(g_1^2 + 0^2)} = g_1$$

and its argument is zero.

* Mathematicians denote $\sqrt{-1}$ by ι, but in electrical engineering it is necessary to use j for this quantity, as otherwise confusion might arise between the symbol ι and that (i) used to denote current.

Similarly for other vectors in the axes of reference,

$$\dot{I}_2 = 0 + j\mathbf{g}_2 = j\mathbf{g}_2\,;\; I_2 = \sqrt{(0^2 + g_2{}^2)} = g_2\,;\; \varphi = \tfrac{1}{2}\pi$$
$$\dot{I}_3 = -\mathbf{g}_3 + j0 = -\mathbf{g}_3\,;\; I_3 = \sqrt{(-g_3{}^2 + 0^2)} = g_3\,;\; \varphi = \pi$$
$$\dot{I}_4 = 0 - j\mathbf{g}_4 = -j\mathbf{g}_4\,;\; I_4 = \sqrt{(0^2 + (-g_4)^2)} = g_4\,;\; \varphi = \tfrac{3}{2}\pi(\text{or} - \tfrac{1}{2}\pi)$$

Polar and Exponential Forms of Expressing Complex Quantities. The method of representing a complex number, or quantity, by its rectangular components is called the *rectangular form of representation.* These components may also be expressed in the polar form Thus, in Fig. 17,

$$\boldsymbol{a}_1 = E_1 \cos\varphi, \qquad \boldsymbol{b}_1 = E_1 \sin\varphi\,;$$

whence

$$\dot{E}_1 = E_1\cos\varphi + jE_1\sin\varphi$$
$$= E_1(\cos\varphi + j\sin\varphi)$$

This is one of the polar forms of expressing complex quantities and is usually called the *trigonometrical form.*

The exponential form, $\dot{E}_1 = E_1\varepsilon^{j\varphi}$, where φ is in circular measure, may be derived by expressing $\sin\varphi$ and $\cos\varphi$ in their trigonometrical series and substituting these in the above expression for $\dot{E}_1$. Thus, if φ is in circular measure,

$$\sin\varphi = \varphi - (\varphi^3/3\,!) + (\varphi^5/5\,!) - (\varphi^7/7\,!) + \ldots$$
$$\cos\varphi = 1 - (\varphi^2/2\,!) + (\varphi^4/4\,!) - (\varphi^6/6\,!) + \ldots$$

Hence

$$\dot{E}_1 = E_1\left[\left(1 - \frac{\varphi^2}{2\,!} + \frac{\varphi^4}{4\,!} - \frac{\varphi^6}{6\,!} + \ldots\right) + j\left(\varphi - \frac{\varphi^3}{3\,!} + \frac{\varphi^5}{5\,!} - \frac{\varphi^7}{7\,!} + \ldots\right)\right]$$

$$= E_1\left(1 + j\varphi - \frac{\varphi^2}{2\,!} - j\frac{\varphi^3}{3\,!} + \frac{\varphi^4}{4\,!} + j\frac{\varphi^5}{5\,!} - \ldots\right)$$

$$= E_1\left(1 + j\varphi + \frac{j^2\varphi^2}{2\,!} + \frac{j^3\varphi^3}{3\,!} + \frac{j^4\varphi^4}{4\,!} + \frac{j^5\varphi^5}{5\,!} + \ldots\right)$$

since numerically $j^2 = -1$, $j^4 = +1$, $j^6 = -1$, etc.

Now

$$1 + j\varphi + \frac{j^2\varphi^2}{2\,!} + \frac{j^3\varphi^3}{3\,!} + \frac{j^4\varphi^4}{4\,!} + \frac{j^5\varphi^5}{5\,!} + \ldots$$

is the expansion for $\varepsilon^{j\varphi}$. Hence the former expression for $\dot{E}_1$ may be written

$$\dot{E}_1 = E_1\varepsilon^{j\varphi}$$

where ε (= 2·71828 . .) is the base of natural logarithms.

Instead of employing the exponential factor $\varepsilon^{j\varphi}$ to express the polar form of a vector a proposal has been made* to employ the factor $J^{\varphi/\frac{1}{2}\pi}$, where J is an operator the application of which to a vector rotates the latter through a right angle with respect to the horizontal axis. The index of J denotes the fraction, or multiple, of a right angle through which the vector is rotated, a plus $(+)$ sign denoting rotation in the counter-clockwise, or positive, direction and a minus $(-)$ sign denoting rotation in the clockwise direction. For example, J^m, $m > 0$, signifies a rotation of $\frac{1}{2}\pi m$ radians, or $90m°$, in the counter-clockwise direction; J^{-m}, a rotation of $\frac{1}{2}\pi m$ radians, or $90m°$, in the clockwise direction. Similarly $J^{\frac{1}{2}}$, $J^{\frac{2}{3}}$, $J^{\frac{1}{3}}$, signify rotations, in the counter-clockwise direction, of $\frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4}\pi$, or 45°, $\frac{2}{3} \times \frac{1}{2}\pi = \frac{1}{3}\pi$, or 60°; $\frac{1}{3} \times \frac{1}{2}\pi = \frac{1}{6}\pi$, or 30°, respectively; and J^{-3}, $J^{-\frac{1}{6}}$, J^{-1}, signify rotations, in the clockwise direction, of $3 \times \frac{1}{2}\pi = \frac{3}{2}\pi$, or 270°; $\frac{1}{6} \times \frac{1}{2}\pi = \frac{1}{12}\pi$, or 15°; $\frac{1}{2}\pi$, or 90°, respectively.

This form of representation enables complex quantities to be expressed in simpler mathematical expressions than the exponential form. It can be shown that the two forms are equivalent† and that J follows the ordinary laws of algebra in the same manner as j.

In addition to the above polar forms for expressing a vector quantity a conventional form, $\dot{E} = E \underline{/\pm\varphi}$, is frequently adopted in practice. This form is purely conventional and does not possess the mathematical significance of the other forms: it simply denotes that the vector quantity $\dot{E}$ has an absolute value equal to E, and is inclined at the angle φ to the axis of reference, the plus $(+)$ sign denoting an angle in the counter-clockwise direction and the minus $(-)$ sign denoting an angle in the clockwise direction. This notation possesses the advantage that numerical results may be expressed directly in the polar form without the introduction of algebraic symbols. For example, a vector $\dot{E}$, of absolute value equal to 100, and inclined at an angle of 60° to the horizontal axis is represented by

$$\dot{E} = 100\underline{/60°}$$

* "Application of a polar form of complex quantities to the calculation of alternating-current phenomena," by Prof. N. S. Diamant. *Transactions of the American Institute of Electrical Engineers*, V. 35, p. 957.

† From De Moivre's theorem we have the identities

$$\cos\varphi + j\sin\varphi \equiv \cos\tfrac{1}{2}\pi m + j\sin\tfrac{1}{2}\pi m \equiv (\cos\tfrac{1}{2}\pi + j\sin\tfrac{1}{2}\pi)^m = J^m,$$

i.e. the operator $J^m(m > 0)$ is identical with $(\cos\varphi + j\sin\varphi)$ where $\varphi = \frac{1}{2}\pi m$.

Summarizing, we have the following ways of expressing symbolically complex numbers and vector quantities—

(1) the rectangular form $\dot{E} = \boldsymbol{a} \pm j\boldsymbol{b}$
(2) the trigonometrical form $\dot{E} = E(\cos\varphi \pm j\sin\varphi)$
(3) the exponential form $\dot{E} = E\varepsilon^{\pm j\varphi}$
(4) the polar form $\dot{E} = EJ^{\pm(\varphi/\frac{1}{2}\pi)}$
(5) the conventional form $\dot{E} = E\underline{/\varphi}$

Conjugate Complex Quantities. When two conjugate quantities have the same absolute values and arguments which are equal in magnitude but of opposite sign, they are called *conjugate quantities* Examples: $\boldsymbol{a} + j\boldsymbol{b}$, $\boldsymbol{a} - j\boldsymbol{b}$; $-\boldsymbol{a} + j\boldsymbol{b}$, $-\boldsymbol{a} - j\boldsymbol{b}$; $E^{j\varphi}$, $E^{-j\varphi}$, $EJ^{(\varphi/\frac{1}{2}\pi)}$, $EJ^{-(\varphi/\frac{1}{2}\pi)}$; $E\underline{/\varphi}$, $E\underline{/-\varphi}$.

Two conjugate complex quantities therefore correspond to two points, in the plane of the co-ordinates, which are images of each other with respect to the horizontal axis.

Addition and Subtraction of Complex Quantities. These operations are effected by the same rules which govern their application to ordinary algebraic quantities, but the in-phase and quadrature components must be treated separately. For example, the *sum of two complex quantities*—$\dot{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$ and $\dot{E}_2 = \boldsymbol{a}_2 + j\boldsymbol{b}_2$ is given by the complex quantity

$$\begin{aligned}\dot{E} = \dot{E}_1 + \dot{E}_2 &= (\boldsymbol{a}_1 + j\boldsymbol{b}_1) + (\boldsymbol{a}_2 + j\boldsymbol{b}_2)\\ &= \boldsymbol{a}_1 + \boldsymbol{a}_2 + j(\boldsymbol{b}_1 + \boldsymbol{b}_2)\end{aligned}$$

The numerical value of this quantity is given by

$$E = \sqrt{[(a_1 + a_2)^2 + (b_1 + b_2)^2]}$$

and its inclination (φ) to the axis of reference is given by

$$\varphi = \tan^{-1}(b_1 + b_2)/(a_1 + a_2)$$

A graphical representation of the process is shown in Fig. 18(a). Similarly with more than two quantities we have

$$\begin{aligned}\dot{E} = \dot{E}_1 + \dot{E}_2 + \dot{E}_3 + \ldots &= (\boldsymbol{a}_1 + j\boldsymbol{b}_1) + (\boldsymbol{a}_2 + j\boldsymbol{b}_2)\\ &\quad + (\boldsymbol{a}_3 + j\boldsymbol{b}_3) + \ldots\\ &= \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 + \ldots\\ &\quad + j(\boldsymbol{b}_1 + \boldsymbol{b}_2 + \boldsymbol{b}_3 + \ldots)\end{aligned}$$

$$E = \sqrt{[(a_1 + a_2 + a_3 + \ldots)^2 + (b_1 + b_2 + b_3 + \ldots)^2]}$$

$$\varphi = \tan^{-1}(b_1 + b_2 + b_3 + \ldots)/(a_1 + a_2 + a_3 + \ldots)$$

The result is obviously independent of the order of the quantities.

If the polar form of representation is employed we have

$$\underset{\bullet}{E} = E_1 J^m + E_2 J^n;$$

where $m = \alpha/\frac{1}{2}\pi$, and $n = \beta/\frac{1}{2}\pi$; α, β being the inclinations, in radians, of the vectors $\underset{\bullet}{E}_1$, $\underset{\bullet}{E}_2$, respectively, to the axis of reference.

The numerical value of this quantity, from the geometry of Fig. 18(a), is

$$E = \sqrt{\{E_1^2 + E_2^2 + 2E_1E_2 \cos(\tfrac{1}{2}\pi n - \tfrac{1}{2}\pi m)\}} \quad [n > m]$$

The inclination (φ) to the axis of reference is given by

$$\varphi = \tan^{-1}\{(E_1 \sin \tfrac{1}{2}\pi m + E_2 \sin \tfrac{1}{2}\pi n)/(E_1 \cos \tfrac{1}{2}\pi m + E_2 \cos \tfrac{1}{2}\pi n)\}$$

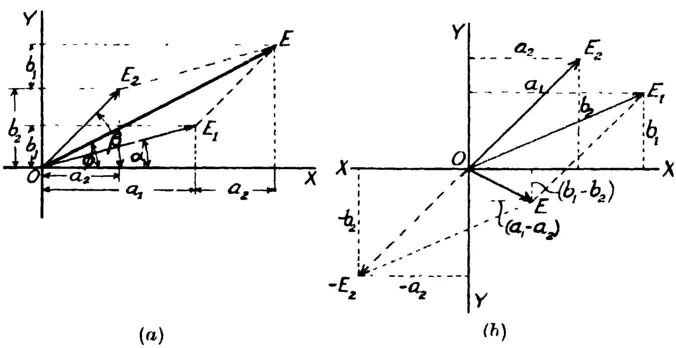


FIG. 18. ADDITION (a) AND SUBTRACTION (b) OF VECTORS BY TREATMENT OF THEIR COMPONENTS ALONG THE AXES OF REFERENCE

When the sum of more than two complex quantities is required it is generally more convenient to employ the rectangular form.

The *difference of two complex quantities*—$\underset{\bullet}{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$, and $\underset{\bullet}{E}_2 = \boldsymbol{a}_2 + j\boldsymbol{b}_2$—is given by the complex quantity

$$\underset{\bullet}{E} = \underset{\bullet}{E}_1 - \underset{\bullet}{E}_2 = (\boldsymbol{a}_1 + j\boldsymbol{b}_1) - (\boldsymbol{a}_2 + j\boldsymbol{b}_2)$$
$$= \boldsymbol{a}_1 - \boldsymbol{a}_2 + j(\boldsymbol{b}_1 - \boldsymbol{b}_2)$$

The numerical value of this quantity is

$$E = \sqrt{[(a_1 - a_2)^2 + (b_1 - b_2)^2]}$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1}(b_1 - b_2)/(a_1 - a_2).$$

A graphical representation of the process is shown in Fig. 18(b). These operations are applicable to vector quantities as well as complex numbers.

Examples. (1) Find the sum and difference of the vectors $\dot{E}_1 = \mathbf{20 + j40}$ and $\dot{E}_2 = \mathbf{15 + j10}$.

The sum is given by the complex expression

$$\dot{E} = \dot{E}_1 + \dot{E}_2 = \mathbf{20 + 15 + j(40 + 10)}$$
$$= \mathbf{35 + j50}$$

Hence $E = \sqrt{(35^2 + 50^2)} = 61$

$\varphi = \tan^{-1} 50/35 = 55°$

The vector difference is given by the complex expression

$$\dot{E}' = \dot{E}_1 - \dot{E}_2 = \mathbf{20 - 15 + j(40 - 10)}$$
$$= \mathbf{5 + j30}$$

Hence $E' = \sqrt{(5^2 + 30^2)} = 30{\cdot}4$

$\varphi' = \tan^{-1} 30/5 = 80{\cdot}6°$

(2) Find the sum of the vectors $\dot{I}_1 = \mathbf{10 - j5}$; $\dot{I}_2 = \mathbf{-2 + j15}$; $\dot{I}_3 = \mathbf{20 + j10}$; $\dot{I}_4 = \mathbf{-4 - j30}$.

The vector sum is given by the complex expression

$$\dot{I} = \dot{I}_1 + \dot{I}_2 + \dot{I}_3 + \dot{I}_4 = \mathbf{(10 - 2 + 20 - 4) + j(-5 + 15 + 10 - 30)}$$
$$= \mathbf{24 - j10}$$

Hence $I = \sqrt{(24^2 + 10^2)} = 26$

$\varphi = \tan^{-1} - 10/24 = -21{\cdot}8°$ or $338{\cdot}2°$

(3) Find the sum of the vectors $\dot{Z}_1 = 10J^{0{\cdot}66}$, $\dot{Z}_2 = 7J^{0{\cdot}26}$

The vector sum is given by

$$\dot{Z} = \sqrt{[10^2 + 7^2 + 2 \times 10 \times 7 \cos\{90(0{\cdot}66 - 0{\cdot}26)\}°]}$$
$$= \sqrt{(100 + 49 + 140 \cos 36°)}$$
$$= \sqrt{262{\cdot}2} = 16{\cdot}2$$
$$\varphi = \tan^{-1}[10 \sin(0{\cdot}66 \times 90)° + 7 \sin(0{\cdot}26 \times 90)°]/[10 \cos(0{\cdot}66 \times 90);$$
$$+ 7 \cos(0{\cdot}26 \times 90)°]$$
$$= \tan^{-1} 11{\cdot}387/11{\cdot}52 = 44°\ 40'$$

(4) Add together the quantities

$5 \sin(2\pi ft - 1)$; $3 \cos(2\pi ft + 1)$; $2 \sin(2\pi ft + 2{\cdot}5)$; $4 \sin(2\pi ft - 1)$

[Note.—The graphical solution to this example is given on p. 23.]

The maximum value of each quantity and its phase difference with respect to an arbitrary quantity of reference—$X \sin 2\pi ft$—are given on p. 22, and from these data we can calculate the rectangular components of each quantity. Thus—

Quantity.		Rectangular components.		Quantity expressed symbolically in rectangular form.
Trigonometrical form.	Conventional form.	Horizontal.	Vertical.	
$5 \sin(2\pi ft - 1)$	$5\underline{/-57{\cdot}3°}$	$5 \cos(-57{\cdot}3)° = 2{\cdot}7$	$5 \sin(-57{\cdot}3)° = -4{\cdot}207$	$\mathbf{2{\cdot}7 - j4{\cdot}207}$
$3 \cos(2\pi ft + 1)$	$3\underline{/147{\cdot}2°}$	$3 \cos 147{\cdot}2° = -2{\cdot}522$	$3 \sin 147{\cdot}2° = 1{\cdot}625$	$\mathbf{-2{\cdot}522 + j1{\cdot}625}$
$2 \sin(2\pi ft + 2{\cdot}5)$	$2\underline{/143{\cdot}2°}$	$2 \cos 143{\cdot}2° = -1{\cdot}6$	$2 \sin 143{\cdot}2° = 1{\cdot}198$	$\mathbf{-1{\cdot}6 + j1{\cdot}198}$
$4 \sin(2\pi ft - 1)$	$4\underline{/-57{\cdot}3°}$	$4 \cos(-57{\cdot}3)° = 2{\cdot}161$	$4 \sin(-57{\cdot}3)° = -3{\cdot}366$	$\mathbf{2{\cdot}161 - j3{\cdot}366}$

Hence the sum is given by the vector quantity

$$\dot{Z} = (2{\cdot}7 - 2{\cdot}522 - 1{\cdot}6 + 2{\cdot}161) + j(-4{\cdot}207 + 1{\cdot}625 + 1{\cdot}198 - 3{\cdot}366)$$
$$= 0{\cdot}739 - j4{\cdot}75$$
$$\therefore \quad Z = \sqrt{(0{\cdot}739^2 + 4{\cdot}75^2)} = 4{\cdot}81$$
$$\varphi = \tan^{-1}(-4{\cdot}75/0{\cdot}739) = -81{\cdot}2°$$

The sum of the above quantities is therefore given by $4{\cdot}81\underline{/-81{\cdot}2°}$.

The results obtained on p. 22 by the graphical construction agree fairly well with these results.

Multiplication and Division of Complex Numbers. These operations are effected by ordinary algebraic rules. For example, the product of the numbers $\dot{X} = \boldsymbol{a}_1 + j\boldsymbol{b}_1$ and $\dot{Y} = \boldsymbol{a}_2 + j\boldsymbol{b}_2$ is given by the complex number $\dot{Z}$, thus

$$\dot{Z} = \dot{X}\dot{Y} = (\boldsymbol{a}_1 + j\boldsymbol{b}_1)(\boldsymbol{a}_2 + j\boldsymbol{b}_2) = \boldsymbol{a}_1\boldsymbol{a}_2 + j\boldsymbol{a}_1\boldsymbol{b}_2 + j\boldsymbol{a}_2\boldsymbol{b}_1 + j^2\boldsymbol{b}_1\boldsymbol{b}_2$$
$$= \boldsymbol{a}_1\boldsymbol{a}_2 - \boldsymbol{b}_1\boldsymbol{b}_2 + j(\boldsymbol{a}_1\boldsymbol{b}_2 + \boldsymbol{a}_2\boldsymbol{b}_1)$$

The absolute value (Z) of the product is

$$Z = \sqrt{[(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2]}$$

and its inclination to the axis of reference is

$$\varphi = \tan^{-1}(a_1b_2 + a_2b_1)/(a_1a_2 - b_1b_2)$$

If the trigonometrical form of representation is employed, and the inclination of $\dot{X}$ and $\dot{Y}$ to the axis of reference are α, β, respectively, then

$$\boldsymbol{a}_1 = X\cos\alpha\,; \quad \boldsymbol{b}_1 = X\sin\alpha\,; \quad \boldsymbol{a}_2 = Y\cos\beta\,; \quad \boldsymbol{b}_2 = Y\sin\beta.$$

Hence,

$$\dot{Z} = \dot{X}\dot{Y} = \boldsymbol{a}_1\boldsymbol{a}_2 - \boldsymbol{b}_1\boldsymbol{b}_2 + j(\boldsymbol{a}_1\boldsymbol{b}_2 + \boldsymbol{a}_2\boldsymbol{b}_1)$$
$$= XY\cos\alpha\cos\beta - XY\sin\alpha\sin\beta + j(XY\cos\alpha\sin\beta + XY\cos\beta\sin\alpha)$$
$$= XY\{\cos(\alpha+\beta) + j\sin(\alpha+\beta)\}$$

This is expressed in the polar form by

$$\dot{Z} = XYJ^{(\alpha+\beta)/\frac{1}{2}\pi}$$

In the exponential form the product is given by

$$\dot{Z} = \dot{X}\dot{Y} = XY\varepsilon^{j(\alpha+\beta)}$$

Thus the product of two complex numbers is a complex number having a magnitude equal to the product of the absolute values of the numbers and an argument equal to the sum of the arguments of the numbers.

It is apparent that the physical meaning of the result is shown better by the polar and exponential forms than by the rectangular form.

The result also holds if one of the numbers is a vector quantity, since in this case the product is another vector quantity. For example, if a sinusoidal alternating quantity, which is represented by a rotating vector, is multiplied by a complex number the result is another alternating quantity of the same frequency but of different magnitude and phase with respect to the original quantity.

If, however, two alternating quantities of the same frequency are multiplied together the result (see p. 70) is an alternating quantity of double frequency.

For example, the product $E \sin \omega t$ and $I \cos \omega t$ is

$$E \sin \omega t \,.\, I \cos \omega t = \tfrac{1}{2} EI \sin 2\omega t$$

The physical meaning of the product is considered in Chapter V.

Division of Complex Numbers and Quantities. The quotient of two complex numbers is another complex number, and that of two vector quantities is a complex number; but if a vector quantity is divided by a complex number the result is another vector quantity of different magnitude and phase to the original vector quantity. For example, the operation on the vector quantity $\underset{\cdot}{E} = \boldsymbol{a} + j\boldsymbol{b}$ by the complex number $\underset{\cdot}{Z} = \boldsymbol{r} + j\boldsymbol{x}$ is given by the vector quantity

$$\underset{\cdot}{I} = \frac{\underset{\cdot}{E}}{\underset{\cdot}{Z}} = \frac{\boldsymbol{a} + j\boldsymbol{b}}{\boldsymbol{r} + j\boldsymbol{x}} = \frac{(\boldsymbol{a} + j\boldsymbol{b})(\boldsymbol{r} - j\boldsymbol{x})}{(\boldsymbol{r} + j\boldsymbol{x})(\boldsymbol{r} - j\boldsymbol{x})} = \frac{(\boldsymbol{a} + j\boldsymbol{b})(\boldsymbol{r} - j\boldsymbol{x})}{r^2 + x^2}$$

$$= \frac{\boldsymbol{ar} + \boldsymbol{bx}}{r^2 + x^2} + j\left(\frac{\boldsymbol{br} - \boldsymbol{ax}}{r^2 + x^2}\right)$$

[Observe that the denominator $\boldsymbol{r} + j\boldsymbol{x}$ is rationalized by introducing the conjugate expression $\boldsymbol{r} - j\boldsymbol{x}$.]

The absolute value of this quantity is

$$I = \sqrt{\left\{\left(\frac{ar + bx}{r^2 + x^2}\right)^2 + \left(\frac{br - ax}{r^2 + x^2}\right)^2\right\}}$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1}(br - ax)/(ar + bx)$$

The physical meaning of the result is shown better when the polar and exponential forms are employed. Thus the operation on the vector quantity $\underset{\cdot}{E} = E\varepsilon^{j\varphi_1}$ by the complex number $\underset{\cdot}{Z} = Z\varepsilon^{j\varphi_2}$ is given by the vector quantity

$$\underset{\cdot}{I} = \frac{\underset{\cdot}{E}}{\underset{\cdot}{Z}} = \frac{E\varepsilon^{j\varphi_1}}{Z\varepsilon^{j\varphi_2}} = \frac{E}{Z}\,\varepsilon^{j(\varphi_1 - \varphi_2)}$$

Similarly, for the polar form, we have

$$\dot{I} = \frac{E\,J^{(\varphi_1/\frac{1}{2}\pi)}}{Z\,J^{(\varphi_2/\frac{1}{2}\pi)}} = \frac{E}{Z}\,J^{(\varphi_1-\varphi_2)/\frac{1}{2}\pi}$$

In the special case when the argument of the vector quantity is zero, the result is given by

$$\dot{I} = \frac{E}{\dot{Z}} = \frac{E}{Z\varepsilon^{j\varphi_2}} = \frac{E}{Z}\,\varepsilon^{-j\varphi_2}$$

or in the polar form by

$$\dot{I} = \frac{E}{Z}\,J^{-(\varphi_2/\frac{1}{2}\pi)}$$

and in the rectangular form by

$$\dot{I} = \frac{E}{\dot{Z}} = \frac{E}{r+jx} = \frac{E(r-jx)}{r^2+x^2} = \frac{Er}{r^2+x^2} - j\left(\frac{Ex}{r^2+x^2}\right)$$

$$I = \sqrt{\left[\left(\frac{Er}{r^2+x^2}\right)^2 + \left(\frac{Ex}{r^2+x^2}\right)^2\right]} = E\sqrt{\left(\frac{1}{r^2+x^2}\right)}$$

$$\varphi_2 = \tan^{-1} - x/r$$

Another special case of interest is where one complex quantity is the reciprocal of the other. Thus let the complex quantity $\dot{Y} = Y\varepsilon^{j\beta}$ be the reciprocal of the complex quantity $\dot{Z} = Z\varepsilon^{j\alpha}$. Then by definition

$$\dot{Y}\dot{Z} = 1 = Y\varepsilon^{j\alpha} \times Z\varepsilon^{j\beta} = YZ\varepsilon^{j(\alpha+\beta)}$$

Hence $\alpha + \beta = 0$

or $\beta = -\alpha$

i.e. the reciprocal of the vector $\dot{Z}$ has a length equal to Z and is inclined at an angle equal to $-\alpha$ to the axis of reference.

If the rectangular form is employed the physical meaning of the result is not so apparent. Thus if the vector $\dot{Z}$ is represented by $\dot{Z} = r + jx$, then the reciprocal vector $\dot{Y}$ is represented by

$$\dot{Y} = \frac{1}{\dot{Z}} = \frac{1}{r+jx} = \frac{r-jx}{(r+jx)(r-jx)}$$

$$= \frac{r}{r^2+x^2} - j\frac{x}{r^2+x^2}$$

The length of the reciprocal vector is therefore equal to

$$Y = \sqrt{\left\{\left(\frac{r}{r^2 + x^2}\right)^2 + \left(\frac{x}{r^2 + x^2}\right)^2\right\}}$$

$$= \sqrt{\left(\frac{1}{r^2 + x^2}\right)}$$

$$= 1/Z$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1} - x/r$$

Examples. (1) Find the quotients of each of the following pairs of numbers—

(a) $3 + j3$; $5 - j2$; (b) 5, $1 + j1$; (c) $-2 + j2$; $-3 - j4$.

Denoting the quotients by $\dot{Y}_a$, $\dot{Y}_b$, $\dot{Y}_c$, and the arguments of these quantities by φ_a, φ_b, φ_c, respectively, we have

$$\dot{Y}_a = \frac{3 + j3}{5 - j2} = \frac{(3 + j3)(5 + j2)}{5^2 + 2^2} = \frac{3 \times 5 - 3 \times 2 + j(3 \times 2 + 3 \times 5)}{29}$$

$$= 0{\cdot}31 + j0{\cdot}724$$

$$Y_a = \sqrt{(0{\cdot}31^2 + 0{\cdot}724^2)} = 0{\cdot}788$$

$$\varphi_a = \tan^{-1} 0{\cdot}724/0{\cdot}31 = 66{\cdot}8^\circ$$

$$\therefore \dot{Y}_a = 0{\cdot}788\underline{/66{\cdot}8^\circ}$$

$$\dot{Y}_b = \frac{5}{1 + j1} = \frac{5(1 - j1)}{1^2 + 1^2} = 2{\cdot}5 - j2{\cdot}5$$

$$Y_b = \sqrt{(2{\cdot}5^2 + 2{\cdot}5^2)} = 3{\cdot}54$$

$$\varphi_b = \tan^{-1} - 2{\cdot}5/2{\cdot}5 = -45^\circ$$

$$\therefore \dot{Y}_b = 3{\cdot}54\underline{/-45^\circ}$$

$$\dot{Y}_c = \frac{-2 + j2}{-3 - j4} = \frac{(-2 + j2)(-3 + j4)}{3^2 + 4^2} = \frac{2 \times 3 - 2 \times 4 + j(-2 \times 4 + 2 \times (-3))}{25}$$

$$= -0{\cdot}08 - j0{\cdot}56$$

$$Y_c = \sqrt{(0{\cdot}08^2 + 0{\cdot}56^2)} = 0{\cdot}566$$

$$\varphi_c = \tan^{-1}(-0{\cdot}56/-0{\cdot}08) = -98{\cdot}1^\circ$$

$$\therefore \dot{Y}_c = 0{\cdot}566\underline{/-98{\cdot}1^\circ}$$

(2) Find the reciprocal of the number $5 + j8$.

The reciprocal number is

$$\dot{Y} = \frac{5}{5^2 + 8^2} - j\frac{8}{5^2 + 8^2} = 0{\cdot}0562 - j0{\cdot}09$$

$$Y = \sqrt{(0{\cdot}0562^2 + 0{\cdot}09^2)} = 0{\cdot}0106$$

$$= \tan^{-1} - 0{\cdot}09/0{\cdot}0562 = -58^\circ$$

$$\therefore \dot{Y} = 0{\cdot}0106\,/\!-58^\circ$$

CHAPTER III

RESISTANCE AND INDUCTANCE

IN direct-current circuits the relationship between E.M.F. and current is a simple one and is given by the equation $E = IR$. Now the resistance (R) of the conductors of any particular circuit is constant provided that the temperature of the conductors is constant. Hence, except for the variation of resistance caused by change of temperature, the resistance of a circuit carrying a direct current is independent of the magnitude of the current.* Thus for a given circuit the ratio E/I is constant.

In alternating-current circuits generally this simple relation between E.M.F. and current is not applicable, as the variations of current and E.M.F. set up magnetic and electrostatic effects, respectively, which must be considered together with the resistance of the circuit when determining the quantitative relations between current and E.M.F. For example, with low-voltage circuits magnetic effects may be very large, especially when the currents are very large, but electrostatic effects are usually negligible. On the other hand, with high-voltage circuits electrostatic effects may be of appreciable magnitude, and magnetic effects are also present. Hence, in obtaining the relation between E.M.F. and current, these effects must be given due consideration.

In this chapter we shall consider the manner in which the magnetic effects due to the current in a circuit affect the relationship between the applied E M.F. and the current, reserving for a later chapter the discussion of electrostatic effects.

Inductance. An electric current in a conductor produces a magnetic field which encircles the conductor. When no other magnetic fields are present the paths of the magnetic lines are circles which are concentric with the conductor. The magnitude of the magnetic field is proportional to the magnitude of the current, and the direction of the field depends on the direction of the current. If the conductor forms part of a circuit carrying an alternating current the flux will be alternating, and the linkage of this flux with the circuit will induce therein an alternating E.M.F. The magnitude of this E.M.F. is, at any instant, proportional to the time rate of change of the flux at that instant. As the induced

* It is assumed that the distribution of the current over the cross section of the conductor remains constant at all currents.

E.M.F. is due *solely* to the magnetic effect of the current, it is called the *E.M.F. of self-induction*, or the *inductive E.M.F.*

Inductance (also called self-induction) is the property of a circuit in virtue of which a varying current causes a variation of the flux interlinked with the circuit and an E.M.F. to be induced therein.

A circuit possessing inductance is called an *inductive circuit* and one devoid of inductance is called a *non-inductive circuit*. Since it is difficult to obtain a circuit absolutely devoid of inductance, the term "non-inductive circuit" usually refers to one in which the inductive effect (or reactance) is negligible in comparison with the resistance. Examples of non-inductive circuits and apparatus: incandescent lamps, liquid and grid rheostats, concentric cables, standard low resistances constructed of concentric tubular conductors. Examples of inductive circuits: solenoids, and all electromagnetic apparatus and machinery, overhead transmission lines.

Coefficient of Inductance. In an alternating-current circuit of constant magnetic reluctance the flux is directly proportional to, and is in phase with, the current. Thus

$$\Phi = \frac{0{\cdot}4\pi i N}{S} = \left(\frac{0{\cdot}4\pi N}{S}\right) i$$

where Φ is the flux corresponding to the current i (amperes), N the number of turns through which the current passes, and S the magnetic reluctance in centimetre units. Hence, in such a circuit the E.M.F. of self-induction (e_L volts) is, at any instant, proportional to the rate of change of the current, thus

$$e_L = -\frac{N}{10^8}\frac{d\Phi}{dt} = -\left(\frac{0{\cdot}4\pi N^2}{S \times 10^8}\right)\frac{di}{dt} = -L\frac{di}{dt} \quad . \quad . \quad . \quad (6)$$

where L is a constant, called the *coefficient of inductance*—or, shortly, the *inductance*—of the circuit. The minus sign is introduced because the direction of the induced E.M.F. must be such as to oppose the flow of current.

Numerically,

$$L = \frac{0{\cdot}4\pi N^2}{S \times 10^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$= \frac{N\Phi}{i \times 10^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$= \frac{e_L}{di/dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The coefficient of inductance is therefore a constant property of all circuits for which the magnetic reluctance is constant.

The *practical unit of inductance* is the *henry*. This unit is 10^9 times the C.G.S. electromagnetic unit, which is a centimetre; inductance having the dimension $[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{3}{2}}\mathsf{T}^{-2}]\mathsf{T}/[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{1}{2}}\mathsf{T}^{-1}] = \mathsf{L}$, where L, M, T denote length, mass, and time respectively.

A circuit possesses an inductance of 1 henry when

10^8 linkages of flux and turns are produced by a current of 1 ampere passing in the circuit; or when

an E.M.F. of 1 volt is induced by a current varying at the rate of 1 ampere per second.

Equation (9) gives L in practical units (i.e. henries) when e_L is expressed in volts and di/dt in amperes per second.

Example. A wooden ring of circular cross section, 20 cm. mean diameter and 5 cm.^{2} cross section, is wound uniformly with 1000 turns of fine wire. Calculate the inductance of the winding. Also calculate the value of the induced E.M.F. when a current varying at the rate of 190 A. per second is sent through the winding.

The reluctance of the magnetic path of the wound ring is equal to

$$\frac{\text{mean magnetic length}}{\text{cross section}} = \frac{20\pi}{5} = 4\pi$$

Hence, substituting in equation (7), we have

$$L = \frac{0{\cdot}4\pi \times 1000^2}{4\pi \times 10^8} = 10^{-3}\text{ H., or 1 milli-henry.}$$

The induced E.M.F. is obtained from equation (9). Thus

$$\begin{aligned} e_L &= L\,di/dt \\ &= 10^{-3} \times 190 \\ &= 0{\cdot}19\text{ V.} \end{aligned}$$

Mutual Inductance. If a circuit (A) carrying an alternating current is in close proximity to another circuit (B), the flux due to the current in the former will interlink with both circuits and induce in them E.M.Fs., the direction of which will be the same for each circuit. Now the E.M.F. induced in circuit A acts in such a direction as to oppose the change of current in it. Hence, if circuit B is closed upon itself, the direction in which the current circulates is the same as that of the induced E.M.F., and is opposite to that of the current in A. The resultant ampere-turns due to the currents in these circuits are therefore smaller than those due to the current in A, and on the assumption of constant reluctance, the flux linked with the circuits under these conditions is smaller than that which is linked with A when B is open or removed, the current in A being the same in each case. Thus the effect of the induced current in B

is equivalent to a reduction of the self-induction of A. This inductive action of one circuit upon another is called ***mutual inductance*** or ***mutual induction***

Suppose the above circuits consist of N_1, N_2 turns, respectively, wound in close proximity upon a common non-magnetic, non-metallic core of reluctance S. Assume alternating current to be supplied to circuit A, and circuit B to be open. Then the flux (Φ_1) due to a current i_1 amperes in A is

$$\Phi_1 = 0{\cdot}4\pi i_1 N_1/S$$

The number of linkages of this flux and the turns in B is

$$\Phi_1 N_2 = 0{\cdot}4\pi i_1 N_1 N_2/S$$

and the E.M.F. (e_2) induced in circuit B is

$$e_2 = -\frac{N_2}{10^8}\frac{d\Phi_1}{dt} = -\frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S}\frac{di_1}{dt} = -M\frac{di_1}{dt}$$

where $M = 0{\cdot}4\pi N_1 N_2/S \times 10^8$, and is a constant.

Similarly, if alternating current is supplied to circuit B and circuit A is open, the flux (Φ_2) due to a current i_2 amperes in B is

$$\Phi_2 = 0{\cdot}4\pi i_2 N_2/S$$

The number of linkages of this flux and the turns of A is

$$\Phi_2 N_1 = 0{\cdot}4\pi i_2 N_1 N_2/S$$

and the E.M.F. (e_1) induced in circuit A is

$$e_1 = -\frac{N_1}{10^8}\frac{d\Phi_2}{dt} = -\frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S}\frac{di_2}{dt} = -M\frac{di_2}{dt}$$

where $M = 0{\cdot}4\pi N_1 N_2/S \times 10^8$

The constant M is called the *coefficient of mutual induction*, or, shortly, the *mutual inductance* of the two circuits, and is a constant property of the circuits provided that the reluctance is constant.

Numerically,

$$M = \frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$= \frac{N_1 \Phi_2}{i_2 \times 10^8} = \frac{N_2 \Phi_1}{i_1 \times 10^8} \quad . \quad . \quad . \quad . \quad (11)$$

$$= \frac{e_1}{(di_2/dt)} = \frac{e_2}{(di_1/dt)} \quad . \quad . \quad . \quad . \quad (12)$$

The *practical unit of mutual inductance* is the *henry*. Thus inductance, whether self or mutual, is expressed in henries.

Two circuits possess a mutual inductance of 1 henry when

10^8 linkages of flux and turns are produced in one circuit due to a current of 1 ampere in the other circuit; or when

an E.M.F. of 1 volt is induced in one circuit by a current varying at the rate of 1 ampere per second in the other circuit.

Equation (12) gives M in henries when the induced E.M.Fs (e_1, e_2) are expressed in volts and the rate of change of current is expressed in amperes per second.

Example. A straight cylindrical wooden core 4 cm. diameter is wound over uniformly with one layer of fine wire, there being 8 turns per cm. for a length of 100 cm. Around the middle of this helix is wound a search coil of 50 turns. Calculate the mutual inductance of the coils and the E.M.F. induced in the search coil when a current of 1A., at a frequency of 50 cycles per second, is sent through the helix.

As the length of the helix is great in comparison with its diameter the flux density (B) at its centre, due to current i, is

$$B = 0{\cdot}4\pi \times \text{amp. turns per cm. length}$$
$$= 0{\cdot}4\pi \times 8i$$

Hence the flux (Φ) at the middle of helix due to unit current is

$$\Phi = BA$$
$$= 0{\cdot}4\pi \times 8 \times \pi \times 2^2$$

This flux is linked with the turns of the search coil, and therefore the mutual inductance of the coils is

$$M = \frac{\Phi \times 50}{10^8} = \frac{0{\cdot}4\pi \times 8 \times \pi \times 2^2 \times 50}{10^8}$$
$$= 0{\cdot}063 \times 10^{-3}$$
$$= 0{\cdot}063 \text{ milli-henries.}$$

The instantaneous value (e_2) of the E.M.F. induced in the search coil is

$$e_2 = -M\, di_1/dt$$

where i_1 is the instantaneous value of the current in the primary coil. If this current is represented by the equation $i = I_m \sin \omega t$, we have

$$e_2 = -M \frac{d}{dt}(I_m \sin \omega t)$$
$$= -MI_m\omega \cos \omega t$$
$$= \omega MI_m \sin(\omega t - \tfrac{1}{2}\pi)$$

The maximum value of the induced E.M.F. is ωMI_m,

and the R.M.S. value (E_2) is ωMI.

Hence $E_2 = 2\pi \times 50 \times 0{\cdot}063 \times 10^{-3} \times 10$
$= 0{\cdot}02$ V.

Relation Between Self-inductance and Mutual Inductance. If L_1, L_2 denote the respective self-inductances of the above circuits,

$$L_1 = 0{\cdot}4\pi N_1^2/S \times 10^8$$
$$L_2 = 0{\cdot}4\pi N_2^2/S \times 10^8$$

Hence,
$$L_1 L_2 = (0{\cdot}4\pi N_1 N_2/S \times 10^8)^2 = M^2$$

or
$$M = \sqrt{(L_1 L_2)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the mutual inductance of two circuits is equal to the square root of the product of their self-inductances.

This equation is only strictly true when the whole of the flux due to a current in one circuit is linked with the whole of the turns of the other circuit, i.e. the turns in both circuits are coincident. In general, these two sets of turns are displaced from each other, and M is less than $\sqrt{L_1 L_2}$. The ratio $M/\sqrt{L_1 L_2}$ is called, in radio-telegraphy, the *coefficient of coupling*, and mutually-inductive circuits are said to be tightly or loosely coupled, according to whether the value of ratio $M/\sqrt{L_1 L_2}$ approaches unity or is considerably less than unity.

Apparent Self-inductance of Mutually-inductive Series Circuits. If the two mutually-inductive circuits A, B considered above are connected in series, the joint, or apparent, inductance is given by $L = L_1 + L_2 \pm 2M$.

Thus the self-induced E.M.F. in A due to the current (i) is equal to $-L_1 di/dt$, and that in B is equal to $-L_2 di/dt$. Again, the E.M.F. induced in A due to mutual induction from B is equal to $\mp M di/dt$, and that induced in B due to mutual induction from A is equal to $\mp M\, di/dt$, the double sign denoting that two combinations of the circuits are possible, i.e. the connections may be such that the magneto-motive forces act either cumulatively or differentially.

Hence, the sum of the induced E.M.Fs. due to the variation of current is

$$-L_1 \frac{di}{dt} - L_2 \frac{di}{dt} \mp 2M \frac{di}{dt} = -\frac{di}{dt}(L_1 + L_2 \pm 2M) = -L \frac{di}{dt}$$

where $L = L_1 + L_2 \pm 2M$

If the mutual inductance is variable between the limits M_{max}, M_{min}, variation of the joint inductance may be obtained over the ranges $L_1 + L_2 - 2M_{max}$ to $L_1 + L_2 - 2M_{min}$, and $L_1 + L_2 + 2M_{min}$ to $L_1 + L_2 + 2M_{max}$. This principle is applied extensively in practice to variable standards of self and mutual inductance.

Relation Between Current and E.M.F. for a Simple Circuit Possessing Resistance. Consider a non-inductive circuit, such as an incandescent lamp, in which an alternating current is passing. Let the resistance of the circuit be R ohms and let the current be represented

by the equation $i = I_m \sin \omega t$. It is required to determine the impressed E.M.F. necessary to maintain this current.

Now in any electric circuit the resultant of all the E.M.Fs. (internal and external) acting in the circuit must be zero, i.e. the internal E.M.Fs. which come into existence with the current, and are due to the resistance or other properties of the circuit, must balance the external, or impressed, E.M.F.

If the circuit possesses only resistance we must have $(e - \Sigma ir) = 0$; or $e = \Sigma ir$, where e is the impressed, or applied, E.M.F., and Σir the internal E.M.Fs. produced by the passage of the current through the resistance of the circuit. It is important to observe that the internal E.M.Fs. (Σir) act in opposition to the current, and at any instant their phase difference with respect to the current is 180°.

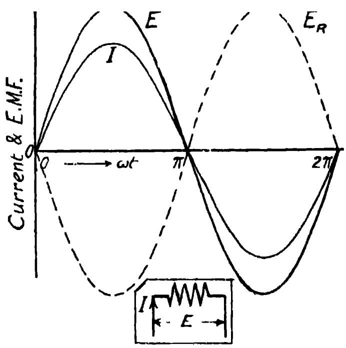


Fig. 19. Representation of Current and E.M.F's. (External and Internal) for Non-inductive Circuit

Hence for the circuit under consideration the internal E.M.F. due to resistance is, at any instant, given by

$$e_R = -Ri = -RI_m \sin \omega t,$$

the minus sign indicating that e_R acts in opposition to the current.

Therefore the equation to the impressed E.M.F. is

$$e = -e_R = Ri = RI_m \sin \omega t \quad . \quad . \quad (14)$$

Thus the impressed E.M.F. must be sinusoidal and of the same frequency as the current. It must also be in phase with the current. These conditions are shown graphically in Fig. 19, in which the sine curve I represents the current, the sine curve E the impressed E.M.F., and the sine curve E_R the internal E.M.F.

Conversely, if a circuit possessing pure resistance be connected to a source of sinusoidal E.M.F. the current in the circuit will be sinusoidal and will have the same frequency and phase as the impressed E.M.F. For example, if the impressed E.M.F. is given by the equation

$$e = E_m \sin \omega t$$

the equation to the current is

$$i = \frac{E_m}{R} \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The maximum value of the current is

$$I_m = E_m/R$$

and the R.M.S. value is

$$I = E/R$$

Hence the relationship between current and E.M.F. is the same as for a continuous-current circuit, and the ratio E/I, or E_m/I_m, gives the true resistance of the circuit, such as would be obtained by a test with continuous current. This statement, however, holds true only in cases where (*a*) the current is uniformly distributed over the cross-section of the conductors, (*b*) the conductors are removed from the influence of external alternating magnetic fields.

Relation Between Current and E.M.F. for Circuits Possessing Inductance Only. Consider a purely inductive circuit of which the inductance is constant and equal to L. Although such a circuit cannot be realized absolutely in practice, it is approximated to by a non-magnetic torus with a low-resistance winding. If the torus is of laminated iron, or of iron wire, a closer approximation to a purely inductive circuit is obtained ; but in this case the inductance will vary with the saturation of the iron core.

Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$. Then the E.M.F. induced in the circuit by the alternations of the current is

$$\begin{aligned} e_{\mathrm{L}} &= -L\frac{di}{dt} = -L\frac{d}{dt}(I_m \sin \omega t) \\ &= -L\,\omega\, I_m \cos \omega t \\ &= \omega L I_m \sin (\omega t - \tfrac{1}{2}\pi), \end{aligned}$$

which is of the same frequency as the current but lags behind it by an angle of $\frac{1}{2}\pi$ radians, or 90°.

Since the resistance of the circuit is zero the impressed E.M.F. (e) must balance the E.M.F. of self-induction. Hence the equation to the impressed E.M.F. is

$$\begin{aligned} e &= -e_{\mathrm{L}} = -\omega L I_m \sin (\omega t - \tfrac{1}{2}\pi) \\ &= \omega L I_m \sin (\omega t + \tfrac{1}{2}\pi) \qquad . \qquad . \quad (16) \end{aligned}$$

Thus the impressed E.M.F. must be sinusoidal and of the same frequency as the current. Moreover, it must lead the current by an angle of 90°.

Conversely, if a purely inductive circuit be connected to a source of sinusoidal E.M.F. the phase difference between E.M.F. and

current will be 90° (lagging). A graphical representation of these conditions is shown in Fig. 20, in which the sine curve I represents the current, the sine curve E the impressed E.M.F., and the sine curve E_L the E.M.F. of self-induction.

From equation (16) the maximum value of the impressed E.M.F. is

$$E_m = \omega L I_m$$

and the R.M.S. value is

$$E = \omega L I = 2\pi f L I$$

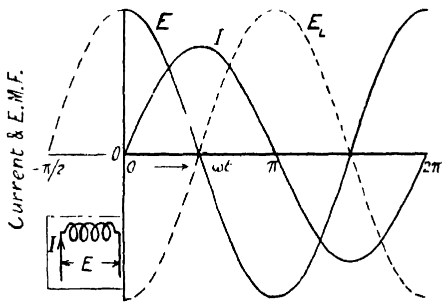


FIG. 20. REPRESENTATION OF CURRENT AND E.M.FS. (EXTERNAL AND INTERNAL) FOR PURELY INDUCTIVE CIRCUIT

The ratio $E/I = \omega L = 2\pi f L$ is called the *reactance* of the circuit, and is denoted by the symbol X. Reactance has the same dimensions as resistance (both having the dimension [L/T] in the electromagnetic system of units) and is accordingly expressed in ohms.

Relation Between Current and E.M.F. for Circuits Possessing Resistance and Inductance. Consider a circuit of resistance R ohms and an inductance (which is assumed to be constant) of L henries. Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$.

The internal E.M.Fs. in the circuit are

(1) the E.M.F. due to resistance ($= e_R = -Ri$) which has a phase difference of 180° with respect to the current;

(2) the E.M.F. of self-induction ($= e_L = -L\,di/dt$) which has a phase difference of 90°, lagging, with respect to the current.

The impressed E.M.F. (e) must balance the internal E.M.Fs. Therefore

$$\begin{aligned} e &= -(e_R + e_L) = Ri + L\,di/dt \\ &= RI_m \sin \omega t + \omega L I_m \cos \omega t \qquad . \quad . \quad (17) \end{aligned}$$

But the sum of two sinusoidal quantities can be expressed as a single sinusoidal quantity. Thus, multiplying and dividing each term of equation (17) by $\sqrt{(R^2 + \omega^2 L^2)}$, we have

$$e = I_m\sqrt{(R^2 + \omega^2 L^2)}\left\{\frac{R}{\sqrt{(R^2 + \omega^2 L^2)}}\sin \omega t + \frac{\omega L}{\sqrt{(R^2 + \omega^2 L^2)}}\cos \omega t\right\}$$

Now if
$$\tan \varphi = \omega L/R$$
$$\cos \varphi = R/\sqrt{(R^2 + \omega^2 L^2)}$$
$$\sin \varphi = \omega L/\sqrt{(R^2 + \omega^2 L^2)}$$

we obtain on substituting these values in the above equation,

$$\begin{aligned} e &= I_m\sqrt{(R^2 + \omega^2 L^2)}\left\{\cos \varphi \sin \omega t + \sin \varphi \cos \omega t\right\} \\ &= I_m\sqrt{(R^2 + \omega^2 L^2)}\sin (\omega t + \varphi) \quad . \quad . \quad . \quad (18) \end{aligned}$$

This equation shows that the impressed E.M.F. is sinusoidal and leads the current by the angle φ, the tangent of which is equal to

$$\frac{\omega L}{R} = \frac{\text{reactance}}{\text{resistance}}$$

Conversely, if the equation to the current had been given as $i = I_m \sin (\omega t - \varphi)$, where $\varphi = \tan^{-1} \omega L/R$, we should have obtained for the impressed E.M.F. the equation

$$e = I_m\sqrt{(R^2 + \omega^2 L^2)}\sin \omega t$$

Hence, if a sinusoidal E.M.F. represented by the equation $e = E_m \sin \omega t$ be applied to the circuit, the current, when the steady or cyclic state* is reached, will be given by

$$i = \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2}}\sin (\omega t - \varphi) \quad . \quad . \quad . \quad . \quad (19)$$

Thus in an inductive circuit, of constant resistance and inductance, supplied by a source of sinusoidal E.M.F., the current is sinusoidal and of the same frequency as the impressed E.M.F., but is lagging with respect to the latter

The maximum value of the current, from equation (19), is

$$I_m = E_m/\sqrt{(R^2 + \omega^2 L^2)},$$

* The general equation to the current must take into account the value of the impressed E.M.F. at the instant of closing the circuit. In general, the first few cycles of the current wave are irregular, but the waves become sinusoidal after a short time (see Chapter XXIII).

and the R.M.S. value is

$$I = E/\sqrt{(R^2 + \omega^2 L^2)}$$

The ratio $E/I = \sqrt{(R^2 + \omega^2 L^2)}$ is called the *impedance,* or "apparent resistance," of the circuit, and is usually denoted by the symbol Z. Impedance has the same dimensions as resistance, and is accordingly expressed in ohms.

The equation connecting E.M.F., current, and impedance in an alternating-current circuit is therefore similar to that connecting E.M.F., current, and resistance in a continuous-current circuit. Hence, by regarding impedance as "apparent resistance," Ohm's law is applicable to alternating-current circuits in which E.M.F. and current are sinusoidal. We shall see later that this law is applicable to all simple series and parallel circuits, even when electrostatic capacity in a concentrated form is present. It should be observed, however, that impedance is not necessarily a constant property of an alternating-current circuit, as this quantity includes resistance, inductance, and frequency. But with constant resistance, inductance, and frequency the impedance will be constant, and therefore the ratio E/I will be constant.

The relationship between the current, the two internal E.M.Fs., and the impressed E.M.F. is shown graphically in Fig. 21. The impressed E.M.F. is represented by the sine curve E, and the current is represented by the sine curve I, lagging $\varphi°$ behind the latter. The internal E.M.Fs. due to resistance and inductance are represented by the sine curves E_R, E_L, respectively, the former having a phase difference of 180° (lagging), and the latter a phase difference of 90° (lagging), with respect to the current curve I.

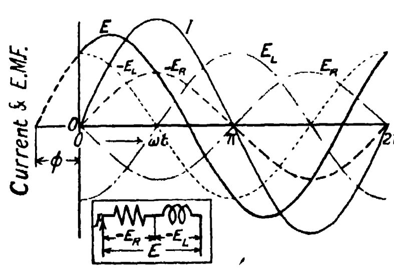


FIG. 21. REPRESENTATION OF CURRENT AND E.M.FS. (EXTERNAL AND INTERNAL) FOR AN INDUCTIVE CIRCUIT

The sum of curves E_R and E_L gives a sine curve which represents the resultant internal E.M.F. and balances the impressed E.M.F. Obviously the impressed E.M.F. curve E may be resolved into two components which balance the internal E.M.F. curves E_R, E_L. These components are represented by the sine curves $-E_R$, $-E_L$, respectively, the former being in phase with the current and of maximum value RI_m, the latter leading the current by 90° and having a

maximum value equal to ωLI_m. The curves $-E_R$, $-E_L$ therefore represent the components of the impressed E.M.F. which are expended against resistance and inductance respectively.

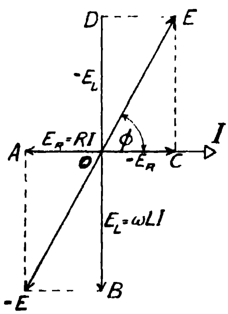


FIG. 22. VECTOR DIAGRAM FOR AN INDUCTIVE CIRCUIT

Vector Diagram for a Series Circuit Containing Resistance and Inductance. The vector diagram for this circuit is shown in Fig. 22, in which the current vector OI is taken as the vector of reference. The internal E.M.Fs., E_R, E_L, are represented by the vectors OA, OB respectively, the lengths of which are proportional to RI and ωLI respectively. The components of the impressed E.M.F. which balance the internal E.M.Fs. are represented by the vectors OC, OD. The impressed E.M.F is therefore represented by OE, which is the resultant of OC and OD, and leads the current vector by the angle φ.

The length of OE, which is proportional to the R.M.S. value of the impressed E.M.F., is given by

$$E = \sqrt{(OC^2 + OD^2)} = \sqrt{[(RI)^2 + (\omega LI)^2]}$$
$$= I\sqrt{(R^2 + \omega^2 L^2)}$$

and $\tan \varphi = OD/DE = \omega LI/RI = \omega L/R$

The vector triangle OCE is a triangle of E.M.Fs. for the circuit *referred to the external source of supply*. Thus OE represents the impressed E.M.F., OC the component which is expended against resistance, and CE the component which is expended against inductance

Now, $$OC : CE : OE = RI : \omega LI : I\sqrt{(R^2 + \omega^2 L^2)}$$
$$= R : \omega L : \sqrt{(R^2 + \omega^2 L^2)}$$

Hence the sides OC, CE, OE of triangle OCE are proportional to the resistance, reactance, and impedance respectively. On account of this feature the triangle OCE, when drawn to an ohm scale, is called the *impedance triangle* of the circuit.

Impedance is therefore a complex quantity, i.e. it is only completely specified when its magnitude and inclination, or alternatively its two perpendicular components with respect to the current, are given. But impedance is not a vector quantity as it is a non-directive quantity.

In symbolic notation impedance is represented by the complex number

$$\mathbf{Z} = R + jX$$

the absolute value of which is

$$Z = \sqrt{(R^2 + X^2)}$$

and its phase or argument with respect to the current axis is

$$\varphi = \tan^{-1} X/R$$

In the polar forms of representation, impedance is given by

$$\underset{\cdot}{Z} = Z\varepsilon^{j\varphi}; \quad \underset{\cdot}{Z} = ZJ^{\varphi}; \quad \underset{\cdot}{Z} = Z/\underline{\varphi}$$

Examples. (1) An inductive circuit has a resistance of 12 Ω. and an inductance of 0·2 H. What current will be taken when a sinusoidal E.M.F

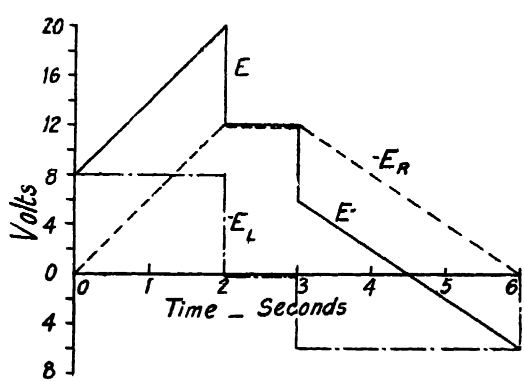


FIG. 23. GRAPHICAL SOLUTION TO EXAMPLE

of 100 V., at 50 frequency, is applied, and what will be the phase difference between E.M.F. and current ?

The reactance of the circuit is

$$X = 2\pi f L = 2\pi \times 50 \times 0{\cdot}2 = 62{\cdot}8\ \Omega$$

and the impedance is

$$Z = \sqrt{(R^2 + \omega^2 L^2)} = \sqrt{(R^2 + X^2)} = \sqrt{(12^2 + 62{\cdot}8^2)} = 63{\cdot}9$$

Hence the current is

$$I = E/Z = 100/63{\cdot}9 = 1{\cdot}56 \text{ A.}$$

and the phase difference between E.M.F. and current is

$$\omega = \tan^{-1} X/R = \tan^{-1}(62{\cdot}8/12) = 79° \ 10'$$

(2) A current passing through a choking coil having an inductance of 3 H. and a resistance of 2 Ω varies according to the following law: At time 0 it is 0; it increases at the rate of 3 A. per second for 2 seconds; it then remains constant for 1 second; it then decreases at the rate of 2 A. per second for 3 seconds. Plot the current curve and also the voltage at the terminals of the choking coil. (C. and G.)

The solution is given in Fig. 23, in which the trapezoidal curve I represents the current and the stepped curve E represents the voltage at the terminals of the coil. The latter is obtained from its components, viz. (1) the E.M.F.

($-E_R$) which is expended against resistance; (2) the E.M.F. ($-E_L$) which balances the E.M.F. of self-induction. The component $-E_R$ is in phase with the current and is equal to $RI = 2I$ volts. The component $-E_L$ is equal to $L\,di/dt = (3 \times$ amperes per second) volts; it is positive—i.e. in the same direction as the current—when the current is increasing (di/dt positive) and negative when the current is decreasing (di/dt negative).

Additional Theory Relating to Mutually-Inductive Circuits

Non-uniform Distribution of Current over Cross-section of Conductor (Skin Effect). The elementary theory of mutually-inductive circuits may be applied to show that the distribution of current over the cross-section of a non-magnetic conductor is not uniform.*

Thus, consider two concentric tubular elements A, B, in the cross-section of a circular conductor, and let the cross-sections of these elements be equal. Further, let i_A, i_B denote the currents in the elements, L_A, L_B, their self-inductances, M their mutual inductance, and R the resistance of each element for a given length of conductor. Then, if the potential difference across this length is denoted by e, we have

$$e = Ri_A + L_A\frac{di_A}{dt} + M\frac{di_B}{dt} = Ri_B + L_B\frac{di_B}{dt} + M\frac{di_A}{dt}$$

whence $$i_A = i_B + \frac{di_B}{dt}\left(\frac{L_B - M}{R}\right) + \frac{di_A}{dt}\left(\frac{M - L_A}{R}\right)$$

Now if A is the outer element, $L_B > L_A$, and $M = L_A$.

Hence $$i_A = i_B + \frac{di_B}{dt}\left(\frac{L_B - L_A}{R}\right)$$

which shows that the current density in the outer element (A) is greater than that in the inner element (B).

Therefore, due to inductive effects, the current tends to concentrate towards the surface of the conductor. This phenomenon is called the *skin effect*. It results in the heating, or I^2R, loss in the conductor being greater than that when the current is uniformly distributed, and, in consequence, the "effective resistance" of a conductor when carrying alternating current is greater than the true resistance of the conductor. The skin effect becomes of considerable importance at high frequencies, and highly stranded conductors, laid up on a hemp core, must be employed for transmitting high-frequency currents, the stranding being necessary for reducing the loss due to eddy currents.

Moreover, on account of the non-uniform distribution of the current, the self-inductance of the conductor will be slightly lower than that calculated upon the assumption of uniform distribution of the current.

* The full treatment involves complex quantities and may be developed along similar lines to those employed in Chapter XV in connection with the calculation of the flux distribution in a magnetic core. In fact the distribution of current over the cross-section of a large conductor may, at high frequencies, be represented by a curve similar to that of Fig. 213.

CHAPTER IV

CAPACITANCE AND CONDENSERS

Electrostatic Potential. The potential of a conductor is defined as the work done by, or against, electric forces in carrying unit positive charge from the conductor to the boundary of the electric field, which is considered to be at zero potential.

For example, consider an isolated spherical conductor, of radius r cm., surrounded by air and charged with Q positive units. Assuming this charge to be distributed uniformly over the surface, the force on unit positive charge at distance x from the centre of the sphere is $(Q \times 1)/x^2$ dynes. Hence the work done in bringing this unit charge from the boundary of the electric field (i.e. infinity) to the surface of the sphere is

$$V = \int_{\infty}^{r} \frac{Q}{x^2}\, dx = \left[\frac{Q}{x}\right]_{\infty}^{r} = \frac{Q}{r} \text{ ergs}$$

which, by definition, is the potential of the sphere.

Energy Stored in an Electric Field. An electrically charged conductor is the seat of an electric (electrostatic) field, the lines of electric force being normal to the surface of the conductor. The work done in charging the conductor to a potential V is given by

$$W = \tfrac{1}{2}QV \text{ ergs,*}$$

where Q and V denote the charge and potential, respectively, in electrostatic units. This energy is stored in the electric field and is released when the conductor is discharged.

Capacitance. The ratio of the charge (Q) on a conductor to its potential (V), when all other conductors in the electric field are at zero potential, is called its *capacitance* or *electrostatic capacity* (C), i.e. $Q/V = C$. With this relationship between Q and V we may express

* From the definition of potential it follows that the work done in carrying a charge dq from the boundary of the electric field to a conductor charged to potential v is $v\,dq$. Now the ratio of charge to potential is constant; hence if dv is the increment of potential due to a charge dq, $dq/dv = C$, a constant; i.e. $dq = C\,dv$. Therefore the total work done in charging the conductor to a potential V is

$$W = \int_0^V v\,dq = C\int_0^V v\,dv = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV \text{ ergs,}$$

where Q is the charge corresponding to a potential V.

the energy (W) stored in the electric field in terms of capacitance and potential, thus

$$W = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

whence

$$C = 2W/V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

i.e. the capacitance of a conductor is equal to twice the energy stored in the electric field when its potential is unity. Capacitance may, therefore, be regarded as a property of a conductor, or a system of conductors, in virtue of which electrical energy can be stored in the surrounding electric field.

Unit of Capacitance. In equation (21) the capacitance will be given in electrostatic units—i.e. centimetres—when W is expressed in ergs and V in electrostatic units of potential. To obtain C in practical units—farads—W and V must be expressed in practical units—i.e. joules or watt-seconds, and volts respectively. Now all practical electrical units are derived from the corresponding absolute electromagnetic units : hence, in deriving the practical unit of capacitance from equation (21), we require the ratio between the electrostatic and electromagnetic units of energy and potential, as well as the ratio between the practical and electromagnetic units of these quantities. As both electrostatic and electromagnetic systems of units are derived from the centimetre-gramme-second fundamental units, the unit of work is the same (viz. the dyne-centimetre or erg) for both systems. But potential, or potential difference, is defined, in the electrostatic system, in terms of work and electric charge—i.e. potential = work/charge—and in the electromagnetic system, in terms of the rate at which work is expended and current, i.e. potential difference = work/(time × current).

According to these definitions the dimensions of potential in terms of length (L), mass (M), and time (T), and the electrostatic and electromagnetic constants κ, μ, respectively, are

$$\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}\kappa^{-\frac{1}{2}} \text{ in the electrostatic system,}$$

and

$$\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-2}\mu^{\frac{1}{2}} \text{ in the electromagnetic system}$$

Whence the ratio of the dimensions is

$$\frac{\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}\kappa^{-\frac{1}{2}}}{\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-2}\mu^{\frac{1}{2}}} = \frac{\mathrm{T}}{\mathrm{L}} \cdot \frac{1}{\sqrt{(\mu\kappa)}}$$

which, since μ and κ are non-dimensional quantities, is the reciprocal of a velocity. The value of this velocity, deduced from measurements of the same quantity (e.g. current or E.M.F.) in electrostatic and electromagnetic units, is 3×10^{10} cm. per second.

Hence, as the ratio between magnitudes of the same kind but in different units, is in the inverse ratio of their dimensions, we have

1 electrostatic unit of potential $= 3 \times 10^{10}$ electromagnetic units of potential.

$$\begin{aligned} \text{Now 1 volt} &= 10^8 \text{ electromagnetic units of potential,} \\ &= 10^8/(3 \times 10^{10}) = 1/300 \text{ of electrostatic unit of potential,} \\ \text{and 1 joule} &= 10^7 \text{ ergs.} \end{aligned}$$

Therefore the quantities W, V, in equation (21) must be divided by 10^7 and $(1/300)$ respectively, to obtain the capacitance in practical units.

$$\text{Hence,} \quad C\,(\text{farads}) = \frac{2W/10^7}{V^2/(1/300)^2} = \frac{2W}{V^2 \times 9 \times 10^{11}}$$

Whence, 1 farad $= 9 \times 10^{11}$ electrostatic units of capacitance.

The farad, however, is too large a unit for commercial purposes, and therefore the commercial unit is chosen equal to one-millionth of a farad– i.e. a microfarad—the symbol for which is μF., μ here being the prefix denoting one millionth.

$$\begin{aligned} \text{Hence,} \quad 1\mu\text{F.} &= 10^{-6}\text{F.} \\ &= 9 \times 10^{11} \times 10^{-6} = 9 \times 10^5 \text{ electrostatic units.} \end{aligned}$$

Capacitance of Isolated Spherical Conductor. Consider an insulated spherical conductor, of radius r cm., surrounded by air and isolated from other conductors. Assume the sphere to be originally uncharged, and let a charge Q be given to it. Then, provided that the charge distributes itself uniformly over the surface, the potential becomes $V = Q/r$. Whence charge/potential $= Q/(Q/r) = r$: i.e. the capacitance of an isolated sphere is, in electrostatic units, equal to its radius in cm

Hence to obtain a capacitance of 1 μF., the radius of the sphere must be 9×10^5 cm. or 9000 metres.

Condenser. The capacitance of a conductor depends on its size and geometrical form; its position relative to other bodies in the electric field; and the specific inductive capacity (see p. 58) of the surrounding insulating medium. With conductors of the forms and dimensions commonly used in practice the capacitance of a single isolated conductor is extremely small. But the capacitance can be increased by bringing the boundary of the field nearer to the conductor, e.g. by placing a second (earthed) conductor near to the charged one. For example, if an insulated conducting sphere of

radius r cm. is surrounded by an earthed concentric conducting shell of radius r_1 cm., the potential of the sphere due to a charge Q becomes $\int_r^{r_1} \frac{Q}{x^2}dx = Q\left(\frac{1}{r} - \frac{1}{r_1}\right)$, and its capacitance is

$$C = Q/V = rr_1/(r_1 - r) = r\{1 + (r/\delta)\}$$
$$= (r + \delta)r/\delta$$
$$= r(r/\delta), \text{ approximately,}$$

where $\delta = r_1 - r$. Hence when δ is small in comparison with r the capacitance will be very much greater than that of the isolated sphere.

Under these conditions it is possible to obtain large charges on the conductors with only a moderate potential difference between them. Such a system of two conductors, insulated from each other and having large surfaces a small distance apart, is called a *condenser*.

In all practical forms of condensers the conductors (called the *plates* of the condenser) consist of sheets of metal foil separated by a thin insulating medium called the *dielectric*. Alternate plates are electrically connected together, so that, by employing a large number of plates, it is possible to obtain the equivalent of a large surface area although the area of the individual plates may be relatively small. The plates are so close together that they always receive equal and opposite charges, and the latter are unaffected by the presence of neighbouring charged or uncharged conductors. In such cases the numerical value of the charge on either plate, when the potential difference between them is unity, is called the *capacitance of the condenser*. [Formerly the term *capacity* was used.]

Practical Uses of Condensers. Condensers, in virtue of their property of storing electrical energy, have a number of practical uses: for example, the stored energy, when discharged under suitable conditions, may be utilized to set up electrical oscillations, the energy being radiated in the form of electric waves—as in radio-telegraphy and telephony—or again, the stored energy may be used to alter, or modify, the characteristics of electric circuits, as discussed in Chapter VI. Condensers are used extensively in telephony: they also form an essential part of electrical ignition systems for internal-combustion engines. A further use is for the protection of electric circuits against high-voltage, high-frequency, surges.

Calculation of Capacitance of Condensers. *General.* In calculating the capacitance of a condenser, or a system of conductors, we first obtain an expression for the potential of the conductors, or plates,

of the condenser due to a given charge. The potential is usually calculated from the summation of the work done on a unit charge when moved from one conductor to the other, and this requires a knowledge of the magnitude and direction of the electric field at all points in the space between the conductors. With the potential known, the capacitance follows directly from the ratio of potential to charge.

Capacitance of Parallel Plate Condenser. Let A denote the area (in square cm.) of each plate and δ the distance (in cm.) between the plates. Then if δ is small in comparison with A, the electric field

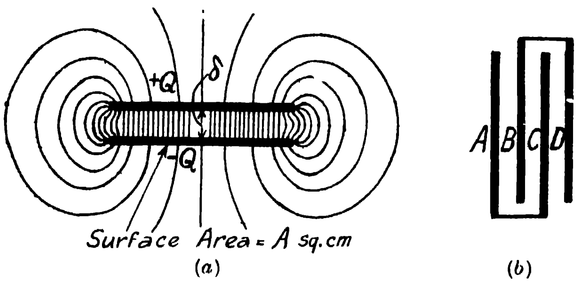


FIG. 24. PERTAINING TO THEORY OF PARALLEL PLATE CONDENSER

between the plates due to charges $+Q$, $-Q$, will be uniform and normal to their surfaces. Near the edges of the plates, however, the field is not uniform owing to the "fringing" of the lines of force (see Fig. 24a). Neglecting fringing effects, the electric force at any point in the air space between the plates is equal to $4\pi\sigma$ dynes, where σ $(= Q/A)$ is the surface density of the charges on the plates. Hence the work done in carrying unit charge from one plate to the other is equal to $4\pi\sigma\delta = 4\pi\delta Q/A$ ergs, which represents the potential difference between the plates. Therefore the capacitance is given by

$$C = Q/(4\pi\delta Q/A) = A/4\pi\delta \text{ electrostatic units} \quad . \quad . \quad (22)$$

$$= A/(4\pi\delta\times 9\times 10^5) = A/(\delta\times 113\times 10^5) \text{ microfarads} \quad . \ (22a)$$

NOTE.—If the dielectric is not air but a material having a dielectric constant equal to κ, the capacitance will be κ times that given by the above equations.

Example. The plates of a parallel-plate condenser are each 30 cm. × 25 cm., and the dielectric is a sheet of paraffined paper 0·015 cm. thick, for which the dielectric constant is 3. Assuming the plates to be in intimate contact with the dielectric, the capacitance of the condenser is

$$C = \frac{\kappa A}{4\pi\delta \times 9 \times 10^5} = \frac{30 \times 25 \times 3}{4\pi \times 0{\cdot}015 \times 9 \times 10^5} = 0{\cdot}013\mu\text{F}.$$

Multiple-plate Condenser. If two additional plates, and sheets of dielectric, are added, and alternate plates are connected together, as shown in Fig. 24*b*, the effective area of each "plate" of the condenser will now be $3 \times 30 \times 25 = 2250$ cm.^{2}, as both sides of the intermediate plates B, C, Fig. 24*b*, are effective, but only one side of the end plates A, D, is effective. Hence the capacitance will now be three times that due to a single pair of plates. Similarly, if two more plates and dielectric are added, the capacitance will be five times that due to a single pair of plates. In general, if n is the number of pairs of similar plates in a multiple-plate condenser, and C is the capacitance due to a single pair of plates, the capacitance of the condenser $= (2n - 1)\,C$.

Hence to obtain a capacitance of 1μF. with plates 30 cm. $\times$ 25 cm. and a dielectric of paraffined paper 0·015 cm. thick, the number of pairs of plates is

$$n = \tfrac{1}{2}\left(\frac{1}{0{\cdot}013} + 1\right) = \tfrac{1}{2}(75 + 1) = 38$$

Capacitance of Cylindrical Condensers, or Concentric Cylinders. Assume the cylinders to be of indefinite length and let r = radius, in cm., of surface of inner cylinder and r_1 = radius, in cm., of the internal surface of the coaxial surrounding cylinder (see Fig. 25*a*). Further, let the charge per cm. length of the inner cylinder be $+ Q$, and let the outer cylinder be earthed. Then the electric force in the air space between the cylinders is normal to their surfaces and acts radially outwards from the inner cylinder. In the case of long cylinders and an air dielectric the force at any point, P, in this space, distant x from the common axis, is equal to $4\pi \times$ density of electric field at that point: i.e. electric force $= 4\pi Q/2\pi x = 2Q/x$ dynes. Therefore, work done in moving unit positive charge from outer to inner cylinder

$$= \int_r^{r_1} \frac{2Q}{x}\,dx = 2Q\,\Big[\log_\varepsilon x\Big]_r^{r_1} = 2Q \log_\varepsilon \frac{r_1}{r} \text{ ergs},$$

which is equal to the potential difference between the cylinders.

Whence the capacitance per cm. length of the cylinders is

$$C = \frac{1}{2 \log_\varepsilon (r_1/r)} \text{ electrostatic units} \quad . \quad . \quad . \quad . \quad (23)$$

$$= \frac{1}{2 \times 2{\cdot}3 \times 9 \times 10^5 \log_{10} (r_1/r)} \text{ microfarads}$$

$$= \frac{1}{41{\cdot}4 \times 10^5 \log_{10} (r_1/r)} \text{ microfarads} \quad . \quad . \quad . \quad . \quad (23a)$$

If $(r_1 - r) = \delta$, equation (23) becomes

$$C = \frac{1}{2 \log_\varepsilon (1 + \delta/r)}$$

Now $\log_\varepsilon (1 + \delta/r)$ may be represented by the series

$$\log_\varepsilon (1 + \delta/r) = \delta/r - \tfrac{1}{2}(\delta/r)^2 + \tfrac{1}{3}(\delta/r)^3 - \tfrac{1}{4}(\delta/r)^4 + \ldots$$

and when δ/r is small, the second and following terms may, for a first approximation, be neglected.

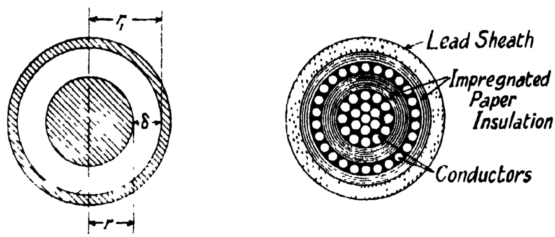


FIG. 25. (a) PERTAINING TO THEORY OF CONCENTRIC CYLINDRIC CONDENSER. (b) CROSS SECTION OF CONCENTRIC CABLE

Hence when δ is small in comparison with r, the capacitance of the concentric cylinders will be given by

$$C = \frac{1}{2\delta/r} = \frac{r}{2\delta} = \frac{2\pi r}{4\pi\delta} = \frac{A}{4\pi\delta}$$

where A is the surface of the inner cylinder per cm. of its length. Thus in this special case the capacitance is approximately equal to that of a parallel-plate condenser of the same equivalent surface.

Applications of Cylindric Condensers and Conductors. Condensers formed of concentric cylinders with air dielectric are occasionally employed as standards of capacitance in electrical measurements, but this form of condenser is not used commercially.

Concentric conductors of the form shown in Fig. 25*b* have, however, a large practical application as distributing mains in single-phase alternating-current systems, as, with this arrangement of conductors, no inductive effects are produced by the alternations of the current; moreover, such cables cannot produce external magnetic fields.

Concentric cables, however, may possess an appreciable capacitance, which may be calculated by means of equation (23*a*). Since in practice we usually require the capacitance per 1000 yd., or per mile,

equation (23*a*) is modified so as to give the capacitance for these lengths instead of unit (cm.) length. Thus

$$C = \frac{1000 \times 36 \times 2{\cdot}54 \times \kappa}{41{\cdot}4 \times 10^5 \log_{10}(r_1/r)} = \frac{0{\cdot}022 \times \kappa}{\log_{10}(r_1/r)} \mu\text{F. per 1000 yd.} \quad (23b)$$

$$= \frac{1760 \times 36 \times 2{\cdot}54 \times \kappa}{41{\cdot}4 \times 10^5 \log_{10}(r_1/r)} = \frac{0{\cdot}039 \times \kappa}{\log_{10}(r_1/r)} \mu\text{F. per mile} \quad . \ (23c)$$

Example. The inner conductor of a concentric cable, designed for a working pressure of 2200 volts, consists of 37 strands of 0·064 in. wire, the overall diameter being 0·45 in. The insulation consists of impregnated paper of a radial thickness of 0·12 in. The outer conductor consists of a singe layer of 29 wires, laid over the insulation.

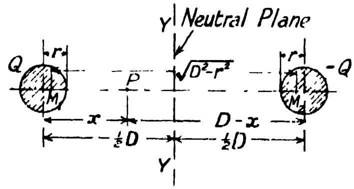


FIG. 26. PERTAINING TO THEORY OF PARALLEL CYLINDRIC CONDENSER

[*Note.*—In practice, the outer conductor of a concentric cable is earthed at the generating station.]

The capacitance of the cable is obtained by substituting in equations (23*b*), (23*c*). The ratio (r_1/r) $(0{\cdot}225 + 0{\cdot}12)/0{\cdot}225 = 1{\cdot}533$ and $\log_{10} 1{\cdot}533 = 0{\cdot}1855$. Hence, assuming the dielectric constant of the impregnated paper to be 3·2, we have

$$C = (0{\cdot}022 \times 3{\cdot}2)/0{\cdot}1855 = 0{\cdot}38\ \mu\text{F. per 1000 yd.}$$

$$= (0{\cdot}0388 \times 3{\cdot}2)/0{\cdot}1855 = 0{\cdot}67\ \mu\text{F. per mile.}$$

Capacitance of Parallel Cylinders. Consider two long, straight, and parallel conductors surrounded by air and removed from other conductors. Let r cm. be the radius of each conductor, D cm. the distance between the axes of the conductors, $+Q$, $-Q$ the charges per cm. length. Then, assuming r to be small in comparison with D, the charges may be considered to be concentrated at the axes of the cylinders. Hence the force acting on unit charge at a point P, distant x cm. from the axis of one cylinder, Fig. 26, is

$$F = \frac{4\pi Q}{2\pi x} + \frac{4\pi Q}{2\pi(D-x)} = \frac{2Q}{x} + \frac{2Q}{D-x} \text{ dynes.}$$

This force is a minimum in the neutral plane YY, which bisects perpendicularly, the plane containing the axes of the conductors.

Therefore the work done in moving unit charge from the neutral plane to the surface of one conductor is equal to

$$\int_r^{\frac{1}{2}D} \left(\frac{2Q}{x}dx + \frac{2Q}{D-x}dx\right) = 2Q\Big[\log_\epsilon x - \log_\epsilon (D-x)\Big]_r^{\frac{1}{2}D}$$

$$= 2Q \log_\epsilon \frac{D-r}{r} \text{ ergs}$$

and the total work done when unit charge is moved from one conductor to the other is

$$2\int_{r}^{\frac{1}{2}D}\left(\frac{2Q}{x}\,dx+\frac{2Q}{D-x}\,dx\right)=4Q\log_{\varepsilon}\frac{D-r}{r}=4Q\log_{\varepsilon}\left(\frac{D}{r}-1\right)\text{ergs},$$

which is equal to the potential difference between the conductors.

Hence the capacitance per cm. length of the conductors is

$$C=\frac{1}{4\log_{\varepsilon}\{(D/r)-1\}}\text{ electrostatic units} \quad . \quad . \quad . \quad (24)$$

$$=\frac{1}{82{\cdot}8\times10^{5}\log_{10}\{(D/r)-1\}}\text{ microfarads} \quad . \quad . \quad (24a)$$

and the capacitance per mile is

$$C=\frac{0{\cdot}0195}{\log_{10}\{(D/r)-1\}}\text{ microfarads} \quad . \quad . \quad . \quad . \quad (24b)$$

If the distance apart of the conductors is not large in comparison with their radii, the charges cannot be assumed to be concentrated at the axes of the conductors. The charges may, however, be considered to be concentrated along axes M_1, M_2, Fig. 26, which are contained by the plane containing the axes of the conductors and are a distance $\sqrt{(D^2-4r^2)}$ apart. In this case the capacitance per cm. length is given by

$$C=\frac{1}{82{\cdot}8\times10^{5}\log_{10}\left\{\dfrac{D+\sqrt{(D^2-4r^2)}}{2r}\right\}}\text{ microfarads} \quad . \quad (24c)$$

Examples. (*a*) Two conductors, each 0·5 in. in diameter, are stretched horizontally in space with their axes parallel and 2 ft. apart.

The capacitance per 1000 yd. of the system is given with sufficient accuracy by

$$C=\frac{0{\cdot}011}{\log_{10}(D/r)}\text{ microfarads}$$

$$=\frac{0{\cdot}011}{\log_{10}(24/0{\cdot}25)}=\frac{0{\cdot}011}{1{\cdot}9823}=0{\cdot}0056\ \mu\text{F}.$$

(*b*) Two conductors, each 1 in. in diameter, are supported horizontally in space with their axes parallel and 4 in. apart.

The capacitance per 1000 yd. of the system is now given by

$$C=\frac{0{\cdot}011}{\log_{10}[\{D+\sqrt{(D^2-r^2)}\}/2r]}\text{ microfarads}$$

$$\frac{0{\cdot}011}{\log_{10}[\{4+\sqrt{(4^2-1^2)}\}/1]}=\frac{0{\cdot}011}{\log_{10}7{\cdot}87}=0{\cdot}01238\ \mu\text{F}.$$

Dielectric Constant (Specific Inductive Capacity). The effect of an insulating medium other than air between the plates of a condenser invariably leads to an increase in the capacitance of the condenser. The ratio of the capacitance of a given condenser, with a given substance as dielectric, to the capacitance of the same condenser with air as dielectric, is called the *specific inductive capacity*, or the *dielectric constant*, of the substance, and is denoted by κ. With ordinary gases κ differs little from unity, but with solids and liquids κ exceeds unity. Approximate values of κ for a number of dielectrics are

Air [0° C. 760 mm. Hg.]	1·000	Paper, manilla (dry)	1·95
Glass, heavy flint	9·9	,, oil impregnated	3·4 to 3·6
,, light flint	6·6		
,, hard crown	6·9	,, paraffin waxed	3·0
Gutta-percha	2·8	Paraffin wax	2·36
Mica	8·0	Rubber, pure para	2·6
Oil, linseed	3·35	,, vulcanized	2·72
,, petroleum	2·13	,, hard (ebonite)	3·15
,, rape seed	2·85	Sulphur	4·2
,, turpentine	2·23	Vacuum	0·99941

Notes.—The above values are based principally upon test results obtained by employing alternating electric forces, the frequency being approximately 1000 cycles per second and the temperature about 15–20° C. With certain substances, e.g. rubber, gutta-percha, the values of κ given above may differ appreciably from those obtained with steady electric force, but with other substances κ varies only slightly with frequency. In general κ is smaller for alternating than for steady electric forces.

Hygroscopic substances show large variations of κ according to the quantity of moisture present. With these substances the variation of κ due to frequency and temperature increases with increase of moisture.

With solid dielectrics the variation of κ with temperature is usually small. In general, an increase of temperature results in a decrease in κ, e.g. the dielectric constant of paraffin wax decreases 0·036 per cent per 1° C. over a range of 11–32° C., but those of ebonite and sulphur increase about 0·1 per cent per 1° C. rise of temperature over a range of 10–20° C.

When a conductor is surrounded by a dielectric other than air, the electric force at a given point in the dielectric, due to a given charge, is $1/\kappa$ of that at the same point when the conductor is surrounded by air, the charge and other conditions remaining unaltered. Hence the potential in the former case will be $1/\kappa$ of that in the latter case.

Commercial Forms of Condensers. When condensers of fixed capacitance are required for commercial purposes the parallel-plate form is generally adopted. The manufacture of these condensers for telephone, radio, and power purposes is carried out by employing continuous sheets of aluminium foil for the "plates" and winding these in a roll together with two or more sheets of thin interleaving paper. The winding is done on specially designed reeling machines,

and during the process narrow strips of foil are inserted at intervals across the "plates," these strips being ultimately connected to the terminals of the condenser.

When the requisite length of foil has been wound, the roll is removed from the machine; it is then dried *in vacuo* to remove moisture and occluded air, impregnated either with paraffin wax or a viscous mineral oil, and finally sealed hermetically in a metal case.

Condensers of this type are called *paper condensers* and are used in large numbers for telephone, radio, and power circuits. For telephone and low-voltage radio circuits the wax-impregnated form is employed, but for power circuits the oil-impregnated form is employed and the oil-impregnated rolls are immersed in a light "filling" oil which acts both as a sealing as well as a cooling medium. In the latter case the rolls are wound to have a capacitance between about 2·5μF. (for a working voltage of 220 V.) and 0·5 μF. (for a working voltage of 900 V.), and the requisite number of rolls are connected in parallel to obtain the required capacitance. Thus a 1000-μF. condenser for a power circuit may consist of from about 500 to 2000 individual condensers according to the working voltage.

Condensers for *extra-high-voltage circuits* may have a dielectric of mica, glass or oil-impregnated paper. Extra-high-voltage condensers with oil-impregnated paper dielectric are a recent development, and such condensers are less costly than mica-dielectric condensers of similar rating. For radio frequencies, however, condensers with mica dielectric are preferred on account of their low dielectric losses.

Electrolytic condensers are not used on A.C. circuits because the oxide film which forms the dielectric possesses insulating properties only when the electrode on which the film is formed is connected to the positive pole of the circuit.

Standard condensers for laboratory purposes have either mica or air dielectric, air being employed when a condenser without dielectric losses and dielectric absorption is required.

Variable-capacitance condensers, with continuous adjustment of capacitance between definite limits, are of the multiple-plate type, with either air or oil dielectric. One "plate" consists of a fixed stack of thin aluminium, or brass, vanes spaced uniformly a small distance apart and electrically connected together; the other "plate" consists of a similar set of vanes mounted on a spindle and so arranged that they can be rotated in the spaces between the fixed vanes so as to present to the latter a variable surface.

Charging Current of a Condenser. Assume the dielectric to be a perfect non-conductor and let the condenser receive a charge q during the time t. Then the mean rate of charge during this

interval is q/t, and is called the *charging current*. To obtain the instantaneous value of this current t must be taken infinitely small, i.e. dt. Then if the corresponding charge which accumulates on the plates of the condenser is dq, the instantaneous charging current (i) is equal to dq/dt. Hence if C is the capacitance of the condenser and e the potential difference between the plates when the charge is q,

$$dq = Cde,$$

whence

$$i = Cde/dt.$$

Thus *the charging current is directly proportional to the rate of change of the potential difference at the plates of the condenser.*

If the applied potential difference varies sinusoidally, and is given by $e = E_m \sin \omega t$, then the current will be given by

$$\begin{aligned} i = C\frac{de}{dt} &= C\frac{d}{dt}(E_m \sin \omega t) \\ &= \omega C E_m \cos \omega t \\ &= \omega C E_m \sin(\omega t + \tfrac{1}{2}\pi) \quad . \quad . \quad . \quad (25) \end{aligned}$$

Thus the current varies sinusoidally with the same frequency as the applied E.M.F., but leads the latter by 90°.

The maximum value of the current is

$$I_m = \omega C E_m$$

and the R.M.S. value is

$$I = \omega C E = 2\pi f C E$$

Thus the charging current is directly proportional to the frequency and the impressed E.M.F. For example, the charging current of a 1μ F. condenser connected to a 100 V., 50-cycle, circuit is

$$I = 2\pi \times 50 \times 1 \times 10^{-6} \times 100 = 0{\cdot}0314 \text{ A.}$$

For the same condenser connected to a 100 V., 25-cycle circuit the charging current is

$$I = 2\pi \times 25 \times 1 \times 10^{-6} \times 100 = 0{\cdot}0157 \text{ A.}$$

and when it is connected to a 600 V., 50-cycle, circuit the charging current is

$$I = 2\pi \times 50 \times 1 \times 10^{-6} \times 600 = 0{\cdot}1884 \text{ A.}$$

The ratio $E/I = 1/\omega C$ is called the *reactance* of the condenser, and is expressed in ohms. As the term reactance is employed in

connection with inductive circuits, it is necessary to distinguish between reactance due to inductance and that due to capacitance. The former is usually called " inductive reactance," and the latter " capacitive reactance."

The reactance of a condenser of given capacitance is therefore inversely proportional to the frequency. Hence a condenser of relatively small capacitance may have an extremely low reactance at exceptionally high frequencies, such as those of lightning discharges. This property of condensers is utilized in practice for protecting apparatus from high-voltage, high-frequency, surges ; a condenser being connected across the circuit, or between the circuit and earth,

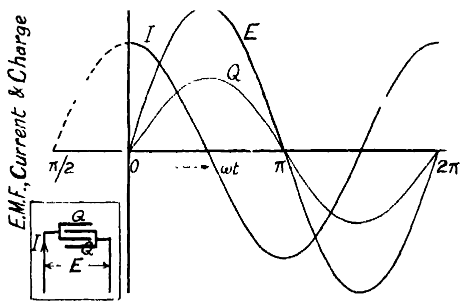


FIG. 27. GRAPHIC REPRESENTATION OF CURRENT, CHARGE AND IMPRESSED E.M.F. FOR A CONDENSER

so that the high-frequency surge may discharge through the condenser. The dielectric of the condenser, however, must be capable of withstanding high voltage.

A *graphical representation* of the conditions expressed in equation (25) is shown in Fig. 27, in which the sine curve E represents the impressed E.M.F., and the sine curve I, 90° in advance of E, represents the charging current. The variation of the charge with respect to time is shown by the sine curve Q, which is the integral of the current curve, since $Q = \int i.dt = \int I_m \cos \omega t \, dt = (I_m/\omega) \sin \omega t = CE_m \sin \omega t$.

A condenser connected to an alternating supply is therefore charged and discharged periodically : it receives a charge during the first positive quarter-period of the E.M.F., discharges during the next quarter-period, and is again charged and discharged successively during the following half-period. During charge the current decreases from its positive maximum value to zero, and during discharge the current rises from zero to its maximum value.

The energy stored in the condenser during charging is given back to the circuit during discharge, so that there is a continual transference, or surge, of energy to and from the condenser. In practice, due to losses in the dielectric, the energy given back to the circuit during discharge is always slightly less than that stored in the condenser during charge. This, however, is not the case when the dielectric is dry air.

Relation Between Current and E.M.F. for Circuits Containing Capacitance and Resistance in Series. Let the current in the circuit be represented by $i = I_m \sin \omega t$. Also let e_1, e_2, denote, at any instant, t, the internal E.M.Fs. due to (1) the current passing through the resistance, and (2) the charging of the condenser. Then if R is the resistance of the circuit and C is the capacitance of the condenser

$$e_1 = -Ri = -RI_m \sin \omega t$$

$$e_2 = -(1/C)\int i.dt = -(I_m/C)\int \sin \omega t.dt = +(I_m/\omega C)\cos \omega t$$

The total internal E.M.F. acting in the circuit at this instant is therefore equal to $(e_1 + e_2)$, and must balance the impressed E.M.F. (e). Hence

$$\begin{aligned} e &= -(e_1 + e_2) = RI_m \sin \omega t - (I_m/\omega C)\cos \omega t \\ &= I_m\sqrt{\{R^2 + (1/\omega C)^2\}}\left\{\frac{R}{\sqrt{\{R^2 + (1/\omega C)^2\}}}\sin \omega t - \right. \\ &\qquad\qquad \left. \frac{1/\omega C}{\sqrt{\{R^2 + (1/\omega C)^2\}}}\cos \omega t\right\} \\ &= I_m\sqrt{\{R^2 + (1/\omega C)^2\}}\sin(\omega t - \varphi) = E_m \sin(\omega t - \varphi) \end{aligned}$$

where $\tan \varphi = \dfrac{1/\omega C}{R} = \dfrac{1}{\omega CR}$

Thus the impressed E.M.F. is sinusoidal and lags behind the current by the angle φ.

Conversely, if a sinusoidal E.M.F., represented by the equation $e = E_m \sin \omega t$, be applied to the circuit, the current, when the steady, or cyclic, state is reached will be given by

$$i = \frac{E_m}{\sqrt{\{R^2 + (1/\omega C)^2\}}}\sin(\omega t + \varphi) \quad . \quad . \quad . \quad . \quad (26)$$

The maximum value of the current is

$$I_m = \frac{E_m}{\sqrt{\{R^2 + (1/\omega C)^2\}}}$$

and the R.M.S. value of the current is

$$I = \frac{E}{\sqrt{\{R^2 + (1/\omega C)^2\}}}$$

Hence the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + (1/\omega C)^2\}}$$

The *vector diagram* for this circuit is shown in Fig. 28, in which OE represents the impressed E.M.F. and OI the current. The vector OE may be resolved into two components: one, OA, in phase with the current vector, and the other, OB, perpendicular to the current vector. OA therefore represents the component of the impressed E.M.F. which is expended against the resistance of the circuit, and OB the potential difference at the terminals of the condenser. Now $OA = RI$; $OB = AE = I/\omega C$; $OE = I\sqrt{[R^2 + (1/\omega C)^2]}$. Hence $OA : AE : OE = R : 1/\omega C : \sqrt{[R^2 + (1/\omega C)^2]}$. Therefore triangle OEA is the impedance triangle for the circuit.

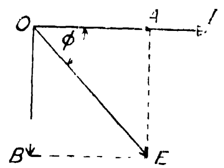


FIG. 28. VECTOR DIAGRAM FOR CIRCUIT CONTAINING RESISTANCE AND CAPACITANCE IN SERIES

Relation Between Current and E.M.F. for a Series Circuit Containing Resistance, Inductance, and Capacitance. Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$. Then if the resistance, inductance, and capacitance are denoted by R, L, C respectively, the equation to the impressed E.M.F. is given by

$$e = RI_m \sin \omega t + \omega L I_m \cos \omega t - (I_m/\omega C) \cos \omega t$$
$$= I_m \{R \sin \omega t + (\omega L - 1/\omega C) \cos \omega t\}$$

which, when simplified by the method given on p. 44, becomes

$$e = I_m \sqrt{[R^2 + \{\omega L - (1/\omega C)\}^2]} \sin (\omega t + \varphi) = E_m \sin(\omega t + \varphi)$$

where $\tan \varphi = [\omega L - (1/\omega C)]/R$.

Conversely, if a sinusoidal E.M.F.—represented by the equation $e = E_m \sin \omega t$—be applied to the circuit, the current, when the steady, or cyclic, state is reached, will be given by

$$i = \frac{E_m}{\sqrt{R^2 + [\omega L - (1/\omega C)]^2\}}} \sin (\omega t - \varphi) \quad . \quad . \quad . \quad (27)$$

The maximum value of the current is

$$I_m = \frac{E_m}{\sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}}$$

and the R.M.S. value of the current is

$$I = \frac{E}{\sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}}.$$

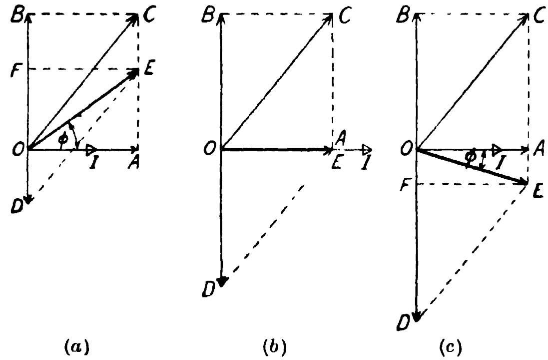


FIG. 29. VECTOR DIAGRAMS FOR A SERIES CIRCUIT CONTAINING RESISTANCE, INDUCTANCE, AND CAPACITANCE;
(a) $\omega L > 1/\omega C$, (b) $\omega L = 1/\omega C$, (c) $\omega L < 1/\omega C$

Hence the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}$$

and the effective reactance by

$$X = \omega L - (1/\omega C)$$

$$= \text{inductive reactance} - \text{capacitive reactance}.$$

The phase difference, φ, between current and E.M.F. may be lagging, zero, or leading, according to whether $\omega L > = < 1/\omega C$. For example, when $\omega L > 1/\omega C$ the current is lagging with respect to the impressed E.M.F., but when $\omega L < 1/\omega C$ the current leads the impressed E.M.F.

The *vector diagrams* for these cases are shown in Fig 29, diagram (a) referring to the case when $\omega L > 1/\omega C$ and φ is positive ; (b) referring to the case when $\omega L = 1/\omega C$ and φ is zero ; (c) referring to the case when $\omega L < 1/\omega C$ and φ is negative. In these diagrams the current vector, OI, is taken as the vector of reference, and the impressed E.M.F. is represented by OE. The component OA, which

is in phase with the current vector, represents the E.M.F. expended against the resistance of the circuit. The component OB, which leads the current vector by 90°, represents the E.M.F. which balances the E.M.F. of self-induction. The resultant, OC, of OA and OB represents the potential difference across the resistance and inductance ; the angle AOC being the phase difference between this potential difference and the current. The potential difference between the terminals of the condenser is represented by OD, which lags 90° with respect to the current vector. Obviously, the vector sum of OC and OD must equal the impressed E.M.F. OE. The component, OF, which is perpendicular to the current vector is equal to the vector difference of OB and OD. Now $OA : OF (= AE) : OE = RI : I(\omega L - 1/\omega C) : I\sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}} = R : (\omega L - 1/\omega C) : \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$. Therefore triangle OFE is the impedance triangle for the circuit. Observe that when $\omega L > 1/\omega C$ the effective reactance of the circuit is positive and the current is lagging, but when $\omega L < \omega 1/C$ the effective reactance is negative and the current is leading.

In the special case when $\omega L = 1/\omega C$, the current is in phase with the impressed E.M.F. and is equal to E/R. The circuit is therefore equivalent, so far as its impedance is concerned, to a non-inductive circuit of resistance R, and is said to be in a *condition of resonance*. Under these conditions the voltages across the condenser and inductance may each be much greater than the impressed E.M.F. (see examples in Chap. VI).

CHAPTER V

POWER IN ALTERNATING-CURRENT CIRCUITS

Instantaneous Power in an Alternating-current Circuit. In an alternating-current circuit the power (p) at any instant is equal to the product of the instantaneous values of current and E.M.F. Thus $p = ei$.

Since current and E.M.F. vary with respect to time, the power will also vary from instant to instant, and may be positive, negative, or zero according to the signs and magnitudes of current and E.M.F. If, at a given instant, the current and E.M.F. have the same sign, the power is positive—indicating that power is being supplied *to* the circuit—but if these quantities have opposite signs the power is negative—indicating that power is being returned *from* the circuit to the generator—while if either, or both, of the quantities are zero, the power is zero

Graphical Representation. In the general case of sinusoidal current and E.M.F. differing in phase, the power has four zero values for each cycle of the current, or E.M.F., and the direction of the power reverses four times in each cycle, as shown graphically in Fig. 30, in which the curves E, I, represent the impressed E.M.F. and current respectively, and curve P represents the power. The shaded areas in this diagram represent energy; areas above the abscissa axis denote that energy is being supplied *to* the circuit, and those below the axis denote that energy is being returned *from* the circuit. For each cycle the difference between the areas above and below the abscissa axis represents the energy expended in the circuit, either in doing useful work or supplying losses.

Two *special cases* of the general case (Fig. 30) are of importance, viz. (1) when the phase difference between current and E.M.F. is zero, (2) when it is 90°. These cases are represented graphically in Figs 31 and 32. In the case represented in Fig. 31—where the current and E.M.F. are in phase—the power pulsates between zero and a definite maximum value, but does not change sign throughout the cycle. Hence in this case power is transmitted always in one direction, viz. from generator to circuit.

In the cases represented in Fig. 32—in which E.M.F. and current have a phase difference of 90°, this being lagging in one case, Fig. 32*a*, and leading in the other, Fig. 32*b*—the power curve (P) alternates at twice the frequency of the current or E.M.F. Hence

for each half-cycle of current, or E.M.F., there are two alternations of power. Therefore during this interval a certain amount of energy is supplied to the circuit and an equal amount is returned to

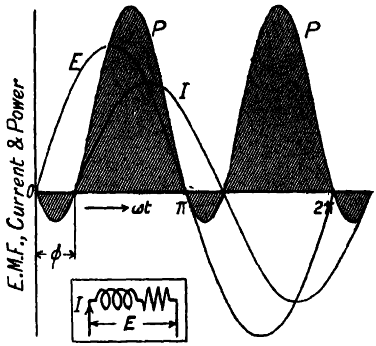


FIG. 30. E.M.F., CURRENT AND POWER IN CIRCUIT CONTAINING RESISTANCE AND INDUCTANCE

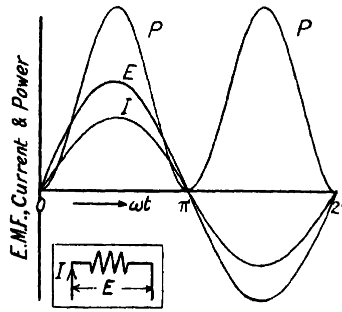


FIG. 31. E.M.F., CURRENT AND POWER IN NON-INDUCTIVE CIRCUIT

the generator. Thus, although energy is continually surging between generator and circuit, no energy is actually expended in the latter.

During the time that the current and E.M.F. have the same sign energy is stored in the circuit—in either the electrostatic or the

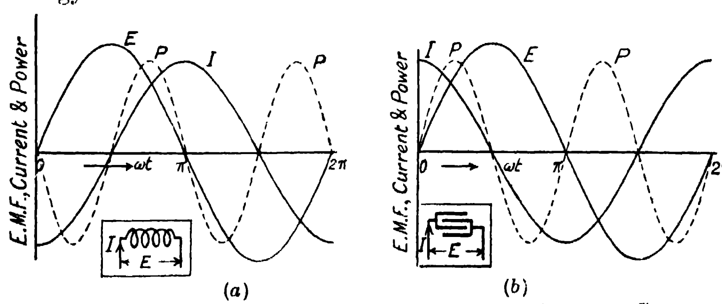


FIG. 32. E.M.F., CURRENT AND POWER IN PURELY REACTIVE CIRCUITS

electromagnetic form, according to whether the circuit is purely capacitive or inductive—and this energy is returned to the generator during the time that the current and E.M.F. have opposite signs.

Components of Power Curve. The power curve (*P*, Fig. 33) for the general case of a circuit containing resistance, inductance and capacitance may be resolved into components representing the

power in the several parts of the circuit. To separate out these components it is necessary to determine the curves for the potential differences across the several parts of the circuit. Now the potential difference across the resistance is in phase with the current; that across the inductance leads the current by 90°, and that across the condenser lags 90° with respect to the current. The curves for these quantities, together with those for the impressed E.M.F. and

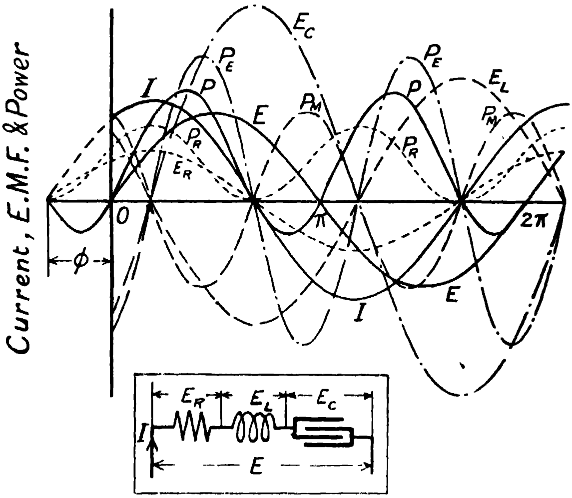


FIG. 33. E.M.F., CURRENT AND POWER IN CIRCUIT CONTAINING RESISTANCE, INDUCTANCE, AND CAPACITANCE

(Capacitive reactance > inductive reactance)

current, are shown in Fig. 33; curves E, I, denoting the impressed E.M F. and current, respectively, and curves E_R, E_L, E_C, denoting the components of the impressed E.M.F. which are expended against resistance, inductance, and capacitance respectively. The instantaneous power in the several parts of the circuit is given by the product of the current and the appropriate E.M.F. curve. The power component curves so obtained are marked P_R, P_M, P_E. P_R therefore represents the power expended in supplying the losses in the resistance; P_M, P_E, represent the power supplied to the inductive and capacitive portions, respectively, of the circuit. The difference between the curves P_E, P_M, represents the component of the power which surges between the generator and the circuit.

Analytical Expressions for Instantaneous Power and Power Components. Let the equation to the impressed E.M.F. be $e = E_m \sin \omega t$,

and that to the current be $i = I_m \sin(\omega t - \varphi)$. Then the instantaneous power is given by

$$p = ei = E_m I_m \sin \omega t \,.\, \sin(\omega t - \varphi) \quad . \quad . \quad . \quad . \quad (28)$$

This is the equation to the power curve P, Fig. 30. The equation to the power curve (P, Fig. 33) for the case when the current is leading the impressed E.M.F. is $p = E_m I_m \sin \omega t \sin(\omega t + \varphi)$.

In a general series circuit containing resistance, inductance and capacitance the equations to the components of the impressed E.M.F. for the several portions of the circuit are

$$e_R = Ri = RI_m \sin(\omega t - \varphi)$$

$$e_L = L\,di/dt = \omega L I_m \cos(\omega t - \varphi)$$

$$e_C = (1/C)\int i.dt = -(I_m/\omega C)\cos(\omega t - \varphi)$$

The components of the power for the several parts of the circuit are therefore,

$$\begin{aligned} p_R &= ie_R = RI_m^2 \sin^2(\omega t - \varphi) = \tfrac{1}{2}RI_m^2\{1 - \cos 2(\omega t - \varphi)\} \\ &= \tfrac{1}{2}RI_m^2\{1 + \sin[2(\omega t - \varphi) - \tfrac{1}{2}\pi] \\ p_M &= ie_L = \omega L I_m^2 \sin(\omega t - \varphi)\cos(\omega t - \varphi) \\ &= \tfrac{1}{2}\omega L I_m^2 \sin 2(\omega t - \varphi) \\ p_E &= ie_C = -(I_m^2/\omega C)\sin(\omega t - \varphi)\cos(\omega t - \varphi) \\ &= -\tfrac{1}{2}(I_m^2/\omega C)\sin 2(\omega t - \varphi). \end{aligned}$$

The first expression represents a pulsating quantity ; the second and third expressions represent alternating quantities of a frequency twice that of the current. The resultant of the two alternating quantities p_M, p_E, is, except in the special case when $p_M = p_E$, another alternating quantity, viz.—

$$\begin{aligned} p_X &= p_M + p_E \\ &= \tfrac{1}{2}I_m^2[\omega L - (1/\omega C)]\sin 2(\omega t - \varphi). \end{aligned}$$

Hence in the general case the instantaneous power consists of a pulsating component equal to

$$p_R = RI_m^2 \sin^2(\omega t - \varphi) = \tfrac{1}{2}RI_m^2\{1 + \sin[2(\omega t - \varphi) - \tfrac{1}{2}\pi]\} \quad (29)$$

and a double-frequency alternating component equal to

$$p_X = \tfrac{1}{2}I_m^2\{\omega L - (1/\omega C)\}\sin 2(\omega t - \varphi) \quad . \quad . \quad . \quad (30)$$

The pulsating component is in phase with the current, i.e. its zero and maximum values occur at the same instants as the corresponding values of the current. The double-frequency alternating

component is symmetrical with respect to the current wave, i.e. the three zero values corresponding to one cycle of the former coincide with the two zero and maximum values for one half-cycle of the current.

Equation to Power Curve. This equation may be obtained from equations (29) and (30), since $p = p_R + p_X$, but it is best obtained by expanding equation (28). Thus

$$\begin{aligned} p &= E_m I_m \sin \omega t \,.\, \sin(\omega t - \varphi) \\ &= E_m I_m \times \tfrac{1}{2}\{\cos[\omega t - (\omega t - \varphi)] - \cos[\omega t + (\omega t - \varphi)]\} \\ &= \tfrac{1}{2} E_m I_m \{\cos \varphi - \cos(2\omega t - \varphi)\} \\ &= \tfrac{1}{2} E_m I_m \cos \varphi + \tfrac{1}{2} E_m I_m \sin[2\omega t - (\tfrac{1}{2}\pi + \varphi)] \quad . \quad (31) \end{aligned}$$

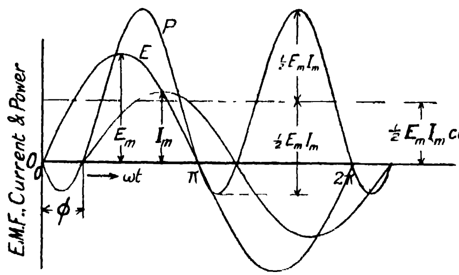


FIG. 34. GRAPHICAL REPRESENTATION OF EQUATION TO POWER CURVE OF FIG. 30

The first term represents a constant quantity of value $\frac{1}{2}E_mI_m\cos\varphi$; the second term represents a sinusoidal quantity of maximum value $\frac{1}{2}E_mI_m$, and of twice the frequency of the current or E.M.F., lagging $(\frac{1}{2}\pi + \varphi)$ with respect to a reference quantity $A \sin 2\omega t$.

Hence the power curve (P, Fig. 30) may be represented by a double-frequency sine curve—of maximum value $\frac{1}{2}E_mI_m$ and displaced $(\frac{1}{2}\pi + \varphi)$ with respect to the time-angle zero—superimposed upon a horizontal axis at a distance $\frac{1}{2}E_mI_m \cos\varphi$ above the abscissa axis, as shown in Fig. 34. When $\varphi = 0$, the axis of the double-frequency curve is at a distance $\frac{1}{2}E_mI_m$ above the abscissa axis, and since the maximum value of this curve is $\frac{1}{2}E_mI_m$, it does not, therefore, cross the abscissa axis. Hence we have the conditions shown in Fig. 31. When $\varphi = \frac{1}{2}\pi$, $\cos\varphi = 0$, and the axis of the double-frequency curve coincides with the abscissa axis. We then have the conditions shown in Fig. 32.

Mean, or True, Power. The mean, or true, power in an alternating current circuit is defined as the algebraic mean of the instantaneous

values of the power taken over a period. In the general case, when the instantaneous power is given by

$$p = \tfrac{1}{2}E_m I_m \cos\varphi + \tfrac{1}{2}E_m I_m \sin[2\omega t - (\tfrac{1}{2}\pi + \varphi)],$$

the mean power is

$$P = \tfrac{1}{2}E_m I_m \cos\varphi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

since the mean value of the double-frequency term, taken over a period, is zero.

Substituting R.M.S. values for current and E.M.F., we have

$$P = \tfrac{1}{2}(\sqrt{2}E \,.\, \sqrt{2}I)\cos\varphi$$

$$= EI\cos\varphi \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32a)$$

Therefore, *the mean power in an alternating-current circuit is given by the product of the impressed E.M.F. and the component of the current which is in phase with the impressed E.M.F.*

Apparent Power or Volt-amperes. The product $E\,I$ may be called the *apparent power*, but in practice the term *volt-amperes* (abbreviated VA.) is employed. Similarly $EI/1000$ is called *kilo-volt-amperes* (abbreviated kVA.). This quantity is important in the rating of all A.C. machines and transformers as the temperature rise depends upon kVA. rather than kW. Again the rating of cables and switchgear for a given circuit voltage depends upon the current, and therefore in these cases also kVA., rather than kW., must be considered.

Power Factor. The term $\cos\varphi$, in the expression for power, is called the "power factor" of the circuit, as it is the factor by which the apparent power must be multiplied to obtain the true power.

In the case of a non-inductive circuit, $\varphi = 0$, $\cos\varphi = 1$, and the power factor is unity. The true power is then given by the product of the impressed E.M.F. and current.

In the case of circuits containing pure inductance or capacitance, $\varphi = 90°$, $\cos\varphi = 0$, and the power factor is zero.

Reactive Volt-amperes. The product of the impressed E.M.F. and the component of the current which is perpendicular to it, i.e. the product $EI \sin\varphi$, is called the *reactive volt-amperes* (abbreviated VAr.). Similarly $(EI/1000)\sin\varphi$ is called *reactive kilo-volt-amperes* (abbreviated kVAr.). It represents the component of the volt-amperes which surges between the generator and the circuit,

and *vice versa*, without doing work. In commercial circuits the reactive volt-amperes supplies the magnetic and electrostatic fields, and the true power ($EI \cos \varphi$) supplies the losses and performs useful, or mechanical, work.

Measurement of Power. The true power in any circuit may be measured either indirectly by measuring each of the quantities E, I, $\cos \varphi$, separately by means of suitable instruments as discussed in Chapter XVI, or directly by means of a wattmeter (see p. 374). The latter method is usually adopted in practice on account of its superior accuracy, but special precautions are necessary in order to obtain accurate readings at low power factors (see p. 378).

Power Component and Wattless Component of Current. In a circuit supplied at constant voltage the true power is proportional to $I \cos \varphi$, and the reactive volt-amperes is proportional to $I \sin \varphi$. Now $I \cos \varphi$ is the component of the current in phase with the impressed E.M.F., and $I \sin \varphi$ is the component at right angles to the impressed E.M.F. The component $I \cos \varphi$ is therefore called the "power component" or the "in-phase component" of the current, and $I \sin \varphi$ is called the "wattless component" (sometimes the terms "idle current," "quadrature component," and "reactive component" are employed).

This method of resolving the current into power and wattless components is useful in the solution of problems and its applications are discussed in Chapter VI. The method is particularly useful in connection with parallel circuits. For example, if a number of circuits are connected in parallel and if I_p, I_w, denote respectively the sums of the power and wattless components of the currents in the several branch circuits, the total, or "line," current supplied to the circuits is given by $I = \sqrt{(I_p^2 + I_w^2)}$, and the phase difference between the impressed E.M.F. and line current is given by $\varphi = \tan^{-1} (I_w/I_p)$.

Example. Three circuits A, B, C, connected in parallel are supplied with power from 220 V. mains. Circuit A consists of a bank of incandescent lamps taking a current of 15 A. at unity power factor; B consists of an inductive resistance taking a current of 20 A. at a power factor of 0·85 lagging; C consists of an apparatus taking a current of 10 A. at a power factor of 0·95 leading. Determine the current and power supplied by the mains; also the power factor.

Let the currents in the several circuits be denoted by I_A, I_B, I_C, respectively, and the phase differences by φ_A, φ_B, φ_C. Let the current in the mains be denoted by I, and its phase difference with respect to the impressed E.M.F. by φ.

Then $I_A = 15 \qquad \cos \varphi_A = 1{\cdot}0 \qquad \sin \varphi_A = 0$

$I_B = 20 \qquad \cos \varphi_B = 0{\cdot}85 \qquad \sin \varphi_B = -\sqrt{(1 - 0{\cdot}85^2)} = -0{\cdot}28$

$I_C = 10 \qquad \cos \varphi_C = 0{\cdot}95 \qquad \sin \varphi_C = \sqrt{(1 - 0{\cdot}95^2)} = 0{\cdot}1$

Resolving each current into its power and wattless components, we have

Power or in-phase components.	Wattless or quadrature components.
$I_A \cos \varphi_A = 15 \times 1{\cdot}0 = 15$	$I_A \sin \varphi_A = 15 \times 0 = 0$
$I_B \cos \varphi_B = 20 \times 0{\cdot}85 = 17$	$I_B \sin \varphi_B = 20 \times (-0{\cdot}28) = -5{\cdot}6$
$I_C \cos \varphi_C = 10 \times 0{\cdot}95 = 9{\cdot}5$	$I_C \sin \varphi_C = 10 \times 0{\cdot}1 = 1{\cdot}0$

$\therefore I \cos \varphi = (15 + 17 + 9{\cdot}5) = 41{\cdot}5 \quad \therefore I \sin \varphi = (-5{\cdot}6 + 1{\cdot}0) = -4{\cdot}6$

Hence $\qquad I = \sqrt{[(I \cos \varphi)^2 + (I \sin \varphi)^2]} = \sqrt{(41{\cdot}5^2 + 4{\cdot}6^2)} = 41{\cdot}8$ A.

$$\cos \varphi = (I \cos \varphi)/I = 41{\cdot}5/41{\cdot}8 = 0{\cdot}993 \text{ (lagging)}$$

$$P = 220 \times 41{\cdot}8 \times 0{\cdot}993 = 9130 \text{ watts}$$

CHAPTER VI

SERIES AND PARALLEL CIRCUITS

I.—Series Circuits

Series Circuits of Constant Impedance. The impedance of a simple series circuit containing resistance and reactance is given by

$$Z = \sqrt{(R^2 + X^2)},$$

and the phase difference between impressed E.M.F. and current is

$$\varphi = \tan^{-1}(X/R).$$

In these equations X is the effective reactance of the circuit : it is equal to the algebraic difference between the inductive reactance (ωL) and the capacitive reactance $(1/\omega C)$, i.e. $X = \omega L - (1/\omega C)$.

In the case of a complex series circuit, such as is represented in Fig. 35, the joint impedance is given by

$$Z = \sqrt{\{(R_1 + R_2 + R_3 + \ldots)^2 + (X_1 + X_2 + X_3 \ldots)^2\}} \quad (33)$$

where R_1, R_2, R_3 . . . , are the resistances of the several parts of the circuit, and X_1, X_2, X_3 . . . . , are the effective reactances. The phase difference between impressed E.M.F. and current is given by

$$\varphi = \tan^{-1}(X_1 + X_2 + X_3 + \ldots)/(R_1 + R_2 + R_3 + \ldots),$$

or by $\quad \varphi = \cos^{-1}(R_1 + R_2 + R_3 + \ldots)/Z.$

These expressions follow directly from the vector diagram for the circuit. Thus, in Fig. 36 the current vector OI is taken as the vector of reference, and the potential differences across the several parts of the circuit are represented by the vectors OA, OB, OC. The geometric sum (OD) of these vectors represents the terminal voltage (E) of the circuit. From the diagram

$$\begin{aligned} E^2 &= (E_1\cos\varphi_1 + E_2\cos\varphi_2 + E_3\cos\varphi_3)^2 + (E_1\sin\varphi_1 + E_2\sin\varphi_2 \\ &\qquad + E_3\sin\varphi_3)^2 \\ &= (IR_1 + IR_2 + IR_3)^2 + (IX_1 + IX_2 + IX_3)^2, \end{aligned}$$

Whence,

$$\begin{aligned} Z &= E/I \\ &= \sqrt{\{(R_1 + R_2 + R_3)^2 + (X_1 + X_2 + X_3)^2\}}, \end{aligned}$$

and

$$\tan\varphi = (X_1 + X_2 + X_3)/(R_1 + R_2 + R_3).$$

Observe that the joint impedance is equal to the *geometric* sum of the separate impedances. In symbolic notation

$$\begin{aligned} \dot{Z} &= \dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34) \\ &= (R_1 + jX_1) + (R_2 + jX_2) + (R_3 + jX_3) \\ &= (R_1 + R_2 + R_3) + j(X_1 + X_2 + X_3) \end{aligned}$$

Graphical Construction for Obtaining the Joint Impedance of a Series Circuit. The construction is similar to that of the E.M.F. vector diagram of Fig. 36. In fact, by a suitable change of scale,

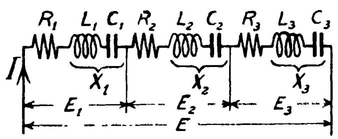


FIG. 35. DIAGRAMMATIC REPRESENTATION OF COMPLEX SERIES CIRCUIT

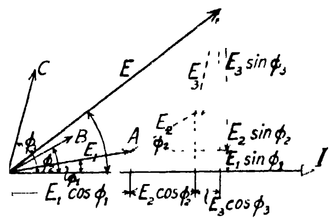


FIG. 36. VECTOR DIAGRAM FOR CIRCUIT REPRESENTED IN FIG. 35

the diagram becomes an impedance diagram, the vector OD now representing the impedance of the circuit. Instead of setting off the magnitudes Z_1, Z_2, Z_3 of the impedances and their phase angles φ_1, φ_2, φ_3, it is usually more convenient to set off their rectangular co-ordinates—resistance as abscissæ, and effective reactance as ordinates. In this case the only calculations required are those for obtaining the reactances of the several parts of the circuit.

Special Cases. When the reactance of the entire circuit is zero, the impedance becomes

$$Z = R_1 + R_2 + R_3 + \ . \ . \ . = R,$$

where R is the joint resistance, of the circuit.

For circuits containing only reactance we have

$$Z = X_1 + X_2 + X_3 \ . \ . \ . = X,$$

where X is the joint reactance of the circuit.

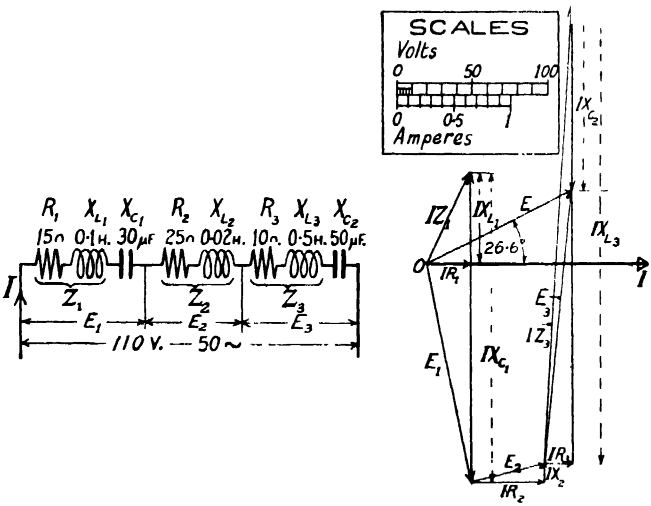
Hence if a number of condensers of capacitances C_1, C_2, C_3, . . are connected in series, the joint reactance will be given by

$$\begin{aligned} X &= X_1 + X_2 + X_3 + \ . \ . \ . \\ &= \frac{1}{\omega C_1} + \frac{1}{\omega C_2} + \frac{1}{\omega C_3} \end{aligned}$$

Whence the joint capacitance is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

Thus the series connection of condensers gives a joint capacitance smaller than the lowest individual capacitance of the group. For example, if two condensers of 10 μF. and 20 μF. are connected in series the joint capacitance is $1/(\frac{1}{10} + \frac{1}{20}) = 6{\cdot}66\ \mu$F.



FIGS. 37, 38. CIRCUIT AND VECTOR DIAGRAM FOR WORKED EXAMPLE

In the case of a circuit containing a number of condensers connected in series, the terminal potential difference of the circuit is divided across the condensers in the inverse ratio of their capacitances and the potential difference across the smallest condenser may be very much higher than that across the largest condenser. For example, if condensers of 5 μF. and 50 μF. are connected in series across 200 V. mains, the potential difference across the 5 μF. condenser will be $\{200 \times 50/(5 + 50)\} = 181{\cdot}8$ V., and that across the 50 μF. condenser will be $\{200 \times 5/(5 + 50)\} = 18{\cdot}2$ V.

Example. Determine, for the circuit shown in Fig. 37, (1) the joint impedance, (2) the current, (3) the phase difference between current and terminal E.M.F., (4) the potential differences across each part of the circuit. The circuit is supplied at a pressure of 110 V. and a frequency of 50 cycles per second.

The total resistance $= 15 + 25 + 10 = 50\ \Omega$.

The inductive reactance $= 2\pi \times 50(0{\cdot}1 + 0{\cdot}02 + 0{\cdot}5) = 195\ \Omega$.

The capacitive reactance $= \frac{10^6}{2\pi \times 50}\left(\frac{1}{30} + \frac{1}{50}\right) = 170\ \Omega.$

Hence the effective reactance $= 195 - 170 = 25\ \Omega.$

Whence the joint impedance $= \sqrt{(50^2 + 25^2)} = 55{\cdot}9\ \Omega.$

Therefore the current $= 110/55{\cdot}9 = 1{\cdot}97$ A.

The phase difference between impressed E.M.F. and current $= \cos^{-1} 50/55{\cdot}9$
$= 26{\cdot}6^\circ$

The potential differences across the several parts of the circuit are

$$
\begin{aligned}
IR_1 &= 1{\cdot}97 \times 15 = 29{\cdot}5 \text{ V.}\\
IX_1 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}1 = 61{\cdot}8 \text{ V.}\\
IZ_1 &= \sqrt{(29{\cdot}5^2 + 61{\cdot}8^2)} = 68{\cdot}5 \text{ V.}\\
I/\omega C_1 &= 1{\cdot}97 \times 10^6/(2\pi \times 50 \times 30) = 209 \text{ V.}\\
IR_2 &= 1{\cdot}97 \times 25 = 49{\cdot}2 \text{ V.}\\
IX_2 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}02 = 12{\cdot}35 \text{ V.}\\
IZ_2 &= \sqrt{(49{\cdot}2^2 + 12{\cdot}35^2)} = 50{\cdot}8 \text{ V.}\\
IR_3 &= 1{\cdot}97 \times 10 = 19{\cdot}7 \text{ V.}\\
IX_3 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}5 = 309 \text{ V.}\\
IZ_3 &= \sqrt{(19{\cdot}7^2 + 309^2)} = 309{\cdot}5 \text{ V.}\\
I/\omega C_2 &= 1{\cdot}97 \times 10^6/(2\pi \times 50 \times 50) = 125{\cdot}3 \text{ V.}
\end{aligned}
$$

A vector diagram drawn to scale is given in Fig. 38.

Series Circuits of Variable Impedance. (1) *Constant reactance, variable resistance.* Two cases are important, viz. (1) circuits in which the resistance is variable and the inductance is constant; (2) circuits in which the resistance is variable and the capacitance is constant. In both cases the phase difference between the E.M.Fs. across the two parts of each circuit is 90°. Hence for any particular value of resistance the vector diagram for the E.M.Fs. may be drawn as a right-angled triangle, of which the hypotenuse represents the supply, or terminal, E.M.F.

With variable resistance and constant reactance the E.M.F. vector diagrams for the varying conditions may, for constant terminal E.M.F., be represented by a series of right-angled triangles having a common hypotenuse, as shown in Fig. 39. The locus of the apex of the vector triangle is therefore a semicircle described on the hypotenuse. The semicircle for the R–L circuit is on one side of the hypotenuse, and that for the R–C circuit is on the opposite side as shown in Fig. 39, in which the semicircle OAE refers to the R–L circuit and the semicircle OBE refers to the R–C circuit.

The loci of the vectors of the currents in the circuits are also semicircles as shown in Fig. 39, but their centres lie on the opposite sides of, and in an axis perpendicular to, the vector (OE) representing the terminal E.M.F.

Proof. Let, for the R–L circuit, the co-ordinates of the current vector with respect to the origin O, Fig. 40, be

$$y = I\cos\varphi = E\frac{R}{R^2+X^2};\quad x = I\sin\varphi = E\frac{X}{R^2+X^2}$$

Then, squaring and adding, we have

$$x^2+y^2 = \left(\frac{ER}{R^2+X^2}\right)^2 + \left(\frac{EX}{R^2+X^2}\right)^2 = \frac{E^2(R^2+X^2)}{(R^2+X^2)^2} = \frac{E^2}{R^2+X^2}$$

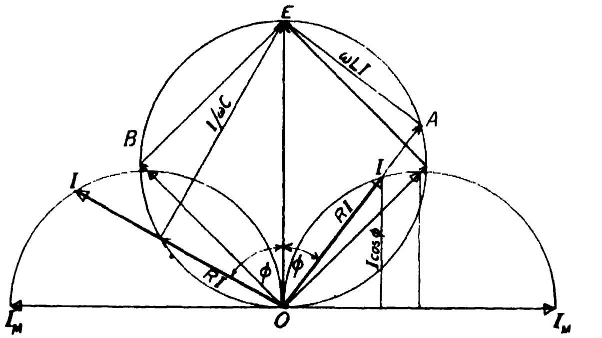


FIG. 39. VECTOR DIAGRAMS FOR SERIES CIRCUITS OF VARIABLE IMPEDANCE

(Constant Reactance, Variable Resistance)

Now $\qquad R^2 + X^2 = EX/x.$

Hence $\qquad x^2 + y^2 = x(E/X).$

This equation may be expressed in the form

$$y^2 + \left(x - \frac{E}{2X}\right)^2 = \frac{E^2}{4X^2}$$

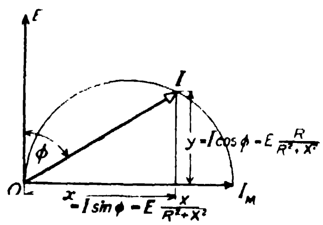


FIG. 40. VECTOR DIAGRAM FOR CIRCUIT OF VARIABLE RESISTANCE AND CONSTANT INDUCTANCE

which is the equation to a circle; the co-ordinates of the centre being $x = E/2X$, $y = 0$, and the radius being $E/2X$.

For the R–C circuit the radius of the semicircle is $E/2X$, and the co-ordinates of its centre are $x = -E/2X$, $y = 0$, where X is the reactance of the condenser.

If, however, X be variable and R constant, the current semicircles have their centres in the E.M.F. vector OE. Thus the sum of the squares of the co-ordinates of any current vector is now written as

$$x^2 + y^2 = y(E/R),$$

whence $$x^2 + \left(y - \frac{E}{2R}\right)^2 = \frac{E^2}{4R^2}$$

The co-ordinates of the centre are therefore $x = 0$, $y = E/2R$, and the radius is $E/2R$.

Properties of Series Circuits Containing Variable Resistance and Constant Reactance. The circle diagram of Fig 39 shows that

these circuits, when supplied at constant voltage and frequency, possess a number of important properties. Thus— (1) the current has a limiting value; (2) the power supplied to the circuit has a limiting value; (3) the power factor when maximum power is being supplied is 0·707.

The maximum current in the circuit is obtained when the resistance is zero: its value for the R–L circuit is $E/\omega L = I_M$.

The power supplied to the circuit is equal to $E\,I\cos\varphi$, and if E is constant the power will be proportional to $I\cos\varphi$. Now ordinates in the current semicircles (Fig. 39) are proportional to this quantity. Hence the maximum ordinate in either semicircle represents the maximum power which can be supplied to the circuit. This ordinate passes through the centre to the semicircle, and, therefore, the current vector makes an angle of 45° to the diameter of the semicircle, and also to the vector of the impressed E.M.F. Thus the power factor corresponding to maximum power is $\cos 45° = 0{\cdot}707$. The maximum power is, therefore, given by

$$P_M = 0{\cdot}707\,EI = 0{\cdot}707\,EI_M \sin 45° = \tfrac{1}{2}EI_M \qquad (35)$$

For the R–L circuit

$$P_M = \tfrac{1}{2}E^2/\omega L,$$

and for the R–C circuit

$$P_M = \tfrac{1}{2}\omega CE^2.*$$

At maximum power ($\varphi = 45°$) the vector triangle of E.M.F.s is an isosceles triangle, and therefore the voltages across the resistance and reactance are each equal to $0{\cdot}707 \times$ supply E.M.F. Hence the *condition for maximum power* is that the resistance of

* These expressions may also be obtained by determining the maximum value of $EI\cos\varphi$. Thus, substituting for I and $\cos\varphi$ in terms of resistance and reactance, differentiating, and equating to zero, we have

$$\frac{dP}{dR} = \frac{d}{dR}\left\{\frac{E^2}{\sqrt{(R^2+\omega^2L^2)}} \cdot \frac{R}{\sqrt{(R^2+\omega^2L^2)}}\right\} = \frac{d}{dR}\left(\frac{E^2R}{R^2+\omega^2L^2}\right)$$
$$= \frac{E^2(R^2+\omega^2L^2) - E^2R\,.\,2R}{(R^2+\omega^2L^2)^2}$$
$$= 0.$$

$$\therefore \quad E^2(R^2+\omega^2L^2) = 2E^2R^2$$

or $$R = \omega L = X.$$

Substituting ωL for R in the power expression we obtain

$$P_M = \frac{E^2\omega L}{\omega^2L^2+\omega^2L^2} = \frac{E^2}{2\omega L} = \frac{E^2}{2X}$$

the circuit must equal the reactance of the circuit, i.e. $R = \omega L$, or $R = 1/\omega C$. The expressions for maximum power may therefore be written $P_M = \frac{1}{2}E^2/R$.

Practical Applications. The current and power-limiting properties of series circuits containing variable resistance and constant inductive reactance are of considerable practical value in connection with *electric smelting furnaces* of the arc type. Thus, by inserting reactance in series with the furnace, (1) the power input to the furnace cannot exceed a predetermined limit; (2) the current

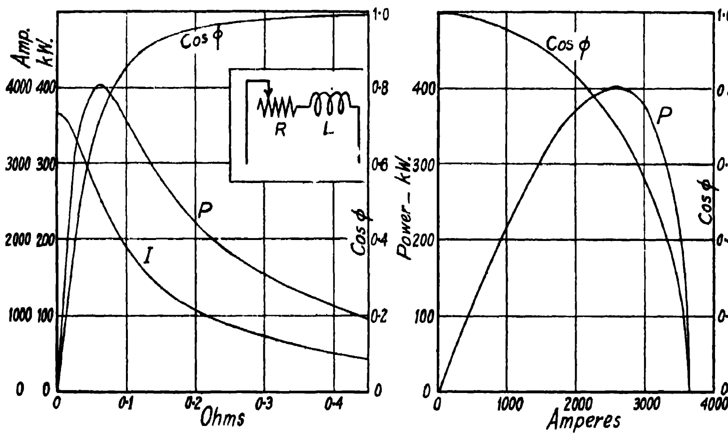


FIG. 41. CHARACTERISTIC CURVES FOR SERIES CIRCUIT CONTAINING VARIABLE RESISTANCE AND CONSTANT INDUCTANCE

Representative of Arc-type Electric Smelting Furnace

taken when "striking" the arc cannot exceed a predetermined value; (3) the furnace is more economical in power consumption than one controlled by a series resistance. These advantages, however, are obtained at the expense of the power factor.

The characteristic curves for a series circuit containing variable resistance and constant reactance are shown in Fig. 41, and refer to a 400 kW. arc-type furnace supplied at 220 V., 50 frequency; the reactance of the furnace circuit being 0·06 ohm.

The current and power-limiting properties of series circuits containing variable resistance and constant capacitance have been applied to *electric lighting* in special cases (e.g. artisan dwellings and cottages) where the average demand is so small that the supply of energy would be unprofitable if house service meters were installed. In such cases meters were dispensed with and a flat weekly charge

for energy was made by installing low-voltage series-connected lamps (each provided with a short-circuiting switch and rated for the same current) together with a condenser of such capacitance that if the normal demand were exceeded an appreciable reduction of the current in the circuit (and therefore the illumination) occurred.*

For an installation consisting of seven 40-W., 2-A. lamps and a 33-μF condenser supplied from 200-V. 50-cycle mains the lamp currents for various switchings are—

No. of lamps in circuit	1	2	3	4	5	6	7
Current (amp.)	2·058	2·026	1·978	1·92	1·86	1·794	1·73

Series Circuits of Variable Impedance. (2) *Constant Resistance and Variable Reactance.* Four cases of these circuits are possible, viz. (1) constant resistance, variable inductance; (2) constant resistance, variable capacitance; (3) constant resistance and inductance, variable capacitance; (4) constant resistance and capacitance, variable inductance. The first two cases have little application in practice, but are of academic interest: the last two cases have a large application in radio-telegraphy and telephony.

Vector Diagram for Circuits of Constant Resistance and Variable Inductance or Capacitance. The combined vector diagram for the R–L and R–C circuits, in which the resistance is constant, is shown in Fig. 42. In this diagram the vector OE, representing the impressed E.M.F., is taken as the vector of reference: OAE is the vector triangle of E.M.Fs. for the inductive circuit for a particular value of inductance; OBE is the corresponding triangle for the capacitive circuit. The loci of the apexes of these triangles are the semicircles $OCAE$, $ODBE$, respectively, their common centre being at the mid-point of OE.

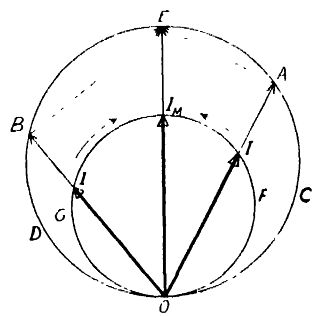


FIG. 42. VECTOR DIAGRAMS FOR SERIES CIRCUITS OF VARIABLE IMPEDANCE (CONSTANT RESISTANCE, VARIABLE REACTANCE)

The maximum current in either circuit is obtained when the reactance is zero: its value is E/R, and it is in phase with the impressed E.M.F. The vector OI_M, in phase with OE, represents this current, on the assumption of equal resistance and impressed E.M.F. in the two cases.

* For further particulars, see "Condensers in series with metal filament lamps," by A. W. Ashton. *Journal I.E.E.* (1912), xlix, 703.

When the inductance is varied the current vector lags with respect to the OE and its locus is the semicircle OFI_M. Similarly, when the capacitance is varied the current vector leads OE, and its locus is the semi-circle OGI_M. The centres of these semicircles are in OE and their radii are equal to $E/2R$ (see p. 78).

In each circuit the power is proportional to the projection of the current vector on the impressed E.M.F. vector: the power is therefore a maximum when the reactance is zero.

Vector Diagram for the General Circuit Containing Resistance and Variable Reactance. Series circuits containing resistance, inductance and capacitance, in which either the inductance or the capacitance is variable, have a large application in radio-telegraphy and telephony, the variable inductance or capacitance being employed to adjust the circuit to resonance at a particular frequency, this adjustment being called "tuning."

The vector diagrams for these circuits are shown in Figs. 43 and 44, the former referring to the case in which the capacitance is variable and the latter to the case when the inductance is variable. In both cases the vector of the impressed E.M.F. is taken as the vector of reference.

The current in either circuit is given, for any particular values of R, L, C, by $I = E/\sqrt{\{R^2 + [\omega L - 1/\omega C]^2\}}$, and its phase difference with respect to the impressed E.M.F. by $\tan\varphi = (\omega L - 1/\omega C)/R$. When the effective reactance is zero, i.e. when $\omega L = 1/\omega C$, the current is in phase with the impressed E.M.F. and its value is equal to E/R, which is the maximum value of the current for the circuit.

When the variable reactance is zero the current is given by $E/\sqrt{(R^2 + \omega^2 L^2)}$ for the circuit with fixed R and L; and by $E/\sqrt{[R^2 + (1/\omega C)^2]}$ for the circuit with fixed R and C. The phase difference between the impressed E.M.F. and the current is $\varphi = \tan^{-1}\omega L/R$, lagging, in the former case, and $\varphi = \tan^{-1}(1/\omega CR)$, leading, in the latter case. These currents are shown by the vectors OA in Figs. 43 and 44.

When the variable reactance is infinite the current in either circuit is zero.

The locus of the current vector when either the inductance in one circuit, or the capacitance in the other circuit, is varied is the arc OAI_MB, the centre of which lies in OE at a distance $\frac{1}{2}I_M = E/2R$ from O. The portion AI_M of this arc corresponds to the condition when the fixed inductive reactance exceeds the variable capacitive reactance, i.e. $\omega L > 1/\omega C$; and the semicircle I_MBO corresponds to the condition when the latter exceeds the former, i.e. $(1/\omega C) > \omega L$.

If a circle be described upon OE as diameter, and any current vector, such as OB, be produced so as to cut this circle at D, then triangle ODE is the triangle of E.M.Fs. for the circuit; OD representing the component of the impressed E.M.F. expended against resistance, and DE that expended against the effective reactance, i.e. DE represents the vector difference between the E.M.Fs. across the inductance and capacitance, these E.M.Fs. being represented by OF (leading the current vector by 90°) and OG (lagging 90° with

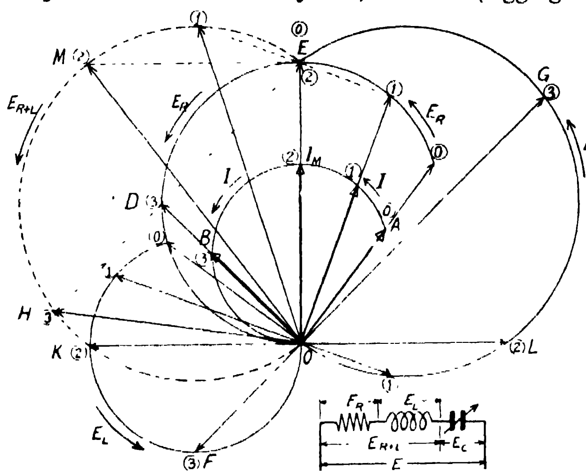


FIG. 43. VECTOR DIAGRAMS FOR SERIES CIRCUIT OF VARIABLE IMPEDANCE (CONSTANT RESISTANCE AND INDUCTANCE, VARIABLE CAPACITANCE)

respect to the current vector) respectively. The voltage across the fixed portion of each circuit is represented in both figures by OH, which is the vector sum of OD and OF.

When the variable portion of the circuits is varied from zero reactance to infinite reactance the loci of the vectors OD, OF, OG, OH, are portions of circles, which are marked E_R, E_L, E_C, E_{R+L}, respectively, in Figs. 43 and 44. The centres of these circles may be determined very simply by considering the conditions occurring at resonance. Thus the effective reactance is zero, the current is a maximum and is in phase with the impressed E.M.F., the E.M.F. across the inductance is balanced by that across the capacitance. These latter E.M.Fs. are represented by OK and OL, respectively. The voltage across the resistance is, at resonance, represented by OE—the vector of the impressed E.M.F.—and the vector sum of OE and OK is represented by OM

For the circuit containing variable capacitance, Fig. 43, the centre of circle E_L lies at the mid-point of OK; the centre of the circle E_{R+L} lies at the mid-point of OM, and the centre of circle E_C is correspondingly situated on the opposite side of OE. Observe that the diameter of circle E_{R+L} is equal to that of circle E_C.

For the circuit containing variable inductance, Fig. 44, the centre of circle E_C lies at the mid-point of OL; the centre of the circle E_L

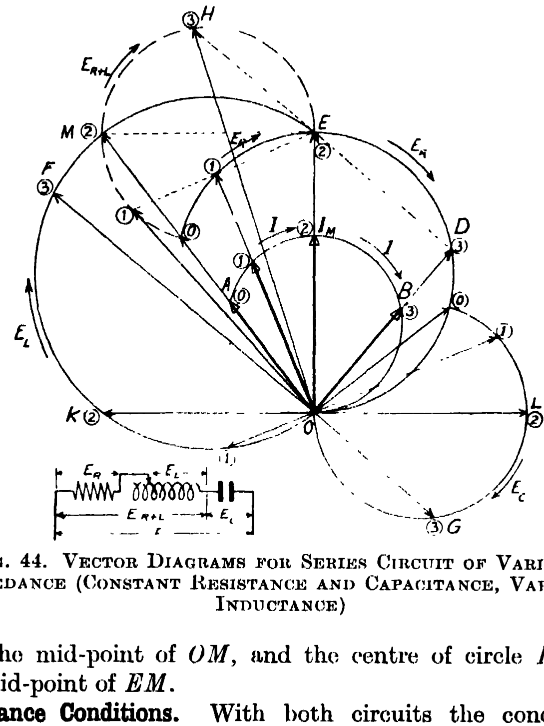


FIG. 44. VECTOR DIAGRAMS FOR SERIES CIRCUIT OF VARIABLE IMPEDANCE (CONSTANT RESISTANCE AND CAPACITANCE, VARIABLE INDUCTANCE)

lies at the mid-point of OM, and the centre of circle E_{R+L} lies at the mid-point of EM.

Resonance Conditions. With both circuits the condition for obtaining resonance is that the inductive reactance must equal the capacitive reactance, i.e. $\omega L = 1/\omega C$. Whence $LC = 1/\omega^2 = 1/(2\pi f)^2$ or $\sqrt{(LC)} = 1/(2\pi f)$, where f is the frequency in cycles per second, and L, C, are the inductance and capacitance, respectively, in practical units (i.e. henries and farads).

If instead of the frequency we are given the wave-length (λ) of the electrical oscillations to which the circuit must be tuned, then $\lambda_{metres} = 3 \times 10^8/f$, where 3×10^8 is the velocity of propagation, in metres per second, of the oscillations. Hence

$$\sqrt{(LC)} = \lambda/(2\pi \times 3 \times 10^8) = \lambda/(1885 \times 10^6).$$

But if L is expressed in micro-henries and C in micro-farads, the relationship between L, C, λ becomes

$$\sqrt{(LC)} = \lambda/1885 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

This, then, is the condition which must be satisfied in a "tuned" radio circuit when the tuning is effected by the use of series capacitance or inductance.

The curves of Fig. 45 show, for a particular circuit (for which $R = 25$ ohms, $L = 0{\cdot}4$ henry (constant), C (variable)), the variation of current, voltage, and power factor when the capacitance is varied and the impressed E.M.F. and frequency are maintained constant.

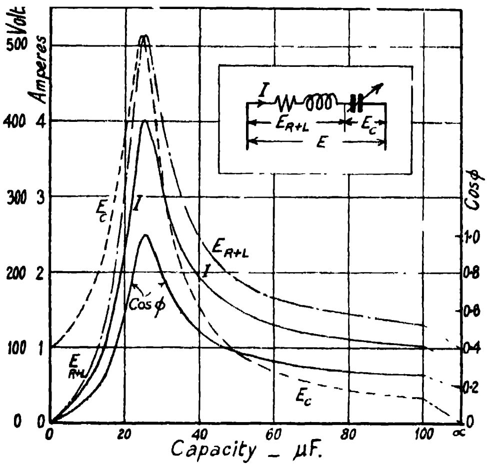


FIG. 45. CHARACTERISTIC CURVES FOR SERIES CIRCUIT OF VARIABLE IMPEDANCE (CONSTANT RESISTANCE AND INDUCTANCE, VARIABLE CAPACITANCE)

It will be observed that the peaks of the curves for the voltages across the condenser and inductive resistance are not coincident, the peak of the condenser voltage occurring when the capacitance has a value which is smaller than that which gives the peak of the voltage across the inductive resistance. This non-coincidence of the voltage peaks is caused by the resistance of the circuit. For a circuit of zero resistance the peaks of voltage are coincident and occur when the current in the circuit is infinite. Data for these curves are given in Table I, and the method of calculating the quantities may be shown best by calculating one or two points. Thus at resonance we must have $\omega L = 1/\omega C$, or $C = 1/\omega^2 L$. Now for a frequency of 50, $\omega = 314$. Hence the capacitance, in μF.,

required to give resonance is $C = 10^6/(314^2 \times 0{\cdot}4) = 25{\cdot}35\ \mu$F. With an impressed E.M.F. of 100 V. the current is $I_M = 100/25 = 4$ A. The voltage across the condenser is $E_C = I/_M\omega C = 4/(314 \times 25{\cdot}35 \times 10^6) = 502{\cdot}4$ V., and the voltage across the inductive resistance is $E_{R+L} = I_M(R^2 + \omega^2 L^2) = 4 \times 128 = 512$ V.

[NOTE.—$R^2 + \omega^2 L^2 = 25^2 + (314^2 \times 0{\cdot}4^2) = 128\ \Omega$.]

For the points corresponding to a capacitance of 20 μF. we have—

Effective reactance $= (\omega L - 1/\omega C) = 314 \times 0{\cdot}4 - [10^6/(314 \times 20)]$
$= -33{\cdot}5\ \Omega$.

Impedance $(Z) = \sqrt{[R^2 + (\omega L - 1/\omega C)^2]} = \sqrt{(25^2 + 33{\cdot}5^2)}$
$= 41{\cdot}7\ \Omega$.

Current $= E/Z = 100/41{\cdot}7 = 2{\cdot}4$ A.

Voltage across condenser $= I/\omega C = 2{\cdot}4 \times 10^6/314 \times 20 = 382$ V.

Voltage across inductive resistance $= I\sqrt{(R^2 + \omega^2 L^2)} = 2{\cdot}4 \times 128$
307 V.

Power factor $= R/Z \quad 25/41{\cdot}7 = 0{\cdot}6$.

TABLE I

Calculations for Fig. 45. Series circuit : $R = 25\ \Omega$., $L = 0{\cdot}4$ H. (constant), C variable. Supply pressure = 100 V. (constant), frequency = 50 (constant)

C (μF.)	$1/\omega C$	$\omega L - 1/\omega C$	$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$	$I = E/Z$.	$\cos\varphi = R/Z$.	$E_C = I/\omega C$.	$E_{R+L} = I\sqrt{(R^2 + \omega^2 L^2)}$.
0	∝	∝	∝	0	0	100	0
5	636	−511	511	0·196	—	124 5	25
10	318	−193	104·6	0·514	0·1285	163·5	66
15	212	−87	90·4	1·106	0·2763	234	141
20	159	−33·5	41·7	2·4	0·6	382	307
23	138 4	−13	28·2	3·45	0·886	477	454
24	132 5	− 7	26	3·85	0·964	511	493
24·5	129·8	− 4·2	25·35	3·945	0·986	512	505
25	127·2	− 1·6	25·04	3 99	0·998	508	511
25·35	125 6	0	25	4	1·0	502·4	512
27	117·8	8·8	26·5	3·77	0·943	445	483
30	106·1	19·5	31·7	3·15	0·788	334	404
35	91	34·6	42·7	2·34	0·585	213	300
40	79 6	46	52·4	1·91	0·477	152	244
50	63·6	62	67·6	1·48	0·37	94	189
100	31·8	94	97·3	1·026	0·257	32·6	131
∞	0	125·6	128	0·781	0·195	0	100

Series Circuits of Variable Impedance. (3) *Constant Resistance, Inductance and Capacitance, Variable Frequency.* This circuit is of practical importance on account of its possessing a " resonance frequency " which is given by

$$f_r = \frac{1}{2\pi\sqrt{(LC)}}$$

in which L and C are in practical units (i.e. henries and farads, respectively) and f_r is in cycles per second.

At resonance frequency the current is a maximum and is given by $I_M = E/R$. This current is in phase with the impressed E.M.F. The vector diagram for these conditions is given in Fig. 29 (*b*), p. 64.

Under resonance conditions, and with suitable values of inductance and capacitance, the voltage across the condenser, and that across

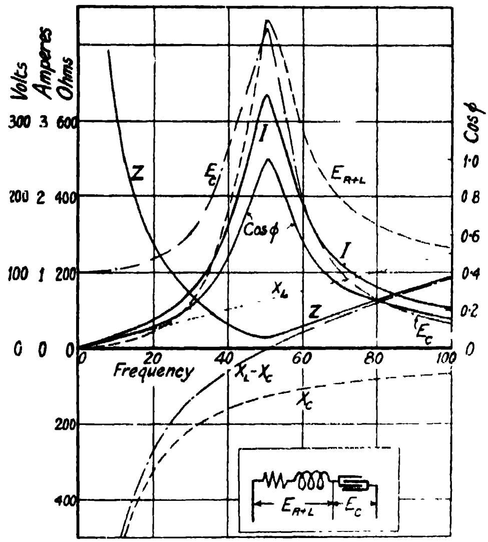


FIG. 46. CHARACTERISTIC CURVES FOR SERIES CIRCUIT SUPPLIED AT CONSTANT VOLTAGE AND VARIABLE FREQUENCY

the inductive resistance, may be considerably greater than the supply voltage. Thus, the voltage across the condenser is given by

$$E_C = I_M/\omega C,$$

which, when $1/\sqrt{(LC)}$ is substituted for ω, and E/R for I_M, becomes

$$E_C = \frac{E}{R}\sqrt{\frac{L}{C}}$$

$$= E\sqrt{\frac{L}{CR^2}}$$

Hence when $L > CR^2$ the voltage across the condenser will be greater than the supply voltage. The ratio of these voltages is given simply by $\sqrt{(L/CR^2)}$.

Similarly the voltage across the inductance and resistance is, at resonance frequency, given by

$$E_{L+R} = I_M \sqrt{(R^2 + \omega^2 L^2)}$$

$$= \frac{E}{R}\sqrt{\left(R^2 + \frac{L}{C}\right)} = E\sqrt{\left(1 + \frac{L}{CR^2}\right)}$$

which is greater than the supply voltage for all values of L and C except zero.

The manner in which these voltages vary with the frequency in the case of a particular circuit, for which $R = 30\ \Omega.$, $L = 0{\cdot}4$ H., $C = 25\ \mu$F., is shown in Fig. 46. Other curves in this figure show the manner in which the current, power factor, inductive reactance, capacitive reactance, and effective reactance vary with the frequency when the supply pressure is maintained constant at 100 V. The calculations for these curves are given in Table II.

TABLE II

Calculations for Fig. 46. Series circuit: $R = 30\ \Omega.$, $L = 0{\cdot}4$ H., $C = 25\ \mu$F Supply pressure = 100 V. (constant). Frequency variable.

	$\omega = 2\pi f$	ωL	$10^6/\omega C$	$\omega L - 1/\omega C$	$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$	$I = 100/Z$	$\cos\varphi = R/Z$	$E_C = I/\omega C$	$E_{L+R} = I\sqrt{(R^2+\omega^2 L^2)}$
0	0	0	∞	∞	∞	—	—	100	0
10	62·8	25·1	637	−612	612	0·1635	0·049	104·2	6·4
20	125·6	50·2	319	−269	270·8	0 369	0·11	117·8	21
30	188·4	75·3	212	−137	140 4	0·713	lead 0·214	151	57·8
40	251	105	159·5	− 54 5	62 2	1·608	0·483	256·5	175·5
45	283	113·2	141·2	− 28	41	2·44	0·732	345	286
50·35	316	126·5	126·5	0	30	3·33	1·0	422	433
55	345	138	116	22	37·2	2·69	0·807	312	380
60	377	151	106	45	54	1·85	0·556	196	285
70	440	176	90·9	85	90 2	1·11	lag 0·333	100	198·5
100	628	251	63·7	187 5	190	0·526	0·158	33·5	133

Resonance in Practice. The equivalent of the above circuit occurs in practice when an alternator is supplying unloaded cables, the inductance and resistance being represented by the armature winding of the alternator, and the condenser by the capacitance between the cores of the cables. But the "constants" of commercial distributing systems are such that the resonance frequency is much higher than the normal frequency. Hence, with a pure sine wave of E.M.F. resonance cannot occur. If, however, the wave-form is non-sinusoidal resonance, due to one of the higher harmonics of the E.M.F., may occur either at normal speed, or at a speed lower than normal if the alternator is run up to speed with its field excited and its armature connected to unloaded cables.

Example. An 11,000-V., 50-cycle, alternator is connected to concentric cables having a total capacitance of 3 μF. The inductance of the armature winding is 0·02 H., and the resistance of the armature winding is 1·3 Ω.

The resonance frequency of the system when the cables are unloaded is

$$f_r = \frac{1}{2\pi\sqrt{(0{\cdot}02 \times 3 \times 10^{-6})}} = 649 \text{ cycles per second}$$

If the wave-form of the no-load E.M.F. contains a thirteenth harmonic (the frequency of which = 13 × 50 = 650), resonance due to this harmonic will occur at normal speed. If the amplitude of this harmonic is 1 per cent of the fundamental, i.e. $\frac{1}{100} \times 11000 \times 1{\cdot}414 = 155$ V., the current due to it at resonance is 155/1·3 = 119 A. (maximum value). Hence if the peak of the E.M.F. wave of this harmonic occurs at the same instant as that of the fundamental, the voltage impressed on the cables is

$$11000 \times 1{\cdot}414 + 119/(2\pi \times 650 \times 3 \times 10^{-6}) = 25{,}250 \text{ V.}$$

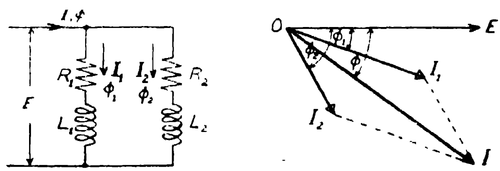


FIG. 47. CIRCUIT AND VECTOR DIAGRAMS FOR PARALLEL CIRCUIT

II.—PARALLEL CIRCUITS

Parallel Circuits of Constant Impedance. The vector diagram for a parallel circuit, consisting of two inductive resistances, is given in Fig. 47. The total, or " line " current, I, is equal to the geometric sum of the branch currents I_1, I_2 : its magnitude and phase may be calculated from Fig. 47 as follows

$$I = \sqrt{\{(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)^2 + (I_1 \sin\varphi_1 + I_2 \sin\varphi_2)^2\}} \quad (37)$$

$$\varphi = \tan^{-1}(I_1 \sin\varphi_1 + I_2 \sin\varphi_2)/(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)$$

The joint impedance, Z, of the circuit may be obtained from Equation (37), and Fig. 47, as follows

$$I = E/Z = \sqrt{\{(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)^2 + (I_1 \sin\varphi_1 + I_2 \sin\varphi_2)^2\}}$$

$$= \sqrt{\left\{\left(\frac{E}{Z_1}\cdot\frac{R_1}{Z_1} + \frac{E}{Z_2}\cdot\frac{R_2}{Z_2}\right)^2 + \left(\frac{E}{Z_1}\cdot\frac{X_1}{Z_1} + \frac{E}{Z_2}\cdot\frac{X_2}{Z_2}\right)^2\right\}}$$

or

$$\frac{1}{Z} = \sqrt{\left\{\left(\frac{R_1}{Z_1^2} + \frac{R_2}{Z_2^2}\right)^2 + \left(\frac{X_1}{Z_1^2} + \frac{X_2}{Z_2^2}\right)^2\right\}}$$

$$= \sqrt{\left\{\frac{1}{Z_1^2} + \frac{1}{Z_2^2} + \frac{2(R_1R_2 + X_1X_2)}{Z_1^2\,Z_2^2}\right\}} \quad . \quad . \quad . \quad . \quad (38)$$

Thus for the general case the reciprocal of the joint impedance is not equal to the sum of the reciprocals of the separate impedances: its calculation therefore requires a knowledge of the values of the separate impedances as well as the resistances and reactances.

Special Cases. In the special case when $X_1 = 0$, $X_2 = 0$, we have

$$\frac{1}{Z} = \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

and in the other case when $R_1 = 0$, $R_2 = 0$, we have

$$\frac{1}{Z} = \frac{1}{X} = \frac{1}{X_1} + \frac{1}{X_2}$$

When the reactances X_1, X_2, consist of condensers,

$$\frac{1}{X_1} = \omega C_1 \; ; \; \frac{1}{X_2} = \omega C_2 \; ; \; \frac{1}{X} = \omega C$$

Hence the joint capacitance of the condensers is

$$C = C_1 + C_2.$$

Thus, when condensers are connected in parallel, the joint capacitance is equal to the sum of the separate capacities.

Admittance. Reverting to the general case, Fig. 47, and employing symbolic notation, the equation to the line current may be written

$$\begin{aligned} \underset{\cdot}{I} &= \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 \\ &= (\underset{\cdot}{E}/\underset{\cdot}{Z}_1) + (\underset{\cdot}{E}/\underset{\cdot}{Z}_2) \end{aligned}$$

whence

$$\frac{1}{\underset{\cdot}{Z}} = \frac{1}{\underset{\cdot}{Z}_1} + \frac{1}{\underset{\cdot}{Z}_2} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or

$$\underset{\cdot}{Y} = \underset{\cdot}{Y}_1 + \underset{\cdot}{Y}_2$$

where $\underset{\cdot}{Y}$ is the reciprocal of the joint impedance and $\underset{\cdot}{Y}_1$, $\underset{\cdot}{Y}_2$, are the reciprocals of the separate impedances. These reciprocals are called *admittance.* Hence the joint admittance of a number of circuits connected in parallel is equal to the geometric sum of their separate admittances. The admittances of these circuits may therefore be compounded in a vector diagram in a manner similar to that adopted with the several impedances of a series circuit. For example, in Fig. 48 the vector OA represents the admittance $\underset{\cdot}{Y}_1$, OB, the admittance $\underset{\cdot}{Y}_2$. The joint admittance is represented by OC, which is the vector sum of OA and OB.

Inversion. The joint impedance Z, corresponding to the joint admittance Y, may be obtained from the vector diagram by

determining the vector which is the reciprocal of the admittance vector, this process being called "inversion." For example, two reciprocal quantities, Y, Z ($= 1/Y$), are represented in a vector diagram by vectors of different lengths—such as OY, OZ, Fig. 49—located on opposite sides of the axis of reference and making equal angles with this axis. The projection on OZ of the point Y gives the point Y', which is called the image of Y.

Two points such as Y' and Z, Fig. 49, are called "inverse points." The origin, O, is called the "centre of inversion," and the value of the product $OY'.OZ$ which corresponds to unit values of Y and Z, is called the "inversion constant," K_i.

If the admittance vector is drawn to such a scale that 1 cm. $= m$ mho (unit of admittance), and the impedance vector is drawn to

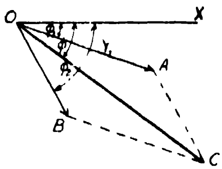


FIG. 48. VECTOR DIAGRAM SHOWING ADDITION OF ADMITTANCES

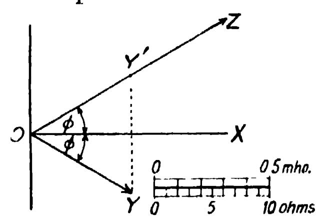


FIG. 49. VECTOR DIAGRAM OF RECIPROCAL VECTOR QUANTITIES

such a scale that 1 cm. $= n$ ohms, then $OY' = Y/m$, $OZ = Z/n$ Whence $OY' \times OZ = YZ/mn = 1/mn$, since $Y.Z = 1$, i.e. the inversion constant is equal to the product of the reciprocals of the scales of the vectors. Hence the inversion of the image of an admittance vector—drawn to a scale of 1 cm. $= m$ mho—to an inversion constant K_i gives an impedance vector having a scale of 1 cm. $= (1/mK_i)$ ohms. Conversely, the inversion of the image of an impedance vector—drawn to a scale of 1 cm. $= n$ ohms—gives an admittance vector having a scale of 1 cm. $= (1/nK_i)$ mho. For example, if OY, Fig. 49, represents an admittance of 0·05 mho, and is drawn to a scale of 1 cm. = 0·025 mho, i.e. $OY = 0{\cdot}05/0{\cdot}025 = 2$ cm., then the length of the impedance vector OZ for inversion constant equal to 8, is $8/(0{\cdot}05/0{\cdot}025) = 4$ cm., and the scale of this vector is $1/(8 \times 0{\cdot}025)$, or 5 ohms = 1 cm. Therefore the magnitude of the impedance is equal to $4 \times 5 = 20$ ohms.

Instead of determining the impedance scale from the inversion constant and the admittance scale, it is more convenient in practice to *select* the impedance scale such that the longest impedance vector is of a convenient length.

The principle of inversion is of considerable value in the graphical solution of problems on parallel and series-parallel circuits, and further applications will be found in this and the following chapters.

Conductance and Susceptance. Since admittance, like impedance, is a complex quantity it must be expressed, when the rectangular form of symbolic notation is employed, in terms of in-phase and quadrature components which are denoted by the symbols g and b, respectively. These components can be expressed in terms of the constants, R, X, of the circuit. For example,

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \left(\frac{R}{R^2 + X^2}\right) - j\left(\frac{X}{R^2 + X^2}\right)$$
$$= \boldsymbol{G} - j\boldsymbol{B}$$

where $G = R/(R^2 + X^2) = R/Z^2$, and $B = X/(R^2 + X^2) = X/Z^2$.

G is called the *conductance** because, for a circuit of zero reactance, we have $G = R/R^2 = 1/R$. B is called the *susceptance* and, for circuits of zero resistance, is equal to the reciprocal of reactance, i.e. $B = X/X^2 = 1/X$.

Hence for the circuit of Fig. 47 we have

$$Y_1 = \boldsymbol{G}_1 - j\boldsymbol{B}_1 ; \quad Y_2 = \boldsymbol{G}_2 - j\boldsymbol{B}_2$$
$$Y = Y_1 + Y_2 = (\boldsymbol{G}_1 - j\boldsymbol{B}_1) + (\boldsymbol{G}_2 - j\boldsymbol{B}_2)$$
$$= (\boldsymbol{G}_1 + \boldsymbol{G}_2) - j(\boldsymbol{B}_1 + \boldsymbol{B}_2)$$

Whence,
$$Y = \sqrt{\{(G_1 + G_2)^2 + (B_1 + B_2)^2\}} \quad . \quad . \quad (40)$$
$$\varphi = \tan^{-1}(B_1 + B_2)/(G_1 + G_2).$$

Parallel Circuits of Variable Impedance. (1) *Constant Resistance, Inductance, and Capacitance ; Variable Frequency.* Let one branch of a parallel circuit consist of a fixed inductive resistance (R, L), and the other a condenser of fixed capacitance (C). The current in the inductive resistance at a frequency f is $I_1 = E/\sqrt{(R^2 + \omega^2 L^2)}$, where $\omega = 2\pi f$. The power component of this current is $I_1 \cos \varphi_1 = ER/(R^2 + \omega^2 L^2)$, and the wattless component is $I_1 \sin \varphi_1 = E\omega L/(R^2 + \omega^2 L^2)$, which lags 90° with respect to the impressed E.M.F. The charging current of the condenser is $I_2 = \omega CE$, and leads the impressed E.M.F. by 90°. Therefore, the phase difference between the wattless component of the current in the

* The term conductance is employed in connection with direct-current circuits to denote the reciprocal of resistance.

inductive resistance and the charging current of the condenser is 180°. Hence the wattless component of the line current is

$$I_W = \frac{E\omega L}{R^2 + \omega^2 L^2} - \omega CE$$

which, if the frequency is variable, becomes zero when

$$\frac{E\omega L}{R^2 + \omega^2 L^2} = \omega CE,$$

i.e. when

$$\frac{\omega L}{\omega C} = R^2 + \omega^2 L^2 = X_C X_L,$$

where X_L ($= \omega L$) is the inductive reactance and X_C ($= 1/\omega C$) is the capacitive reactance. Thus the condition for the wattless component of the line current to be zero is that the product of the inductive and capacitive reactances must equal the square of the impedance of the inductive branch.

The frequency corresponding to this condition is obtained by solving the equation for ω. Thus

$$\omega^2 = (2\pi f)^2 = \frac{1}{L^2}\left(\frac{L}{C} - R^2\right) = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)$$

whence $$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Resonance. Every parallel circuit consisting of a condenser, of fixed capacitance, in one branch and a fixed inductive resistance in the other branch has, therefore, a particular frequency at which the charging current of the condenser balances the wattless current in the inductive resistance. This frequency, which is given by Equation (41), is called the "resonance frequency" of the (parallel) circuit. It is not, however, the natural frequency of the local circuit—consisting of the condenser and inductive resistance—as this frequency is given by

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} \quad . \quad . \quad . \quad . \quad (42)$$

When $R = 0$ the resonance frequency becomes equal to the natural frequency of the circuit: moreover, the resonance frequency for the parallel circuit is then equal to that for the series circuit containing the same values of inductance and capacitance. Under these hypothetical conditions—i.e. with zero resistance and no losses—electric oscillations, when once started in the local circuit,

may be maintained without the further supply of energy to this circuit.

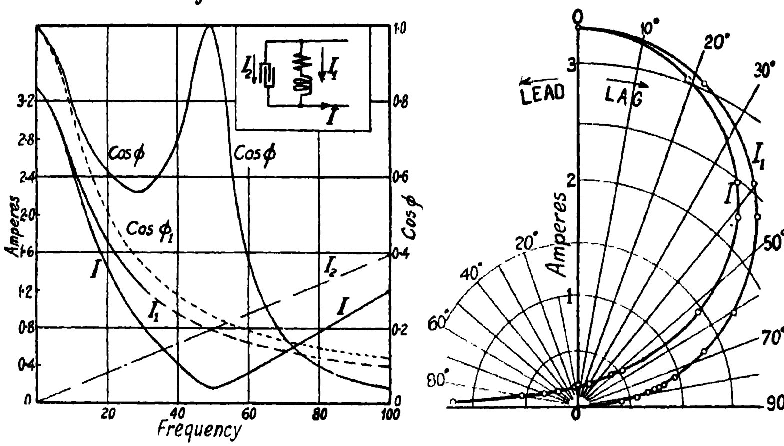
Variation of Line Current with Frequency. Reverting to the practical case, in which the inductive branch contains resistance, the line current at a frequency f is given by

$$I = \sqrt{\{(I_1 \cos \varphi_1)^2 + (I_1 \sin \varphi_1 - \omega CE)^2\}}$$

$$= \sqrt{\left\{\left(\frac{ER}{Z^2}\right)^2 + \left[\omega E\left(\frac{L}{Z^2} - C\right)\right]^2\right\}}$$

$$= E\sqrt{\left\{\frac{1}{R^2 + \omega^2 L^2} - \omega^2\left(\frac{2LC}{R^2 + \omega^2 L^2} - C^2\right)\right\}} \quad . \qquad (43)$$

where $\omega = 2\pi f$.



(a) (b)

FIG. 50. (a) CHARACTERISTIC CURVES FOR PARALLEL CIRCUIT SUPPLIED AT CONSTANT VOLTAGE AND VARIABLE FREQUENCY. (b) PLOT OF CURRENTS I, I_1, IN POLAR CO-ORDINATES

At zero frequency $\omega = 0$, and $I = E/R$. As the frequency is increased from zero the line current decreases in value and becomes a minimum at a frequency slightly above the resonance frequency: it then increases as the frequency increases. Hence if line current be plotted against frequency a V curve is obtained as shown in Fig. 50 (a), which refers to a particular circuit for which $R = 30\ \Omega$, $L = 0{\cdot}4$ H, $C = 25\ \mu$F.

Other curves in this figure show the manner in which the resultant power factor, the branch-circuit currents, and the wattless and power components of the current in the inductive branch vary

when the frequency is varied and the supply pressure is maintained constant at 100 V. The calculations for these curves are given in Table III.

The line current and the current in the inductive branch are also plotted to polar co-ordinates in Fig. 50 (*b*), from which it will be observed that the curve for the latter is a semicircle.

It will be observed that as the frequency is increased from zero, the wattless component of the current in the inductive branch increases from zero to a definite maximum value and then decreases with increasing frequency. The frequency corresponding to the maximum wattless current in the inductive branch is obtained by differentiating, with respect to ω, the expression for this current, equating the result to zero and solving for ω. Thus

$$I_1 \sin \varphi_1 = E\omega L/(R^2 + \omega^2 L^2) = E\omega L/Z^2$$

$$\frac{d}{d\omega}(I_1 \sin \varphi_1) = \frac{ELZ^2 - 2E\omega^2 L^3}{Z^4}$$

$$\therefore \quad ELZ^2 = 2E\omega^2 L^3$$

or $$R^2 + \omega^2 L^2 = 2\omega^2 L^2$$

Whence $$\omega = R/L.$$

TABLE III

Calculations for Fig. 50. Parallel circuit: inductive branch, $R = 30\ \Omega$., $L = 0{\cdot}4$ H.; condenser branch, $C = 25\ \mu$F. Supply pressure = 100 V. (constant). Frequency variable.

f	$\omega = 2\pi f$	ωL	$Z = \sqrt{(R^2 + \omega^2 L^2)}$	$I_1 = 100/Z$	$I_1 \sin \varphi_1$	$I_2 = \omega CE$	$I_1 \sin\varphi_1 - I_2 (= I \sin \varphi)$	$I_1 \cos\varphi_1 (= I \cos \varphi)$	$I = \sqrt{\{(I \sin \varphi)^2 + (I \cos \varphi)^2\}}$	Cos φ	Power Supplied $EI_1 \cos \varphi$
											Watts
0	0	0	30	3·33	0	—	—	3·33	3·33	1·0	333
5	31·4	12·5	32·5	3·07	1·19	0·08	1·11	2·84	3·05	0·938	284
10	62·8	25·1	39·1	2·56	1·65	0·16	1·49	1·96	2·46	0·8	196
12	75	30	42·4	2·36	1·67	0·19	1·48	1·67	2·23	0·747	167
20	125·6	50·2	58·5	1·71	1·46	0·31	1·15	0·88	1·44	0·61	88
30	188·4	75·3	81·1	1·23	1·14	0·47	0·67	0·46	0·51	0·56	46
40	251	100·4	105	0·95	0·91	0·63	0·28	0·294	0·42	0·7	29·4
45	283	113·2	117	0·85	0·82	0·71	0·11	0·218	0·244	0·894	21·8
48·9	307	122·8	126·4	0·791	0·768	0·768	0	0·188	0·188	1·0	18·8
50	314	125·6	129·3	0·773	0·751	0·786	−0·035	0·18	0·183	0·984	18
55	345	138	141·2	0·71	0·69	0·86	−0·17	0·15	0·23	0·654	15
60	377	151	154	0·65	0·64	0·94	−0·3	0·127	0·33	0·385	12·7
70	440	176	178·5	0·56	0·55	1·1	−0·55	0·094	0·55	0·171	9·4
100	628	993	252·8	0·395	0·393	1·57	−1·177	0·047	1·18	0·04	4·7

Analytical Investigation of Variation of Line Current with Frequency. The shape of the line-current/frequency curve for the general case may be ascertained analytically by investigating the law of variation, with respect

to frequency, of the several terms of Equation (43). Thus the value of the first term $[1/(R^2 + \omega^2L^2)]$ is a maximum at zero frequency and decreases as the frequency increases; the rate of decrease being at first rapid, then diminishing gradually, and finally becoming zero at an infinite frequency. The value of the second term $[\omega^2(2LC/Z^2 - C^2)]$ is zero when the frequency is zero: it increases as the frequency increases, until it attains its maximum value at a frequency below the resonance frequency; it then decreases, becomes zero at a particular frequency—which is higher than the resonance frequency — and then changes sign and increases continually as the frequency is still further increased. The variation is shown graphically in Fig. 51, the curves of which refer to the circuit calculated in Table III.

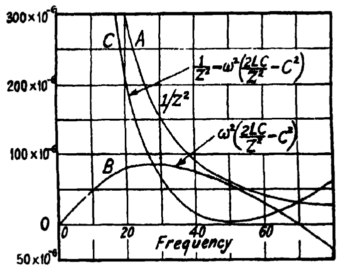


FIG. 51. VARIATION OF QUANTITIES WITH FREQUENCY, FOR PARALLEL CIRCUIT
($R = 30\,\Omega$., $L = 0{\cdot}4$ H., $C = 25\,\mu$F.)

The general shape of these curves is the same for all circuits, but the frequencies at which the maximum and zero values of the second term occur depend upon the "constants" of the circuits. For example, the frequency corresponding to the maximum value of $[\omega^2(2LC/Z^2 - C^2)]$ is obtained by equating the first differential coefficient of this quantity to zero and solving for ω^2. Thus

$$\frac{d}{d\omega}\left\{\omega^2\left(\frac{2LC}{Z^2} - C^2\right)\right\} = \frac{Z^2(4\omega LC) - 2\omega L^2(2\omega^2 LC)}{Z^4} - 2\omega C^2 = 0$$

whence $\qquad 2(Z^2 - \omega^2L^2) = Z^4C/L,$

which, when $(R^2 + \omega^2L^2)$ is substituted for Z^2, reduces to

$$\omega^2 = \left(\frac{R}{L}\sqrt{\frac{2}{LC}}\right) - \frac{R^2}{L^2}$$

The quantity $[\omega^2(2LC/Z^2 - C^2)]$ is zero when $2LC/Z^2 = C^2$, and the frequency corresponding to this condition is obtained from the equation

$$\omega_1^2 = \frac{2}{LC} - \frac{R^2}{L^2}$$

Substituting numerical values for R, L, C, from Table III, we have

$$\omega^2 = \left(\frac{30}{0{\cdot}4}\sqrt{\frac{2 \times 10^6}{0{\cdot}4 \times 25}}\right) - \left(\frac{30}{0{\cdot}4}\right)^2 = 27{,}875,$$

whence $\qquad f = \omega/2\pi = \sqrt{27\,875}/2\pi = 26{\cdot}6$ cycles per second.

Similarly, $\qquad \omega_1^2 = \dfrac{2 \times 10^6}{0{\cdot}4 \times 25} - \left(\dfrac{30}{0{\cdot}4}\right)^2 = 194{,}375,$

whence $\qquad f_1 = \omega_1/2\pi = \sqrt{194\,375}/2\pi = 71{\cdot}2$ cycles per second

Now the difference between the quantities $[1/(R^2 + \omega^2L^2)]$ and $[\omega^2(2LC/Z^2 - C^2)]$ is proportional to the square of the current [see Equation (43)].

Hence intercepts between ordinates of the curves in Fig. 51 are proportional to the current. As the frequency is increased from zero, these intercepts diminish on account of the curves approaching each other, but after a particular frequency is reached the curves diverge and the intercepts

increase. In the case of a hypothetical circuit without resistance, the curves touch at resonance frequency, but for practical circuits the minimum intercept occurs at a frequency slightly higher than the resonance frequency.

The frequency corresponding to the minimum value of the line current is determined by equating the first differential coefficient of Equation (43) to zero and solving for ω. Thus, writing Equation (43) in the form

$$I = E\sqrt{\left(\frac{1 - 2\omega^2 LC}{R^2 + \omega^2 L^2} + \omega^2 C^2\right)}$$

and denoting the quantity in the bracket by u, we have

$$\frac{dI}{d\omega} = \frac{E}{2\sqrt{u}} \cdot \frac{du}{d\omega} = \frac{E}{2\sqrt{u}}\left\{\frac{-4\omega LC(R^2 + \omega^2L^2) - 2\omega L^2(1 - 2\omega^2 LC)}{(R^2 + \omega^2 L^2)^2} + 2\omega C^2\right\} = 0$$

whence

$$\omega^2 = \left\{\frac{1}{LC}\sqrt{\left(2R^2\frac{C}{L} + 1\right)}\right\} - \frac{R^2}{L^2}$$

This is greater than the value of ω^2 at resonance frequency—see Equation (41)—as the coefficient of the first term is greater than unity. The difference between these frequencies depends upon the values of R, L, C. For example, in the case of the circuit calculated in Table III the resonance frequency is equal to 48·9, and the frequency at which the line current is a minimum is equal to 50·3 cycles per second.

The minimum value of the line current may be obtained by substituting this value of ω^2 in Equation (43). For example, substituting the numerical value of 99875 $[= (2\pi \times 50{\cdot}3)^2]$ and the values of R, L, C, from Table III, in Equation (43) we have

$$I_{\text{min.}} = 100\sqrt{\left(\frac{1 - 2 \times 99875 \times 0{\cdot}4 \times 25 \times 10^{-6}}{30^2 + 99875 \times 0{\cdot}4^2} + 99875 \times 25^2 \times 10^{-12}\right)}$$
$$= 0{\cdot}1825 \text{ A}$$

The value of the line current at resonance frequency is obtained by substituting the appropriate value of ω in Equation (43). Thus, at resonance frequency, $\omega^2 = \{(1/LC) - (R/L)^2\}$—*see* Equation (41)—and Equation (43) reduces to

$$I = ER(C/L).$$

Hence for the circuit calculated in Table III the line current at resonance frequency is

$$I = 100 \times 30 \times 25/(0{\cdot}4 \times 10^6)$$
$$= 0{\cdot}1875 \text{ A}.$$

Joint Impedance at Resonance Frequency. A parallel circuit consisting of an inductive resistance connected in parallel with a condenser possesses zero susceptance at resonance frequency, i.e. the expression for the joint impedance of the circuit contains no "reactance" term.

In the case of the circuit considered above, the line current at resonance frequency is given by

$$I = ER(C/L)$$

and therefore the joint impedance at this frequency is

$$Z = \frac{E}{I} = \frac{1}{R} \cdot \frac{L}{C}$$

For the special case when $L = C$, we have

$$Z = 1/R.$$

Another case of interest is where the condenser branch contains resistance. In this case, let R_o denote the resistance in series with the condenser, and R the resistance in the inductive branch, as above. Then the wattless component of the current in the condenser branch is now given by

$$I_2 \sin \varphi_2 = E/\{\omega C[R_o^2 + (1/\omega C)^2]\}$$

The wattless component of the current in the inductive branch is

$$I_1 \sin \varphi_1 = E\omega L/(R^2 + \omega^2 L^2)$$

Hence the condition for resonance is

$$I_1 \sin \varphi_1 = I_2 \sin \varphi_2$$

i.e.
$$\frac{\omega L}{1/\omega C} = \frac{R^2 + \omega^2 L^2}{R_o^2 + (1/\omega C)^2} = \frac{Z^2}{Z_o^2}$$

Solving for ω, we have

$$\omega^2 = \frac{R^2 - L/C}{CL(R_o^2 - L/C)}$$

The line current at resonance frequency is equal to the sum of the power components of the branch-circuit currents, i.e.

$$I = I_1 \cos \varphi_1 + I_2 \cos \varphi_2$$
$$= E\left(\frac{R}{R^2 + \omega^2 L^2} + \frac{R_o}{R_o^2 + (1/\omega C)^2}\right)$$

The special cases are (1) when $R_o = R$; (2) when $R = \sqrt{(L/C)}$, $R_o = \sqrt{(L/C)}$; (3) when $R_o = R$, $L = C$. The values of ω corresponding to the resonance frequencies in these cases are (1) $\omega = \sqrt{(1/CL)}$; (3) $\omega = 1/L = 1/C$.

Hence, substituting for the values of R_o and ω corresponding to the above special cases, we have for the line currents at the resonance frequencies—

(1)	$\omega^2 = 1/LC$	$I = E\,\{2R/[R^2 + (L/C)]\}$
(2)	At all frequencies	$I = E\,\sqrt{(C/L)}$
(3)	$\omega^2 = 1/L^2 = 1/C^2$	$I = E\,[R/(R^2 + 1)]$

Whence the impedances at the resonance frequencies are

(1)
$$Z = \frac{R}{2} + \frac{1}{R}\left(\frac{1}{2}\frac{L}{C}\right)$$

(3)
$$Z = \frac{R}{2} + \frac{1}{2R}$$

Phase Difference Between Branch-circuit Currents for Special Case. In the special case where $R_o = R = \sqrt{(L/C)}$, the line current at all frequencies is directly proportional to the line voltage, just as for the case of a simple circuit containing resistance. If φ_1 is the phase difference between the line voltage and the current in the

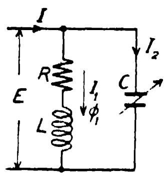


FIG. 52

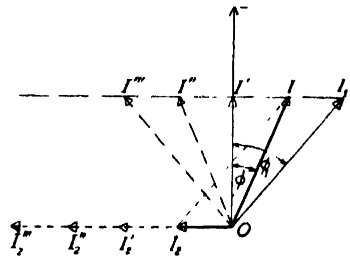


FIG. 53

CIRCUIT AND VECTOR DIAGRAMS FOR PARALLEL CIRCUIT OF VARIABLE IMPEDANCE (VARIABLE CONDENSER IN ONE BRANCH)

inductive branch ($R - L$) and φ_2 that for the capacitive ($R_o - C$) branch, then

$$\sin \varphi_1 = \frac{\omega L}{\sqrt{[(L/C) + \omega^2 L^2]}} = \frac{1}{\sqrt{(1 + 1/\omega^2 LC)}}$$

$$\cos \varphi_1 = \frac{1}{\sqrt{(1 + \omega^2 LC)}}$$

$$\sin \varphi_2 = \frac{1}{\omega C\sqrt{[(L/C) + (1/\omega^2 C^2)]}} = \frac{1}{\sqrt{(1 + \omega^2 LC)}}$$

$$\cos \varphi_2 = \frac{\sqrt{(L/C)}}{\sqrt{[(L/C) + 1/\omega^2 C^2)]}} = \frac{1}{\sqrt{(1 + 1/\omega^2 LC)}}$$

Hence $\sin \varphi_1 = \cos \varphi_2$, and $\cos \varphi_1 = \sin \varphi_2$. Therefore the phase difference between the branch currents is 90° at all frequencies.

If, then, such a circuit is supplied at constant voltage but variable frequency, the branch-circuit currents will each vary in magnitude, and the phase difference between them will always be 90°. Hence the vectors representing the line and branch-circuit currents will form a series of right-angled triangles on a common hypotenuse.

Parallel Circuits of Variable Impedance. (2) *Constant Resistance and Inductance in One Branch, Variable Reactance in the Other Branch.* The case which is of practical importance and which will be considered here is where the variable reactance consists of a condenser of adjustable capacitance. The circuit diagram is shown in Fig. 52, and the vector diagram in Fig. 53. Assuming constant

supply voltage and frequency, the current, I_1, in the branch of constant impedance will be constant, while that, I_2, in the branch containing the condenser will be directly proportional to the capacitance, e.g. $I_2 = \omega EC$. The phase differences between these currents and the supply E.M.F. are constant: that between the supply E.M.F. and I_1 is $\varphi_1 = \tan^{-1}\omega L/R$ (lagging); that between the supply E.M.F. and I_2 is 90° (leading).

The line current, I, is equal to the vector sum of I_1 and I_2. Since I_1 is constant in magnitude and direction, and I_2 is of variable magnitude but fixed direction, the locus of the extremity of the line current vector will be a straight line drawn through I_1 parallel to OI_2 (Fig. 53). Thus, as the capacitance is varied from zero, the line current is brought more into phase with the supply E.M.F., i.e. the effective reactance of the circuit as a whole is diminished and the power factor is improved. At a particular value of the capacitance the line current is exactly in phase with the supply E.M.F., and has its minimum value*: at higher values of the capacitance the line current increases in magnitude and leads the supply E.M.F.

If values of line current and capacitance are plotted, we obtain a V curve, as shown in Fig. 54, and if the power factor is also plotted we obtain an inverted V curve (see Fig. 54).

The curves of Fig. 54 are calculated for a parallel circuit similar to that of Fig. 52, supplied at a constant E.M.F. of 200 V., and a constant frequency of 50 cycles per second. The inductive branch has a resistance of 40 ohms and a constant inductance of 0·54 henries: the capacitive branch contains a condenser, the capacitance of which is adjustable between zero and 30 μF.

The method of calculating these curves is as follows—

The impedance of the inductive branch is

$$Z = \sqrt{\{40^2 + (2\pi \times 50 \times 0{\cdot}54)^2\}} = 174{\cdot}4 \text{ ohms}$$

Whence

$$I_1 = 200/174{\cdot}4 = 1{\cdot}146 \text{ A.}$$

$$\cos \varphi_1 = R/Z = 40/174{\cdot}4 = 0{\cdot}23.$$

The line current at unity power factor is

$$\begin{aligned} I &= I_1 \cos \varphi_1 \\ &= 1{\cdot}146 \times 0{\cdot}23 = 0{\cdot}264 \text{ A} \end{aligned}$$

The capacitance required to give unity power factor is determined from the condition that the charging current of the condenser must

* The condition which gives unity power factor is the same as that which gives the resonance frequency, for the parallel circuit, equal to the supply frequency. Under these conditions the stored energy of the circuits is transferred from the condenser to the inductance, and *vice versa*, during successive quarter periods.

be equal to the wattless current taken by the inductive resistance, i.e. $I_2 = I_1 \sin \varphi_1 = \omega CE$,

whence $C = I_1 \sin \varphi_1 / \omega E$

$$= \{1{\cdot}146\sqrt{(1 - 0{\cdot}23^2)}\}/(2\pi \times 50 \times 200)$$

$$= 1{\cdot}116/62800 = 17{\cdot}75 \times 10^{-6} \text{ F.}$$

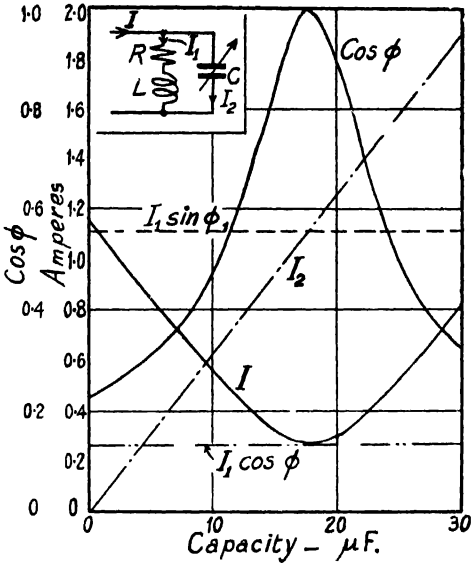


FIG. 54. CHARACTERISTIC CURVES FOR PARALLEL CIRCUIT OF VARIABLE IMPEDANCE

($R = 40\ \Omega$., $L = 0{\cdot}54$ H., C variable)

For any other value of capacitance, such as 10 μF., we have

Wattless component of line current $= I_1 \sin \varphi_1 - \omega CE$

$= 1{\cdot}116 - 62800 \times 10 \times 10^{-6}$

$= 0{\cdot}487$ A.

Power component of line current $= I_1 \cos \varphi_1$

$= 0{\cdot}2635$ A

Line current $= \sqrt{(0{\cdot}2635^2 + 0{\cdot}487^2)}$

$= 0{\cdot}554$ A.

Power factor = power component/line current

$= 0{\cdot}2635/0{\cdot}554$

$= 0{\cdot}475$ (lagging).

Other points are calculated in a similar manner, and are given in Table IV.

Practical Applications. The curves of Fig. 54 show that a circuit consisting of a fixed inductance connected in parallel with a condenser possesses the property that if the capacitance of the condenser be chosen suitably, the power factor of the combined circuit is higher than that of the inductive branch alone. This property is of considerable value in practice. For example, if a particular circuit possesses a low (lagging) power factor, the effects of this low power factor on the supply system may be avoided by connecting in parallel with the circuit a condenser of suitable capacitance. Condensers used in this manner for improving the power factor of power circuits are of the oil-immersed paper type (see p. 59).

In the application of condensers to such circuits it is important to observe that the capacitance required for corrective purposes is proportional to $\cos \varphi_1 (\tan \varphi_1 - \tan \varphi)$, where φ_1 is the phase difference for the inductive circuit alone and φ is the resultant phase difference for the combined circuits. Thus, if the resultant power factor is to be unity (i.e. $\varphi = 0$), the capacitance required will be proportional to $\sin \varphi_1$, but if a lower resultant power factor is required, the capacitance will be smaller. For example, if the original power factor is 0·707, corresponding to $\varphi_1 = 45°$, and the resultant power factor is to be unity, the capacitance required will be proportional to $\cos 45° = 0{\cdot}707$. If, however, the resultant power factor is to be 0·95 (lagging), the capacitance will be proportional to $\cos 45° (\tan 45° - \tan 18{\cdot}2°) = 0{\cdot}707\ (1 - 0{\cdot}3288) = 0{\cdot}475$, which is about two-thirds of that required to obtain a power factor of unity. Hence, in cases where a resultant power factor of, say, 0·95 is satisfactory, the cost of the condenser required for the purpose will be considerably lower than that which would be necessary for correcting the power factor to unity.

Example. A 40 H.P., 550 V., 50 cycle, alternating-current motor has a power factor at full load (which corresponds to an input of 34 kW.) of 0·85 (lagging). What capacitance of condenser, connected in parallel with the motor, is required to obtain a resultant power factor of 0·98 (lagging). If this condenser is permanently connected in parallel with the motor, what will be the resultant power factors when the motor is operating at half load and quarter load, assuming the power input to the motor at these loads to be 17·5 and 9·7 kW. respectively, and the corresponding power factors to be 0·78 and 0·63 ? Neglect losses in the condenser.

The current input to the motor at full load $= 34000/(550 \times 0{\cdot}85) = 72{\cdot}7$A.
" " " " " half " $= 17500/(550 \times 0{\cdot}78) = 40{\cdot}7$A.
" " " " " quarter " $= 9700/(550 \times 0{\cdot}63) = 28$ A.
Wattless component of current input at full load $= 72{\cdot}7\sqrt{(1 - 0{\cdot}85^2)} = 38{\cdot}4$A.
" " " " half " $= 40{\cdot}7\sqrt{(1 - 0{\cdot}78^2)} = 25{\cdot}5$A.
" " " " quarter " $= 28\sqrt{(1 - 0{\cdot}63^2)} = 21{\cdot}7$A.

Power component of full-load current = 34000/550 = 61·8 A.

∴ Line current at 0·98 power factor and full load on motor = 61·8/0·98 = 63·1 A.

Wattless component of this current = $63 \cdot 1\sqrt{(1 - 0 \cdot 98^2)}$ = 12·6 A.

∴ Charging current of condenser = Difference between wattless components at power factors 0·85 and 0·98
= 38·4 − 12·6
= 25·8 A.

Hence capacitance of condenser = $25 \cdot 8 \times 10^6/(2\pi \times 50 \times 550)$ = 149 μF.

NOTE.—Capacitance of condenser required to give a power factor of unity with full load on motor = $38 \cdot 4 \times 10^6/(2\pi \times 50 \times 550)$ = 222 μF.

Assuming constant line voltage and frequency, the charging current of the condenser will remain constant.

Hence the wattless component of the line current when the motor is operating at half load = 25·5 − 25·8 = − 0·3 A.

[The minus sign indicates that this component is leading the impressed E.M.F.]

Power component of line current when the motor is operating at half load
= 17500/550 = 31·8 A.

∴ Line current = $\sqrt{(31 \cdot 8^2 + 0 \cdot 3^2)}$ = 31·8 A.

Power factor = 31·8/31·8 = 1·0

[NOTE.—Actually the phase difference is about 0·6° (leading).]

Wattless component of line current when the motor is operating at quarter load = 21·7 − 25·8 = − 4·1 A.

Power component of line current at this load
= 9700/550 = 17·65 A.

∴ Line current = $\sqrt{(17 \cdot 65^2 + 4 \cdot 1^2)}$ = 18·1 A.

Power factor = 17·65/18·1 = 0·986 (leading)

TABLE IV

Calculations for Fig. 54. Parallel circuit: inductive branch, R = 40 Ω., L = 0·54 H.; condenser branch, C variable, 0 to 30 μF. Supply pressure = 200 V. (constant), frequency = 50 (constant).

$C(\mu F)$	$I_2 = \omega CE$	$I_1 \sin \varphi_1 - I_2$	$I_1 \cos \varphi_1$	$I = \sqrt{\{(I_1 \cos \varphi_1)^2 + (I_1 \sin \varphi_1)^2\}}$	$\cos \varphi$
0	0	1·116	0·2635	1·146	↑ 0·23
5	0·314	0·801		0·844	lag 0·312
10	0·628	0·487		0·554	0·475
15	0·942	0·173		0·315	↓ 0·836
17·75	1·116	0		0·2635	1·0
20	1·256	− 0·141		0·29	↑ 0·909
25	1·57	− 0·455		0·526	lead 0·5
30	1·885	− 0·77	↓	0·814	0·323 ↓

Parallel Circuits of Variable Impedance. (3) *Constant Impedance in One Branch, Variable Impedance in Other Branch.* Consider the

circuits represented in Fig. 55, in which one branch contains a variable resistance in series with a fixed reactance, X_2. Denoting the conductances, susceptances, and admittances of the circuits by G_1, G_2; B_1, B_2; Y_1, Y_2, respectively, we have

$$G_1 = R_1/(R_1^2 + X_1^2) \qquad G_2 = R_2/(R_2^2 + X_2^2)$$
$$B_1 = X_1/(R_1^2 + X_1^2) \qquad B_2 = X_2/(R_2^2 + X_2^2)$$
$$\dot{Y}_1 = \boldsymbol{G}_1 - j\boldsymbol{B}_1 \qquad \dot{Y}_2 = \boldsymbol{G}_2 - j\boldsymbol{B}_2$$

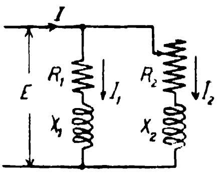


FIG 55

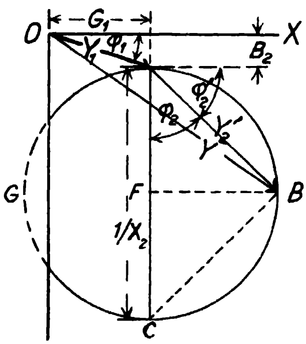


FIG. 56

CIRCUIT AND VECTOR DIAGRAMS FOR PARALLEL CIRCUIT OF VARIABLE IMPEDANCE (VARIABLE RESISTANCE IN ONE BRANCH)

The joint admittance, Y, is equal to the sum of the separate admittances, i.e. $\dot{Y} = \dot{Y}_1 + \dot{Y}_2$. Now $\dot{Y}_2$ is variable, being zero when $R_2 = \infty$, and equal to $j(1/X_2)$ when $R_2 = 0$. The joint admittance, $\dot{Y}$, may be obtained either by calculation or graphically, the graphical construction possessing the advantage that the magnitudes and phase differences of the branch and line currents may be obtained at the same time as the joint admittance.

The admittance diagram is shown in Fig. 56. The vector OA, in the fourth quadrant, represents the admittance $\dot{Y}_1 = \boldsymbol{G}_1 - j\boldsymbol{B}_1$. The vector AC—parallel to the vertical axis and of length equal to $1/X_2$ on the admittance scale—represents the maximum value of the variable admittance $\dot{Y}_2$, this value corresponding to $R_2 = 0$. A semicircle, ABC, described on AC, therefore, gives the locus of the admittance vector $\dot{Y}_2$ when the resistance, R_2, is varied from 0 to ∞.

Proof. Let the vector AB, Fig. 56, represent any particular value, $\dot{Y}'_2$, of the admittance Y_2, corresponding to a particular value, R'_2 of the variable

resistance. Then $Y'_2 = G'_2 + jB'_2$, where $G'_2 = R'_2/(R'_2 + X_2^2)$, and $B'_2 = X_2/(R'^2_2 + X_2^2)$. Now if AF is the projection of AB on AC

$$AF = Y'_2 \cos\varphi_2 = B'_2 = \frac{X_2}{R'^2_2 + X_2^2}\,;\; FB = Y'_2 \sin\varphi_2 = G'_2 = \frac{R'_2}{R'^2_2 + X_2}$$

$$Y'^2_2 = AB^2 = AF^2 + FB^2 = G'^2_2 + B'^2_2 = \left(\frac{R'_2}{R'^2_2 + X_2^2}\right)^2 + \left(\frac{X_2}{R'^2_2 + X_2^2}\right)^2 = \frac{1}{R'^2_2 + X_2^2}$$

But $Y'_2 \cos\varphi_2 = X_2/(R'^2_2 + X_2^2)$

Hence $\quad 1/(R'^2_2 + X_2^2) = (Y'_2 \cos\varphi_2)/X_2$

i.e. $\quad Y'^2_2 = (Y'_2 \cos\varphi_2)/X_2$

or $\quad Y'_2 = (1/X_2)\cos\varphi_2$

Whence $\quad Y'_2/\cos\varphi_2 = 1/X_2 =$ a constant.

If BC be drawn perpendicular to AB, then

$$AC = Y'_2/\cos\varphi_2 = 1/X_2.$$

Hence AC is of constant value, and since angle ABC is a right angle, point B lies on a semicircle described on AC as diameter.

The joint admittance, Y, is therefore represented by the vector OB, Fig. 56, and the locus of this vector when the resistance, R_2, is varied is the semicircle ABC referred to the pole O.

When the variable branch contains a condenser of fixed capacitance, instead of a fixed inductance, the locus of the joint impedance vector is the semicircle AGC, the pole being at O, as before.

The vector, OB, Fig. 56, representing the joint admittance, also represents the line current to a scale E times the admittance scale, where E is the line voltage. Moreover, the vectors OA, AB, represent the branch currents, I_1, I_2, to this scale. Thus, *by a suitable change of scale, an admittance diagram is converted into a current diagram.*

Determination of the Diagram for the Joint Impedance. Since the reciprocal of admittance is equal to impedance, the inversion of the locus of the joint admittance vector, in Fig. 56, will give the locus of the image of the vector of the joint impedance. Now the inversion of a circle with respect to a pole, or centre of inversion, external to the circle is another circle having its centre on the line joining the pole and the centre of the original circle. The position of the centre of this new circle is determined from the condition that the common tangent drawn from the pole to both circles must be divided at the points of contact such that the product of the lengths from the pole to the point of contact of each tangent must be equal to the inversion constant, K_i.

Proof. Let $ABCD$, Fig. 57, be a circle of which an inversion is required with respect to the pole, or inversion centre, O. From O draw the tangent OA and divide it at A_1 such that $OA.OA_1$ is equal to the inversion constant, K_i. At A_1 draw a perpendicular, A_1Q_1, to meet the line joining the centre of the circle $ABCD$ and the pole O. Then the point of intersection, Q_1, is the centre of the circle which is the inversion of circle $ABCD$.

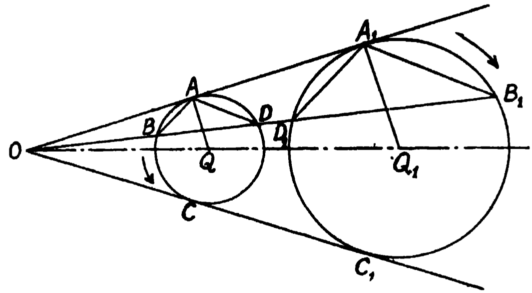


FIG. 57. INVERSE CIRCLES

From O draw any line, such as $OBDD_1B_1$, to cut both circles at points D, B, B_1, D_1. Join AD, AB, A_1B_1, A_1D_1. Then AD is parallel to A_1B_1; AB is parallel to A_1D_1. Therefore triangles OAD, OA_1B_1 are similar; also triangles OAB, OA_1D_1 are similar. Hence $OD : OA = OB_1 : OA_1$: $OB : OA = OD_1 : OA_1$. Whence $OB.OB_1$ $OD.OD_1 = OA.OA_1$. Points A, A_1; B, B_1; D, D_1; are therefore inverse points, and the arc ABC is inverse to the arc $A_1B_1C_1$. Similarly the arc $A_1D_1C_1$ is inverse to the arc ADC.

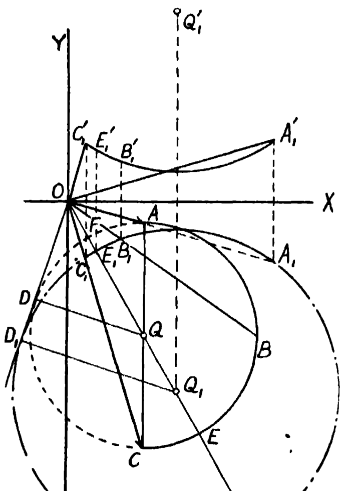


FIG. 58. IMPEDANCE DIAGRAM CORRESPONDING TO ADMITTANCE DIAGRAM OF FIG. 56

To construct the impedance diagram corresponding to the admittance diagram of Fig. 56, which is reproduced with the same lettering in Fig. 58, draw the line OQ, joining the centre, Q, of the semicircle ACB and the pole O: draw the tangent OD and determine the point D_1 such that $OD.OD_1 = K_i = 1/mn$, where K_i is the inversion constant and m, n, are the scales for admittance and impedance respectively. From D_1 draw a perpendicular to meet the line OQ, produced, if necessary, at Q_1, which is the centre of the inverse circle.

An alternative method of construction, which does not require a

knowledge of the inversion constant and which permits the scale for the inverse circle to be selected according to the space available and to the diameter required for this circle, is as follows: Join OQ and produce to the circumference of the admittance circle $ABECF$. Measure the lengths of the intercepts OF, OE, for the points where this line intersects the admittance circle, and calculate from the scale for admittance, the values, in mhos, of the admittances thus represented. Then the reciprocal of the admittance represented by OF gives the maximum value, in ohms, of the joint impedance. Hence, select the scale for impedance such that this quantity is represented by a vector, OF_1, of convenient length. Point F_1 is therefore inverse to point F. Determine similarly, to the impedance scale, the point E_1 (on OQ) representing the minimum value of the joint impedance, corresponding to the maximum value of the joint admittance OE. Hence, points F_1, E_1, are on the diameter of the inverse circle, the centre of which, Q_1, is obtained by bisecting the line joining F_1E_1.

The arc $A_1B_1E_1C_1$ is the inversion, with respect to the pole O, of the semicircle $ABEC$; the points $A_1B_1E_1C_1$ being the inverse points to $ABEC$ respectively. Thus, as the joint admittance vector moves along the semicircle $ABEC$ from A to C, the image of the joint impedance vector moves along the arc $A_1B_1E_1C_1$ from A_1 to C_1. Finally, by determining the image of this arc in the first quadrant we have the locus of the joint impedance vector. This locus is shown in Fig. 58 by the arc $A_1'B_1'E_1'C_1'$. Thus the vector OC_1' represents the joint impedance when the variable resistance, R_2, is zero, and the vector OA_1' represents the impedance when the variable resistance is infinite.

The diagram of Fig. 56 has a more extensive application than that of obtaining the joint admittance and joint impedance of the circuits represented in Fig. 55. We shall show in the following chapter that all quantities relating to the performance of the circuit when the resistance R_2 is varied may be obtained from this diagram.

Worked Examples on Series and Parallel Circuits

General Remarks Concerning the Solution of Problems on Series and Parallel Circuits. *Series Circuits.* In problems on series circuits it is necessary to determine the impedance of the circuit for the purpose of calculating the current. The calculations are of a simple nature and are easily effected by algebraic methods.

Thus the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$$

where R is the total resistance of the circuit, L the total inductance, and C the total capacitance.

The current is given by

$$I = E/Z,$$

and the phase difference between terminal E.M.F. and current is given by

$$\varphi = \tan^{-1}(\omega L - 1/\omega C)/R,$$

or by $$\varphi = \cos^{-1}R/Z.$$

In connexion with the first expression for the phase difference, it should be observed that a plus (+) sign for φ, corresponding to $\omega L > 1/\omega C$, denotes "lag," and a minus (–) sign, corresponding to $\omega L < 1/\omega C$, denotes "lead," of the current with respect to the impressed E.M.F.

When the second expression, or cosine, is used for obtaining φ, no indication is given as to whether the current is leading or lagging with respect to the impressed E.M.F. In this case it will be necessary to determine whether $\omega L >< 1/\omega C$. Hence when the actual angle φ is required it should be calculated from the "tangent" expression, but when only the power factor of the circuit is required, the "cosine" expression should be used.

For the special cases of series circuits the equations—some of which were deduced in Chapter III—follow directly from the general equation and are tabulated below—

Special Case.	Impedance.	Current.	Tan φ	φ
R only	R	E/R	0	0
L only	ωL	$E/\omega L$	∞	90° (lag)
C only	$1/\omega C$	ωCE	$-\infty$	90° (lead)
R and L in series	$\sqrt{(R^2+\omega^2L^2)}$	$E/\sqrt{(R^2+\omega^2L^2)}$	$\omega L/R$	$>0<90°$ (lag)
R and C in series	$\sqrt{\{R^2+(1/\omega C)^2\}}$	$E/\sqrt{\{R^2+(1/\omega C)^2\}}$	$-1/\omega CR$	$>0<90°$ (lead)
L and C in series	$\sqrt{(\omega L-1/\omega C)^2}$	$E/\sqrt{(\omega L-1/\omega C)^2}$	∞ or 0	$\pm 90°$ or 0, according to whether $\omega L >=< 1/\omega C$

Parallel Circuits. In the majority of problems on parallel circuits it is generally simpler to obtain the line current from the joint admittance of the circuits rather than from their joint impedance, and in these cases the results are best calculated by the symbolic method.

The numerical value of the joint admittance is given by

$$Y = \sqrt{\{(G_1 + G_2 + G_3 + \ldots)^2 + (B_1 + B_2 + B_3 + \ldots)^2\}}$$

where $G_1, G_2, G_3 \ldots, B_1, B_2, B_3, \ldots$ are the conductances and admittances, respectively, of the several branch circuits.

The line current is given by

$$I = EY,$$

and its phase difference with respect to the impressed E.M.F. is given by

$$\varphi = \tan^{-1}(B_1 + B_2 + B_3 + \ldots)/(G_1 + G_2 + G_3 + \ldots).$$

Series-parallel Circuits. In all problems on series-parallel circuits the parallel portions of the circuit are treated separately and the joint impedances determined. The series-parallel circuit is then replaced by an equivalent series circuit and the total impedance is readily determined.

Example 1. A coil having a resistance of 5 ohms and an inductance of 0·02 henry is arranged in parallel with another coil having a resistance of 1 ohm and an inductance of 0·08 H. Calculate the current flowing through each coil when a pressure of 100 volts at 50 cycles is applied to them. Find the total current passing, and estimate the resistance of a single coil which will take the same current at the same power factor. [*L.U.*]

Solution by the Trigonometric Method

The steps in this method of solution are—

1. Calculate the current in each branch of the parallel circuit, and the phase difference between each of these currents and the impressed E.M.F.
2. Compound these currents to obtain the line current, determining also its power and energy components.
3. Calculate the joint impedance of the parallel circuits.
4. Thence, from the impedance and the power and wattless components of the line current, determine the resistance and reactance of the single coil which will take the same current, at the same power factor, as the parallel circuit.

Thus, the impedance of the 5 Ω., 0·02 H. coil is

$$Z_1 = \sqrt{[5^2 + (314 \times 0{\cdot}02)^2]} = 8{\cdot}02\ \Omega.$$

Whence,
$$I_1 = 100/8{\cdot}02 = 12{\cdot}46\ \text{A}.$$
$$\cos\varphi_1 = R_1/Z_1 = 5/8{\cdot}02 = 0{\cdot}624$$
$$\sin\varphi_1 = X/Z_1 = 314 \times 0{\cdot}02/8{\cdot}02 = 0{\cdot}783$$

The impedance of the 1 Ω. 0·08 H. coil is

$$Z_2 = \sqrt{[1^2 + (314 \times 0{\cdot}08)^2]} = 25{\cdot}12\ \Omega.$$

Whence
$$I_2 = 100/25{\cdot}12 = 3{\cdot}98\ \text{A}.$$
$$\cos\varphi_2 = R_2/Z_2 = 1/25{\cdot}12 = 0{\cdot}0398$$
$$\sin\varphi_2 = X_2/Z_2 = 314 \times 0{\cdot}08/25{\cdot}12 = 1{\cdot}0\ \text{approx.}$$

The line current is

$$I = \sqrt{(I_1^2 + I_2^2 + 2I_1I_2\cos(\varphi_2 - \varphi_1)} = 15{\cdot}85\ \text{A.},$$

and the joint impedance of the parallel circuit is

$$Z = 100/15{\cdot}85 = 6{\cdot}31\ \Omega.$$

Power component of line current $= I\cos\varphi = I_1\cos\varphi_1 + I_2\cos\varphi_2$
$= 12{\cdot}46 \times 0{\cdot}624 + 3{\cdot}98 \times 0{\cdot}0398$
$= 7{\cdot}94$ A.

Wattless component of line current $= I\sin\varphi = I_1\sin\varphi_1 + I_2\sin\varphi_2$
$= 12{\cdot}46 \times 0{\cdot}783 + 3{\cdot}98 \times 1{\cdot}0$
$= 13{\cdot}74$ A.

$\therefore \quad \sin\varphi = I\sin\varphi/I = 13{\cdot}74/15{\cdot}85 = 0{\cdot}866,$

and $\quad \cos\varphi = I\cos\varphi/I = 7{\cdot}94/15{\cdot}85 = 0{\cdot}5.$

Now if R and X are the resistance and reactance, respectively, of the single coil, we have

$$R = Z\cos\varphi = 6{\cdot}31 \times 0{\cdot}5 = 3{\cdot}16\ \Omega.$$

$$X = Z\sin\varphi = 6{\cdot}31 \times 0{\cdot}866 = 5{\cdot}46\ \Omega.$$

Whence, $\quad L = 5{\cdot}46/(2\pi \times 50) = 0{\cdot}0174$ H.

Solution by the Symbolic Method.

Taking $\omega = 2\pi \times 50 = 314$, the admittance of the 5 Ω. 0·02 H. coil is

$$\dot{Y}_1 = \frac{5}{5^2 + (314 \times 0{\cdot}02)^2} - j\left(\frac{314 \times 0{\cdot}02}{5^2 + (314 \times 0{\cdot}02)^2}\right)$$
$$= 0{\cdot}0777 - j\,0{\cdot}0975$$

Whence, $\quad Y_1 = \sqrt{(0{\cdot}0777^2 + 0{\cdot}0975^2)} = 0{\cdot}1246$ mho.

The admittance of the 1 Ω. 0·08 H. coil is

$$\dot{Y}_2 = \frac{1}{1^2 + (314 \times 0{\cdot}08)^2} - j\left(\frac{314 \times 0{\cdot}08}{1^2 + (314 \times 0{\cdot}08)^2}\right)$$
$$= 0{\cdot}00158 - j\,0{\cdot}0397$$

Whence, $\quad Y_2 = \sqrt{(0{\cdot}00158^2 + 0{\cdot}0397^2)} = 0{\cdot}0398$ mho.

Hence, the joint admittance of the two coils connected in parallel is

$$\dot{Y} = \dot{Y}_1 + \dot{Y}_2$$
$$= (0{\cdot}0777 + 0{\cdot}00158) - j(0{\cdot}0975 + 0{\cdot}0397)$$
$$= 0{\cdot}0793 - j\,0{\cdot}1372$$

Whence, $\quad Y = \sqrt{(0{\cdot}0793^2 + 0{\cdot}1372^2)} = 0{\cdot}1585$ mho.

The branch currents are

$$I_1 = 100Y_1 = 12{\cdot}46 \text{ A.},$$
$$I_2 = 100Y_2 = 3{\cdot}98 \text{ A.},$$

and the line current is

$$I = 100Y = 15{\cdot}85 \text{ A.}$$

The admittance of a single coil which will take this (line) current at the same power factor must equal the joint admittance of the above coils connected in parallel.

Hence if G and B denote the conductance and susceptance, respectively, of the coil,

$$\dot{Y} = G - jB = 0{\cdot}0793 - j\,0{\cdot}1372$$

Whence, $\quad G = 0{\cdot}0793$
$B = 0{\cdot}1372$
$Y = \sqrt{(0{\cdot}0793^2 + 0{\cdot}1372^2)} = 0{\cdot}1584$

Now $\quad G = R/(R^2 + X^2)\,; \quad B = X/(R^2 + X^2)$

and $\quad G^2 + B^2 = Y^2 = \dfrac{R^2}{(R^2 + X^2)^2} + \dfrac{X^2}{(R^2 + X^2)^2} = \dfrac{1}{R^2 + X^2}$

From these equations we obtain

$$
\begin{aligned}
R^2 + X^2 &= 1/Y^2 = R/G = X/B. \\
\therefore \quad R &= G/Y^2 \\
&= 0{\cdot}0793/0{\cdot}1584^2 \\
&= 3{\cdot}16\,\Omega. \\
X &= B/Y^2 \\
&\ 0{\cdot}1372/0{\cdot}1584^2 \\
&= 5{\cdot}46\,\Omega. \\
L &= 5{\cdot}46/(2\pi \times 50) \\
&= 0{\cdot}0174\ \text{H.}
\end{aligned}
$$

Example 2. If an inductive resistance, for which $R = 2$ ohms, $L = 0{\cdot}005$ henry, is connected in series with the circuit of the inductive coils in Example (1) above, to what value must the line voltage be raised in order that the currents through these coils may be the same as in the above case, the frequency being remaining unchanged? What will be the resultant power factor under these conditions?

Adopting the values of $3{\cdot}16\ \Omega$. and $5{\cdot}46\ \Omega$. obtained above for the joint resistance and reactance, respectively, of the parallel-connected inductive coils, the series-parallel circuit of this example may therefore be replaced by an equivalent series circuit having a total resistance of

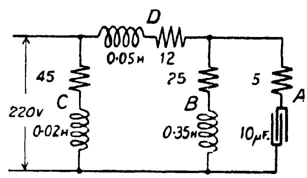


FIG. 59. CIRCUIT DIAGRAM FOR WORKED EXAMPLE NO. 3

$$3{\cdot}16 + 2 = 5{\cdot}16\,\Omega.,$$

and a total reactance of

$$5{\cdot}46 + (2\pi \times 50 \times 0{\cdot}005) = 7{\cdot}03\,\Omega$$

Hence the total impedance $= \sqrt{(5{\cdot}16^2 + 7{\cdot}03^2)} = 8{\cdot}72\,\Omega.$

Therefore the impressed E.M.F. $= 15{\cdot}85 \times 8{\cdot}72 = 138{\cdot}2$ V.

Resultant power factor $= 8{\cdot}72/\sqrt{(5{\cdot}16^2 + 7{\cdot}03^2)}$

$= 5{\cdot}16/8{\cdot}72 = 0{\cdot}592.$

Example 3. The circuit shown in Fig. 59 consists of three parallel branches A, B, C, the branches A and B being connected to branch C through an inductive resistance D. Determine the current in each branch circuit, the line current, the resultant power factor, and the power supplied to the circuits. Branch A consists of a condenser of 10 μF. capacitance in series with a resistance of 50 Ω; branch B consists of an inductive resistance, for which $R = 25\,\Omega$, $L = 0{\cdot}35$ H.; branch C consists of an inductive resistance, for which $R = 45\,\Omega$, $L = 0{\cdot}02$ H. The series inductive resistance D has a resistance of 12 Ω and an inductance of 0·05 H. The supply pressure is 220 V. and the frequency 50 cycles per second.

Denoting the admittances of the branches A, B, C, by Y_A, Y_B, Y_C, respectively, and the conductances and susceptances by G_A, G_B, G_C, and B_A, B_B, B_C, respectively, we have

$$G_A = \frac{50}{50^2 + (10^6/314 \times 10)^2} = 0{\cdot}482 \times 10^{-3}$$

$$G_B = \frac{25}{25^2 + (314 \times 0{\cdot}35)^2} = 1{\cdot}965 \times 10^{-3}$$

$$G_C = \frac{45}{45^2 + (314 \times 0{\cdot}02)^2} = 21{\cdot}8 \times 10^{-3}$$

$$B_A = -\frac{10^6/(314 \times 10)}{50^2 + (10^6/(314 \times 10))^2} = -3{\cdot}07 \times 10^{-3}$$

$$B_B = \frac{314 \times 0{\cdot}35}{25^2 + (314 \times 0{\cdot}35)^2} = 8{\cdot}65 \times 10^{-3}$$

$$B_C = \frac{314 \times 0{\cdot}02}{45^2 + (314 \times 0{\cdot}02)^2} = 3{\cdot}04 \times 10^{-3}$$

Whence,
$$Y_A = G_A - jB_A = \boldsymbol{0{\cdot}482 \times 10^{-3} + j3{\cdot}07 \times 10^{-3}}$$
$$Y_B = G_B - jB_B = \boldsymbol{1{\cdot}965 \times 10^{-3} - j8{\cdot}65 \times 10^{-3}}$$
$$Y_{A+B} = \boldsymbol{(0{\cdot}482 + 1{\cdot}965)10^{-3} - j(8{\cdot}65 - 3{\cdot}07)10^{-3}}$$
$$= \boldsymbol{2{\cdot}447 \times 10^{-3} - j5{\cdot}58 \times 10^{-3}}$$

The joint admittance, Y_1, of the series-parallel branch ABD is therefore

$$Y_1 = \frac{1}{Z_1 + (1/Y_{A+B})} = \frac{1}{R_D + jX_D + (1/(G - jB))}$$

$$= \frac{1}{R_D + \dfrac{G}{G^2 + B^2} + j\left(X_D + \dfrac{G}{G^2 + B^2}\right)}$$

where G and B are the joint conductances and susceptances, respectively, of the parallel branches A and B, and R_D, X_D, are the resistance and reactance, respectively, of the series portion, D.

Substituting numerical values for these quantities, we obtain

$$\frac{G}{G^2 + B^2} = 65{\cdot}9 \qquad\qquad \frac{B}{G^2 + B^2} = 150{\cdot}3$$

$$Y = \frac{1}{\boldsymbol{(12 + 65{\cdot}9) + j(314 \times 0{\cdot}05 + 150{\cdot}3)}} = \frac{1}{\boldsymbol{77{\cdot}9 + j166}}$$

$$= \frac{\boldsymbol{77{\cdot}9}}{77{\cdot}9^2 + 166^2} - j\left(\frac{\boldsymbol{166}}{77{\cdot}9^2 + 166^2}\right)$$

$$= \boldsymbol{2{\cdot}315 \times 10^{-3} - j4{\cdot}93 \times 10^{-3}}$$

Whence the joint admittance of the complete circuit shown in Fig. 59 is

$$Y = Y_1 + Y_C = \boldsymbol{(2{\cdot}315 \times 21{\cdot}8)10^{-3} - j(4{\cdot}93 \times 3{\cdot}04)10^{-3}}$$
$$= \boldsymbol{24{\cdot}115 \times 10^{-3} - j7{\cdot}97 \times 10^{-3}}$$

Therefore the line current $= 220 . Y$

$= 220\sqrt{(0{\cdot}0241^2 + 0{\cdot}00797^2)} = 5{\cdot}59$ A.

Power factor $= 0{\cdot}0241/\sqrt{(0{\cdot}0241^2 + 0{\cdot}00797^2)} = 0{\cdot}95$ (lagging)

Power supplied $= 220 \times 5{\cdot}59 \times 0{\cdot}95 = 1169$ W.

Current in branch C $\quad 220.Y_C$

$$= 220\sqrt{(0{\cdot}0218^2 + 0{\cdot}00304^2)} = 4{\cdot}84 \text{ A.}$$

Current (I_D) in the series-parallel branch ABD

$$= 220.Y_1$$

$$= 220\sqrt{(0{\cdot}0023^2 + 0{\cdot}00493^2)} = 1{\cdot}2 \text{ A.}$$

Potential difference across the parallel-connected branches A, B

$$= 220 - I_D Z_D$$

$$= 220 - [\{220 \times 10^{-3}(2{\cdot}315 - j4{\cdot}93)\}\{12 + j(314 \times 0{\cdot}05)\}]$$

$$= 220 - (23{\cdot}13 - j5)$$

$$= 197 + j5$$

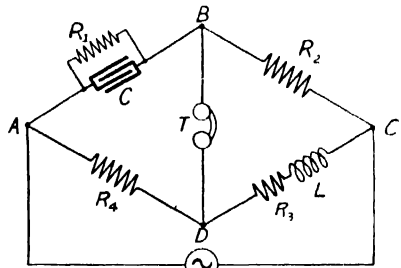


FIG. 60. CIRCUIT DIAGRAM OF WHEATSTONE BRIDGE FOR ALTERNATING CURRENT MEASUREMENTS

Current in branch A—$I_A = Y_A(197 + j5)$

$$= (0{\cdot}482 \times 10^{-3} + j3{\cdot}07 \times 10^{-3})(197 + j5)$$

$$= 0{\cdot}079 + j0{\cdot}607$$

$$I_A = \sqrt{(0{\cdot}079^2 + 0{\cdot}607^2)} = 0{\cdot}612 \text{ A.}$$

Current in branch B—$I_B = Y_B(197 + j5)$

$$= (1{\cdot}965 \times 10^{-3} - j8{\cdot}65 \times 10^{-3})(197 + j5)$$

$$= 0{\cdot}43 - j1{\cdot}695$$

$$\therefore \quad I_B = \sqrt{(0{\cdot}43^2 + 1{\cdot}695^2)} = 1{\cdot}75 \text{ A.}$$

[Note.—All currents are referred to the line voltage which is considered as the vector of reference.]

Example 4. The arms of a Wheatstone bridge, taken in order, consist of (i) a condenser shunted by a non-inductive resistance, (ii) a non-inductive resistance, (iii) an inductive resistance, (iv) a second non-inductive resistance. The bridge is supplied with alternating current of sine wave-form and a telephone is used to indicate balance. Find the condition for which there will be no sound in the telephone. (*L.U.*)

[Note.—The Wheatstone bridge adapted for alternating currents is employed in practice for measuring inductance, capacitance, and dielectric losses at commercial and higher frequencies. A telephone may be used as a detector for balance at audio-frequencies (800 to 2000 cycles), but for other frequencies a vibration galvanometer must be employed.

The circuit of the bridge is represented in Fig. 60.

In balancing the bridge the currents in the adjacent arms must be adjusted to equality of phase as well as magnitude. Therefore when the bridge is balanced the ratio of the impedances of the several arms must be the same as that of the resistances of the corresponding arms in a bridge supplied with steady currents. Hence if Z_1, Z_2, Z_3, Z_4 denote the impedances of the arms AB, BC, CD, DA, respectively (Fig. 60), the condition for balance is

$$Z_1 Z_3 = Z_2 Z_4.$$

Now $Z_1 = \dfrac{1}{\dfrac{1}{R_1} + j\omega C}$ $\quad Z_2 = R_2 \quad Z_3 = R_3 + j\omega L \; ; \; Z_4 = R_4.$

$$\therefore \qquad \left(\frac{1}{\dfrac{1}{R_1} + j\omega C}\right)(R_3 + j\omega L) = R_2 R_4$$

i e.
$$R_3 + j\omega L = R_2 R_4 \Big/ \left(\frac{1}{\dfrac{1}{R_1} + j\omega C}\right)$$
$$= \frac{R_2 R_4}{R_1} + j\omega C R_2 R_4$$

For this equation to be true we must have equality between both the in-phase and quadrature parts. Thus

$$R_3 = R_2 R_4 / R_1$$

or
$$\frac{R_1}{R_2} = \frac{R_4}{R_3}$$

which is the condition for balance with steady currents.

Also
$$\omega L = \omega C R_2 R_4 = \omega C R_1 R_3$$

or
$$\frac{\omega L}{R_3} = \omega C R_1$$

Now $\tan^{-1}\omega L/R_3$ is the phase difference, when the bridge is balanced, between the impressed E.M.F. and the current in the branches CDA; and $\tan^{-1}\omega C R_1$ is the phase difference between the impressed E.M.F. and the current in the branch ABC. Therefore the bridge is balanced for magnitude and phase.

Example 5. A parallel circuit consists of two branches; one branch contains a fixed impedance ($24\,\Omega./\underline{30°}$); the other consists of a variable inductive resistance, of which the reactance is constant and equal to $10\,\Omega.$, and the resistance is variable between 1 and $50\,\Omega$. Determine (1) the variation of line current, (2) the variation of the joint impedance, when the resistance is varied and the circuit is supplied at a constant pressure of 100 V., 50 frequency.

This problem will be solved graphically in order to illustrate the application of the principle of inversion. A reproduction, on a considerably reduced scale, of the diagram for the complete graphical solution is shown in Fig. 61, and the student is advised to re-draw this diagram, step by step, as explained below, on a sheet of drawing paper (half imperial size).

The steps in the construction are—

Draw rectangular axes XOX, YOY. Select the scale for impedance as 1 cm. = 1 ohm, and draw in the first quadrant the vector OA having a length of 24 cm. and inclined at an angle of 30° to the horizontal axis. OA therefore represents the impedance of the non-variable branch. The admittance of this branch = 1/24 = 0·0417 mho, and therefore a convenient scale for

admittance will be 1 cm. = 0·005 mho. Divide OA at A' such that OA' = 0·0417/0·005 = 8·34 cm., and determine the image, B, of A'. Hence, OB will represent, to a scale of 1 cm. = 0·005 mho, the admittance vector corresponding to the impedance vector OA.

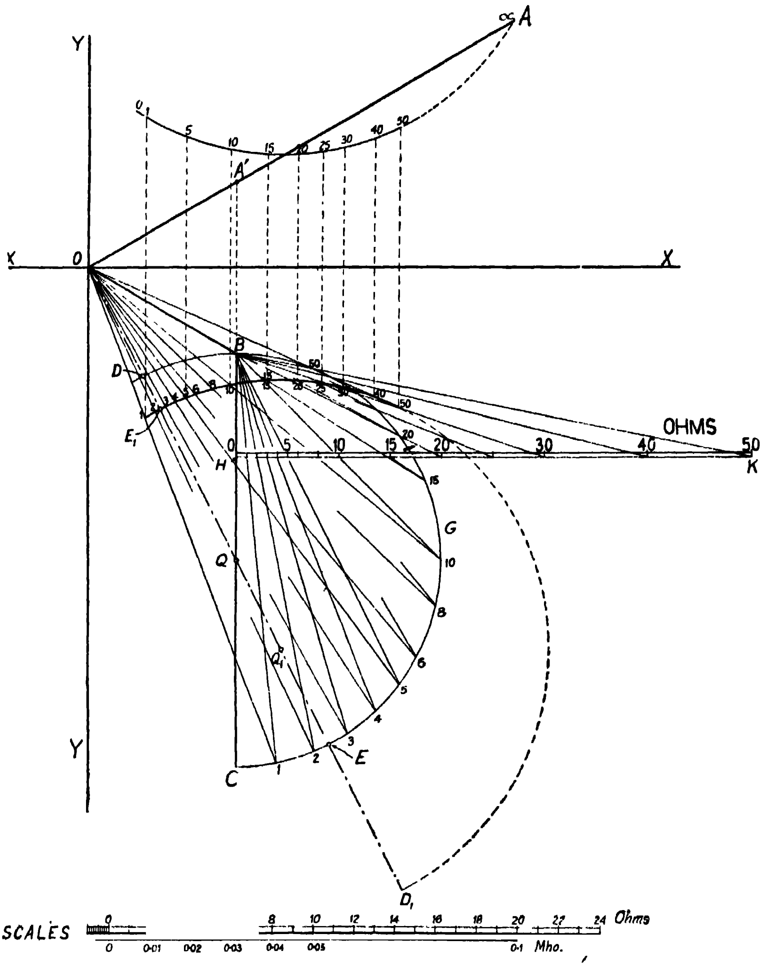


FIG. 61. GRAPHICAL SOLUTION TO EXAMPLE NO. 5

Neglecting for the moment the limits of the resistance of the variable branch, the admittance of this branch is 1/10 = 0·1 mho when R = 0, and is zero when $R = \infty$. Hence from B draw BC parallel to the vertical axis, and of length = (0·1/0·005 =) 20 cm., to represent the admittance of the

variable branch when $R = 0$. Upon BC describe a semicircle, BCG, the centre of which is at Q. Then this semicircle is the locus of the admittance vector, drawn from B, for the variable branch circuit when the resistance is varied from zero to infinity.

To determine the positions of the vector for different values of the variable resistance we must determine the inversion of semicircle BCG with respect to the pole B. Now the inverse point corresponding to C lies along BC, or BC produced, the actual position of the point depending upon the scale adopted for resistance. For the present purpose adopt a scale such that 1 cm. = 2 ohms. Then the inverse point corresponding to C is at H, a distance of $10/2 = 5$ cm. from B The inverse point corresponding to B lies at an infinite distance along a perpendicular drawn through H. Draw this perpendicular, HK, of length equal to $50/2 = 25$ cm. (representing the maximum value of the variable resistance) and construct upon it a scale to represent values of resistance from 1 to 50 O. Joint point 1 on this scale to B

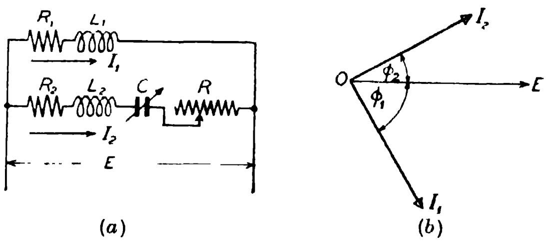


FIG. 62. CIRCUIT AND VECTOR DIAGRAMS FOR WORKED EXAMPLE No. 6

and produce so as to cut the semicircle BGC at 1. Similarly, join point 50 on the resistance scale to B and obtain the point 50 where this line intersects the semicircle. Then the arc 1 G 50 is the locus of the admittance vector for the variable branch when the resistance of this branch is varied between 1 and 50 ohms. Vectors drawn from O to the points 1 and 50 represent the joint admittance of the parallel circuit when the resistance of the variable branch has values of 1 and 50 Ω. respectively.

The joint admittance corresponding to any other value of resistance, say 10 Ω., is obtained by first joining the appropriate point (10) on the resistance scale to B, determining the intersection of this line with the semicircle BGC, and then joining the latter point (10 on semicircle) to O. The line O–10 represents the joint admittance, and the value of this quantity is obtained by multiplying the length of this line by the appropriate scale. The lengths of the joint admittance vectors corresponding to various values of the variable resistance are given in tabular form below.

The line current is obtained by multiplying the joint admittance by the line pressure (100 V.).

To obtain the joint impedance graphically we must determine the inversion of the arc 1 G 50 with respect to the pole O. Thus, join OQ and produce so as to cut the semicircle CGB, produced, at D and E. Measure the lengths OD and OE, which should be 5·91 cm. and 25·9 cm. respectively. Now OD represents an admittance of $5{\cdot}91 \times 0{\cdot}005 = 0{\cdot}0295$ mho, and OE represents an admittance of $25{\cdot}9 \times 0{\cdot}005 = 0{\cdot}1295$ mho. The values of the impedances corresponding are therefore $1/0{\cdot}0295 = 33{\cdot}83$ ohms and $1/0{\cdot}1295 = 7{\cdot}72$ ohms respectively. These impedances are represented on the original impedance scale (viz. 1 cm. = 1 ohm) by OD_1(= 33·83 cm.) and OE_1(= 7·72) cm., both points D_1 and E_1 lying in OQ or its extension. The diameter of the inverse circle is therefore equal to D_1E_1, and its centre is at Q_1. In Fig. 61 only a portion of this circle is shown, and the arc corresponding to the inversion of

the arc 1 G 50 is shown in full line. This arc is the image of the locus of the impedance vector when the resistance of the variable branch is varied between 1 and 50 ohms. By transferring this arc to the first quadrant the impedance vectors and their loci are shown in the correct position with respect to the vector of reference.

The lengths of the impedance vectors corresponding to values of the variable resistance between 1 and 50 ohms inclusive, and the values of impedance deduced therefrom are given below—

R_2 (ohms)	1	2	3	4	5	6	8	10	15	20	30	40	50
Length of joint admittance vector (cm.)	25·68	25·84	25·86	25·6	25·14	24·9	23·55	22·28	19·4	17·25	14·55	13·05	12·05
Joint Admittance (mho.)	0·1284	0·1292	0·1293	0·128	0·1257	0·1245	0·1177	0·1114	0·097	0·0862	0·0727	0·652	0·602
Line current (amp.)	12·84	12·92	12·93	12·8	12·57	12·45	11·77	11·14	9·7	8·62	7·27	6·52	6·02
Length of joint impedance vector (cm.)	7·78	7·74	7·73	7·8	7·97	8·03	8·48	8·98	10·3	11·6	13·8	15·3	16·6
Joint impedance (ohms)	7·78	7·74	7·73	7·8	7·97	8·03	8·48	8·98	10·3	11·6	13·8	15·3	16·6

Example 6. One branch of a parallel circuit contains an inductive coil; the other branch also contains an inductive coil in series with which is connected a condenser of adjustable capacitance and an adjustable non-inductive resistance. Determine the values of capacitance and resistance such that the currents in the inductive coils are equal and have a phase difference of 90° with respect to each other. The circuits are supplied at constant voltage and frequency, and the wave-form of the E.M.F. is sinusoidal.

A diagram for the circuit is shown in Fig. 62 (a), and a vector diagram showing the required conditions is shown in Fig. 62 (b).

Adopting the symbols shown in Fig. 62 (a), we must have, for the above conditions to be satisfied,

(i) $\quad I_1 = I_2$, or $Z_1 = Z_2$

(ii) $\quad I_1 = jI_2$, or $-\tan\varphi_1 = \cot\varphi_2$

Now $\quad Z_1^2 = R_1^2 + \omega^2 L_1^2$

and $\quad Z_2^2 = (R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2$

Hence, from (i),

$$R_1^2 + \omega^2 L_1^2 = (R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2$$

i.e.
$$\omega^2 L_1^2 - \left(\omega^2 L_2^2 - \frac{2\omega L_2}{\omega C} + \frac{1}{\omega^2 C^2}\right) = (R_2 + R)^2 - R_1^2$$

whence
$$C^2\{\omega^2(L_1^2 - L_2^2) + R_1^2 - (R_2 + R)^2\} + 2L_2 C - (1/\omega^2) = 0 \qquad (a)$$

From (ii), we have $\quad E(G_1 - jB_1) = jE(G_2 - jB_2)$

where G_1, G_2, B_1, B_2, are the conductances and susceptances respectively, of the branch circuits.

Hence, $\quad G_1 - jB_1 = jG_2 + B_2$

i.e. $\quad G_1 = B_2; \qquad G_2 = -B_1$

or
$$\frac{R_1}{Z_1^2} = \frac{X_2}{Z_2^2} \qquad \frac{R_2 + R}{Z_2^2} = -\frac{X_1}{Z_1^2}$$

Whence $$\frac{R_1}{X_2} = -\frac{X_1}{R_2 + R}$$

i.e. $R_1(R_2 + R) = -X_1X_2 = \omega L_1(-\omega L_2 + 1/\omega C)$

from which we obtain

$$(R_2 + R) = \frac{\omega L_1}{R_1}\left(\frac{1}{\omega C} - \omega L_2\right) \quad . \quad . \quad . \quad . \quad (b)$$

Substituting this value of $(R_2 + R)$ in equation (a) and re-arranging terms, we have

$$C^2\left\{\omega^2(L_1^2 - L_2^2) - \frac{\omega^4 L_1^2 L_2^2}{R_1^2} + R_1^2\right\} + C\left\{2L_2\left(\frac{\omega^2 L_1^2}{R_1^2} + 1\right)\right\} - \frac{1}{\omega^2} - \frac{L_1^2}{R_1^2} = 0$$

from which C can be calculated when the constants of the inductive coils are known. When C is determined, the value of R is obtained by substituting in equation (b).

Whence. $$C = \frac{R_1/\omega + \omega L_1(L_1/R_1) - L_2[1 + (\omega L_1/R_1)^2]}{\omega^2\{(L_1^2 - L_2^2) - L_2^2(\omega L_1/R_1)^2\} + R_1^2} \quad . \quad . \quad (c)$$

If $L_1 = L_2 = 0{\cdot}2$ H., $R_1 = R_2 = 10\,\Omega.$, and $\omega = 314$; then, on substituting these values in equations (c) and (b), we obtain $C = 43{\cdot}7\,\mu$F., $R = 52{\cdot}8\,\Omega$.

A check against these values for C and R is obtained by calculating the impedances of the branch circuits. Thus

$$Z_1 = \sqrt{(R_1^2 + \omega^2 L_1^2)} = \sqrt{[10^2 + (314 \times 0{\cdot}2)^2]} = \sqrt{(10^2 + 62{\cdot}8^2)} = 70{\cdot}3\,\Omega.$$

$$Z_2 = \sqrt{\{(R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2\}} = \sqrt{\{(10 + 52{\cdot}8)^2 + (314 \times 0{\cdot}2 - 10^6/314 \times 43{\cdot}7)^2\}}$$

$$= \sqrt{(62{\cdot}8^2 + 10^2)} = 70{\cdot}3\,\Omega.$$

$$\varphi_1 = \tan^{-1}\omega L_1/R_1 = \tan^{-1}(314 \times 0{\cdot}2/10) = 80{\cdot}9^\circ$$

$$\varphi_2 = \tan^{-1}\frac{\omega L_2 - 1/\omega C}{R_2 + R} = \tan^{-1}\frac{(314 \times 0{\cdot}2) - 10^6/(314 \times 43{\cdot}7)}{10 + 52{\cdot}8}$$

$$= \tan^{-1} - 10/62{\cdot}8 = -9{\cdot}1^\circ$$

$$\varphi_1 - \varphi_2 = 80{\cdot}9^\circ + 9{\cdot}1^\circ = 90^\circ.$$

[Note.—This method of obtaining a phase difference of 90° between currents in the two branches of a parallel circuit is called "phase splitting"; it has a number of practical applications, an important application being in the phase-shifting transformer for use with the alternating-current potentiometer.]

CHAPTER VII

CIRCLE (OR LOAD) DIAGRAMS FOR SIMPLE CIRCUITS

THE load diagram is an extended vector diagram and is so called because all quantities relating to the circuit (such as current, pressure, power, power factor), as well as the performance of the circuit (i.e. voltage regulation, efficiency, and losses) can be obtained by graphical construction.

The load diagram for a circuit is constructed from the no-load and short-circuit diagram for that circuit, and the latter is obtained from the admittance diagram by a suitable change of scale (see p. 105). In order that the final diagram may appear in a convenient position the admittance diagram is drawn in the first quadrant, instead of in the fourth quadrant, as hitherto, and the vector of reference is vertical. Impedances, or, more correctly, their images, therefore appear in the same quadrant as admittances.

In this chapter we shall deduce load diagrams for series, parallel, and series-parallel circuits, but as the no-load and short-circuit diagram for the parallel circuit has already been deduced (p. 104), the load diagram for this circuit will be obtained before considering series and series-parallel circuits. Moreover, the load diagram for the parallel circuit is more easily deduced than the diagrams for the other types of circuits.

PARALLEL CIRCUITS

Load Diagram for a Parallel Circuit Containing Variable Impedance in One Branch. Let the variable impedance consist of a constant reactance and a variable resistance. The admittance diagram for this circuit is shown in Fig. 56, p. 104, and was constructed with the vector of reference (i.e. the vector of the impressed E.M.F.) horizontal. If the diagram be reconstructed as a current diagram, with the vector of reference vertical, we obtain the diagram shown in Fig. 63. In this diagram the point I_0 (corresponding to $R_2 = \infty$) is called the *no-load point*, and the point I_s (corresponding to $R_2 = 0$) is called the *short-circuit point*. The line currents corresponding to these points are represented by OI_0 and OI_s, and are called the no-load and short circuit currents respectively. Any intermediate point G on the semicircle I_0GI_s corresponds to a particular value

of the variable resistance R_2 intermediate between zero and ∞, and the vector OG represents the line current under these conditions.

The Complete Performance of the Circuit (e.g. the current, power input, power factor, and losses) may be obtained graphically as follows—

The *line current*, I, as already shown, is represented by the vector drawn from O to the semicircle ; the angle which this

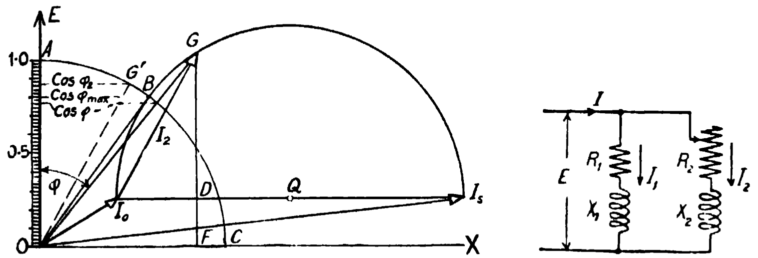


FIG. 63. LOAD AND CIRCUIT DIAGRAMS FOR A SIMPLE PARALLEL CIRCUIT

vector makes with the vertical axis is equal to the phase difference, φ, between line current and impressed E.M.F., and the cosine of this angle is equal to the power factor of the circuit. Hence if a quadrant ABC be drawn with O as centre, and the vertical radius OA be divided into 100 equal parts, then the vertical projection of the point of intersection of any line-current vector with the quadrant gives the power factor as a percentage. Obviously the power factor attains its maximum value when the line-current vector is tangential to the semicircle I_0GI_s.

The *current*, I_2, *in the branch of variable impedance*, corresponding to any particular value of resistance, is represented by a chord in the semicircle, I_0I_s, the chord being drawn from I_0 to the appropriate point, say G, on the circumference.

The *power*, P_2, *supplied* to this portion of the circuit is therefore represented by the ordinate GD in the semicircle I_0GI_s.

The *power factor*, cos φ_2, of this branch may be obtained from the quadrant ABC and scale OA, by drawing through O a parallel to I_0G, and projecting the point of intersection, G', on to the scale OA.

The *power*, P_1, *supplied to the non-variable branch* is represented by the vertical distance, DF, between the horizontal axis and the diameter, I_0I_s, of the semicircle I_0GI_s.

The *total power*, P, *supplied* to the circuits is represented by the

ordinate, GF, drawn from the horizontal axis to the circumference of the semicircle. Since in this case the whole of the power supplied is expended against losses the ordinates representing power will also represent the losses in the circuits.

The *scale for the power ordinates* is readily obtained from the scale of the current vectors. Thus if these vectors are drawn to a scale of 1 cm. $= p$ amp., the current corresponding to a vector of length OG cm. is $OG.p$ amp. If E is the line voltage and $\cos\varphi$ the power factor, the power supplied is equal to $E.I \cos\varphi$. Now the ordinate GF, corresponding to the vector OG, has a length equal to $OG.\cos\varphi$, cm., and represents the power $E.I\cos\varphi$. Hence the scale of the power ordinates, viz. 1 cm. $= q$ watts, must be such that $q.GF = E.I\cos\varphi$, i.e. $q(OG.\cos\varphi) = E(p.OG)\cos\varphi$, whence $q = p.E$, or 1 cm. $= p.E$ watts.

Series Circuits

Load Diagram for a Series Circuit Containing Both Fixed and Variable Impedance. Two cases of series circuits will be considered, viz. (1) a circuit in which the variable impedance consists of a non-inductive resistance, (2) a circuit in which a constant ratio exists between the resistance and reactance of the variable impedance, i.e. the power factor is constant but is less than unity. These cases have a practical application, as the conditions are representative of a simple transmission line supplying power to a single load of variable magnitude but of constant power factor, which may be either equal to, or less than, unity.

The circuit diagrams are shown in Fig. 64(a) and (b). In these diagrams the fixed impedance, Z_1, represents the impedance of the transmission line, and the variable resistance, R_2, Fig. 64(a), or the variable impedance, Z_2, Fig. 64(b), represents the impedance of the "load," which may consist of a bank of lamps in the former case, and a motor, or other electromagnetic apparatus, in the latter case.

The construction of the no-load and short-circuit diagrams for these cases is fairly simple, as only one inversion is necessary, but the deduction of the load diagram differs in a number of respects from that for the parallel circuit.

Construction of the Diagram for a Load Having a Power Factor of Unity. The "line" impedance, Z_1, Fig. 64(a), is represented in Fig 65 by the image vector OA, the co-ordinates of the point A being R_1, X_1. The "load" impedance is represented by a straight line, $AB\infty$, drawn upwards from A parallel to the vertical axis. If this line is extended below the point A, then the lower portion

represents the load impedance for the case when the resistance is negative, i.e. when the load consists of a generator operating at unity power factor. The joint impedance of the circuit, corresponding to a particular value, R_2, of the load, is therefore given by a vector drawn from O to the appropriate point on the line $AB\infty$.

To obtain the current in the circuit the impedance diagram must be inverted with respect to the pole O. Now the inverse point corresponding to A is at A_1, in OA, or OA produced, according to the scales adopted for impedance and admittance. The inversion of the line AB with respect to O is a portion of a semicircle which passes through the origin (O) and has its centre in the horizontal

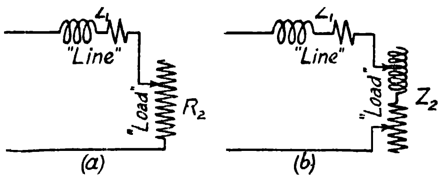


FIG. 64. CIRCUIT DIAGRAMS FOR SERIES CIRCUITS

axis. The diameter of this semicircle corresponds to the inversion of OD, where D is the point of intersection of BA produced, with the horizontal axis. Therefore the scale for admittance must be selected so as to give a convenient diameter for this semicircle.

The inversion of the line DAB is therefore the semicircle OA_1H, and the inversion of the line $DC\infty$ is the semicircle OFH, the centre being at Q in both cases. The arc OG_1A_1 therefore represents the locus of the joint admittance vector when the resistance R_2 is varied from zero to ∞, and the arc $OFHA_1$ represents the locus of this vector when the resistance is negative and is varied between zero and ∞. By a suitable change of scale these arcs also represent the locus of the line current when the resistance is varied, O being the no load point (corresponding to $R_2 = \infty$ and zero current) and A_1, the short-circuit point (corresponding to $R_2 = 0$).

The performance of the circuit is obtained from the diagram as follows—

The *line current*, corresponding to a particular value, R_2, of the load resistance (which is represented by AG on the resistance scale) is given, on the current scale (which is E times the admittance scale, E being the line voltage), by the length of the chord, OG_1, drawn from O through G, G_1 being the inverse point to G.

The *phase difference* between line current and impressed E.M.F. is given by the inclination of OG_1 to the vertical axis (which contains

the vector of the impressed E.M.F.). The power factor may therefore be obtained directly by constructing a power-factor scale and quadrant, as shown in Fig. 63, and projecting the point of intersection of the current vector and the quadrant on to this scale.

The *power taken from the supply system* is given, on the power

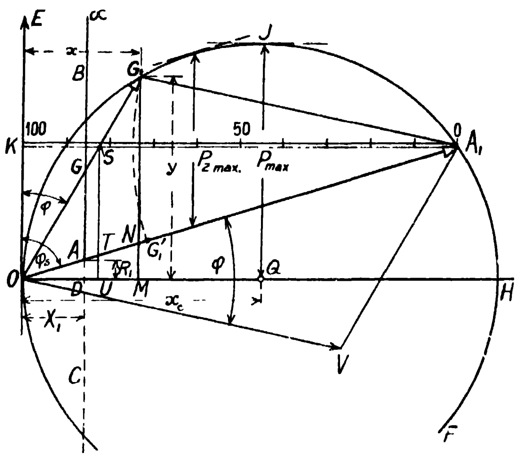


FIG. 65. LOAD DIAGRAM FOR A SERIES CIRCUIT [FIG. 64 (*a*)] HAVING A VARIABLE LOAD OF UNITY POWER FACTOR

scale (which is E times the current scale), by the ordinate, G_1M of the appropriate current vector.

The *power supplied to, and utilized by, the load* is represented by G_1N, and that expended in losses in the line is represented by NM.

Proof. The diameter, OH, of the current semicircle OG_1A_1H corresponds to a current $I_M = E/X_1$, where X_1 is the reactance of the "line." Hence, if the scale for current is 1 cm. $= p$ amp., the length of OH (in cm.) is equal to $I_m/p = E/X_1p$. Similarly the length of any radius-vector, such as OG_1, drawn from O to the semicircle, which corresponds to a current of I, is I/p cm.

If x, y, are the co-ordinates, with respect to rectangular axes passing through the origin O, of any point, G_1, on the semicircle, the equation to the circle is

$$\left(x - \frac{E}{2X_1p}\right)^2 + y^2 = \left(\frac{E}{2X_1p}\right)^2 \quad . \quad . \quad . \quad . \quad (45)$$

since the co-ordinates of the centre of the circle are $x_c = E/2X_1p$, $y_c = 0$.

Whence $\qquad x^2 + y^2 = x(E/X_1p)$,

i.e. $\qquad OG_1{}^2 = x(E/X_1p)$.

Now if φ is the inclination of OG_1 to the vertical axis, the ordinate at G_1 is given by $G_1M = y = OG_1.\cos\varphi = (I/p)\cos\varphi$. Multiplying each side by

the line voltage, E, we have $E.G_1M = (EI \cos \varphi)/p$, whence $pE.G_1M = EI \cos \varphi$. Therefore the ordinate G_1M represents the power taken from the supply system to the scale 1 cm. $= pE$ watts.

The power, P_2, supplied to the load is given by

$$\begin{aligned} P_2 &= EI \cos \varphi - I^2R_1 \\ &= pE.G_1M - p^2R_1(xE/X_1p) \\ &= pE.G_1M - pEx \cot \varphi_s \\ &= pE.G_1M - pE.MN \\ &= pE.G_1N \end{aligned}$$

i.e. for any current represented by OG_1 the power supplied to the load is represented by the intercepted length of the ordinate at this point between the semicircle and the straight line joining the origin and the short-circuit point, the scale being 1 cm. $= pE$ watts.

The line OA_1 is therefore the datum line from which the output power, P_2, is measured. Similarly the line OH, i.e. the abscissa axis, is the datum line from which the input power, P_1, and the losses are measured.

The losses are also represented, for any particular value (such as OG_1) of the line current, by the perpendicular distance of that point from the vertical axis (i.e. by the abscissa of the point G_1), but in this case the scale is 1 cm. $= (pE.\cot \varphi_s)$ watts.

The *maximum power taken from the supply system* is represented by the maximum ordinate, QJ, in the semicircle, and the maximum power supplied to the load is obtained by drawing a tangent parallel to the short-circuit line (OA_1) and determining the length of the intercept, between the circumference and the short-circuit line, of the ordinate drawn from the point of contact of the tangent.

The *efficiency of the circuit*—i.e. the ratio: (power supplied to the load/power taken from the supply system)—is given by the ratio G_1N/G_1M. Instead of calculating this ratio for each value of the current the efficiency may be obtained directly as follows—

Draw through the point A_1, a perpendicular A_1K, to the vertical axis and divide this into 100 parts, placing the zero at A_1. Then the point, S, at which the current vector, OG_1, intersects this scale gives the percentage efficiency directly.

Proof. From S, the point of intersection of OG_1 (or OG_1 produced) with KA_1, draw the ordinate SU, and let T be the point of intersection of this ordinate with the short-circuit line, OA_1. Then, since $SU = KO$, we have the following pairs of similar triangles—

triangle G_1NO is similar to triangle STO,
triangle G_1MO is similar to triangle SUO,
triangle A_1ST is similar to triangle A_1KO.

Hence,

$$\frac{\text{Power supplied to load}}{\text{Power taken from supply}} = \frac{P_2}{P} = \frac{G_1N}{G_1M} = \frac{ST}{SU} = \frac{ST}{KO} = \frac{A_1S}{A_1K}$$

i.e. the distance A_1S expressed as a fraction of A_1K is equal to the efficiency of the transmission. Therefore, if A_1K is divided into 100 equal parts, with zero at A_1, the point of intersection of the line OG_1, or OG_1 produced, with this scale gives the efficiency directly as a percentage.

Voltage Regulation. The voltage drop in the impedance of the line, and the voltage available at the load, may, on the assumption of constant supply voltage, be obtained directly from the diagram as follows, the proof being given below—

Join the points A_1 and G_1. Then the triangle OG_1A_1 is the triangle of voltages for the system; OA_1 representing the supply voltage to a scale Z_1 times the current scale (since at short circuit $I_s = E/Z_1$), OG_1 representing the voltage drop in the "line" to this scale, and G_1A_1 representing the voltage at the load to the same scale. Hence, since the percentage voltage regulation is given by: 100 × (arithmetic difference between "supply" and "load" voltages/supply voltage), this quantity is represented in the diagram by the ratio 100 (OG_1'/OA_1), where A_1G_1', along A_1O, is made equal to A_1G_1.

The phase difference between the "load" and "supply" voltages is given by the angle OA_1G_1.

Proof. Let E denote the supply, or impressed, E.M.F., I_s the short-circuit current, φ_s the phase difference between E and I_s, and Z_1 the impedance of the line. Also let E_1 denote the voltage drop in the line and E_2 the voltage at the load, respectively, for a line current I, which corresponds to a load resistance R_2, the phase difference between this current and the impressed E.M.F. being φ. Then, taking the vector of the impressed E.M.F., OE, Fig. 65, as the vector of reference and employing the exponential form of symbolic notation, we have

$$\dot{Z}_1 = Z_1\varepsilon^{j\varphi_s}$$

$$\dot{I}_s = \frac{E}{\dot{Z}_1} = \frac{E}{Z_1}\varepsilon^{-j\varphi_s}$$

$$\dot{I} = I\varepsilon^{-j\varphi}$$

$$\dot{E}_1 = \dot{I}\dot{Z}_1 = I\varepsilon^{-j\varphi} Z_1\varepsilon^{j\varphi_s} = IZ_1\varepsilon^{j(\varphi_s-\varphi)}$$

$$\dot{E}_2 = \dot{I}R_2 = IR_2\varepsilon^{-j\varphi}$$

Let the vector of reference be now rotated through an angle φ_s in the clockwise direction, so that it coincides with the vector representing the short-circuit current, and let the scale for the voltage vectors be Z_1 times that of the current vectors, i.e. 1 cm. $= pZ_1$ volts. Then this change of position and scale is equivalent to multiplying the original voltage vectors by the quantity $1/\dot{Z}_1 = (1/Z_1)\varepsilon^{-j\varphi_s}$.

Hence the impressed E.M.F. is now represented by the vector quantity

$$\dot{E}' = \frac{E}{\dot{Z}_1} = \frac{E}{Z_1}\varepsilon^{-j\varphi_s}$$

i.e. by the vector OA_1 in Fig. 65.

The voltage drop in the line impedance is represented by the vector quantity

$$\dot{E}'_1 = \frac{\dot{E}_1}{\dot{Z}_1} = \frac{IZ_1\varepsilon^{j(\varphi_s-\varphi)}}{Z_1\varepsilon^{j\varphi_s}} = I\varepsilon^{-j\varphi}$$

i.e. by the vector OG_1,

and the voltage across the load is represented by the vector quantity

$$\dot{E}'_2 = \frac{\dot{E}_2}{\dot{Z}_1} = \frac{IR_2\varepsilon^{-j\varphi}}{Z_1\varepsilon^{j\varphi_s}} = I\left(\frac{R_2}{Z_1}\right)\varepsilon^{-j(\varphi + \varphi_s}$$

i.e. by the vector OV drawn at an angle φ below OA_1.

Since in triangle OG_1A_1 the angle $OA_1G_1 = \varphi$, the side G_1A_1 is parallel to OV. Also since G_1 is the inverse point to G, and A_1 is the inverse point to A, we have $OG.OG_1 = OA.OA_1$, so that triangle OAG is similar to triangle OA_1G_1. Therefore

$$G_1A_1 : OG_1 :: GA : OA$$

i.e. $G_1A_1 = OG_1(GA/AO)$.

Hence, since $GA/AO = R_2/Z_1$, G_1A_1 represents the quantity $I(R_2/Z_1)$.

Therefore the triangle OG_1A_1 is the voltage triangle for the system, OA_1 representing the supply voltage, OG_1 the voltage drop in the line impedance, and G_1A_1 the voltage at the load, the scale of these quantities being 1 cm. $= pZ_1$ volts.

Construction of the Diagram for a Load Having a Power Factor Less Than Unity. Let the power factor of the load be cos φ_2. Then in constructing the no-load and short-circuit diagram we set off OA, Fig. 66, to represent the image vector of the impedance of the "line," as before, and from A draw the line $AB\infty$, at an angle φ_2 to the vertical axis, to represent the image vector of the impedance of the load. Observe that if the power factor is lagging, φ_2 is set off in the clockwise direction, but if the power factor is leading, φ_2 is set off in the counter clockwise direction.

We now obtain the inversion of the lines OA and $AB\infty$ with respect to the pole O. The inversion of the line $AB\infty$ gives a portion of a semicircle which passes through the origin. The centre of this semicircle lies in a line, drawn through the origin, perpendicular to $AB\infty$, and the diameter is obtained in the same manner as in the above case.

The semicircle is shown in Fig. 66 by OG_1A_1H, the centre being at Q. The point A_1 on the circumference is the inverse point to A, and the arc OG_1A_1 is inverse to the line $AB\infty$. Thus the arc OG_1A_1 is the locus of the joint admittance vector when the load is varied from zero to short circuit. By a suitable change of scale this arc also will represent the locus of the line-current vectors when the load is varied, O being the no-load point and A the short-circuit point.

On comparing Fig. 65 with Fig. 66 it will be observed that in the former diagram the centre of the semicircle lies in the horizontal axis, but that in the latter diagram the line containing the centre is displaced from the horizontal axis by the angle φ_2, the displacement being in the counter-clockwise (or positive) direction for a leading power factor, and in the clockwise direction for a lagging power factor. The short-circuit line, OA_1, occupies the same position in each diagram, if the constants of the "line" are the same for the two cases.

The datum lines for input power and output power are the horizontal axis, OX. and the short-circuit line OA_1, respectively.

The *power input* corresponding to a current represented by the vector OG_1 is given by the ordinate G_1M at this point; the power scale being E times the current scale, as in the case above when the power factor of the load was unity. The power input is also given by the distance, G_1M', of the point G_1 from the horizontal axis measured in a direction perpendicular to the diameter of the

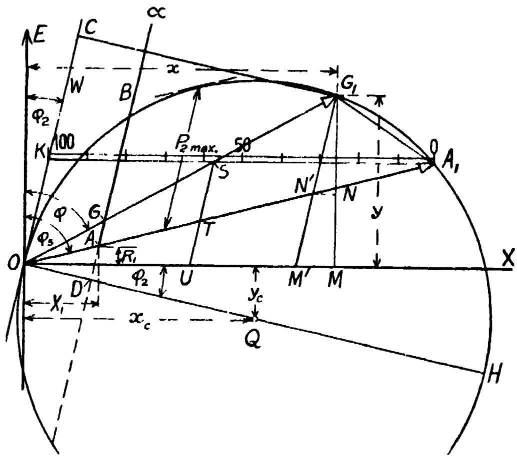


FIG. 66. LOAD DIAGRAM FOR A SERIES CIRCUIT [FIG. 64 (b)] HAVING A VARIABLE LOAD OF CONSTANT POWER FACTOR

semicircle, i.e. in a direction parallel to the tangent at O, but the scale is now $(E \cos \varphi_2)$ times the current scale.

The *phase difference* between the current and the supply voltage is given by the angle which the current vector, OG_1, makes with the vertical axis.

The *power output* is given by the portion, G_1N', of the line G_1M' which is intercepted between the circumference and the short-circuit line OA_1; and the losses in the "line" are given by the portion, $N'M'$, intercepted between the short-circuit line and the horizontal axis, the scale in both cases being $(E \cos \varphi_2)$ times the current scale.

The *I^2R loss in the "line" impedance*, Z_1, is given by the perpendicular distance of the point G_1 from the tangent OW.

The *voltage regulation* and the voltage at the load are obtained as before from the triangle OG_1A_1; OA_1 representing the supply voltage, OG_1 the voltage drop in the line, and A_1G_1 the voltage at the load, the scale in each case being Z_1 times the current scale.

Proof. The line current is a maximum when the reactance of the whole circuit has a value represented by OD, Fig. 66, which is the perpendicular distance of the line $AB\infty$, produced backwards, from the origin O. This value of reactance is only obtained practically by operating the load as a generator, and is equal to $Z_1 \sin(\varphi_s - \varphi_2)$, where Z_1 is the impedance of the "line." Whence the maximum current is $I_M = E/Z_1 \sin(\varphi_s - \varphi_2)$, and the diameter of the current circle, for a scale of 1 cm. $= p$ amp. is $I_M/p = E/(pZ_1 \sin(\varphi_s - \varphi_2))$. The co-ordinates of the centre of the circle are therefore

$$x_c = (\tfrac{1}{2}I_M/p)\cos\varphi_2 = E\cos\varphi_2/2pZ_1\sin(\varphi_s - \varphi_2)\,;$$

$$y_c = -(\tfrac{1}{2}I_M/p)\sin\varphi_2 = -E\sin\varphi_2/2pZ_1\sin(\varphi_s - \varphi_2).$$

Hence the equation to the current circle is

$$\left(x - \frac{E\cos\varphi_2}{2pZ_1\sin(\varphi_s-\varphi_2)}\right)^2 + \left(y + \frac{E\sin\varphi_2}{2pZ_1\sin(\varphi_s-\varphi_2)}\right)^2 = \left(\frac{E}{2pZ_1\sin(\varphi_s-\varphi_2)}\right)^2$$

or
$$x^2 + y^2 = \frac{E}{pZ_1}\left\{x\frac{\cos\varphi_2}{\sin(\varphi_s-\varphi_2)} - y\frac{\sin\varphi_2}{\sin(\varphi_s-\varphi_2)}\right\}$$

$$= \frac{E}{pZ_1\sin(\varphi_s-\varphi_2)}\sqrt{(x^2+y^2)}\,.\sin(\varphi-\varphi_2) \quad . \quad . \quad (46)$$

where φ is the inclination to the vertical axis of the line joining the point x, y, to the origin. The I^2R loss in the "line," due to a current which is represented by OG_1 $(=\sqrt{(x^2+y^2)}$, where x, y, are the co-ordinates of the point G_1), is given by

$$R_1(p.OG_1)^2 = p^2R_1(x^2+y^2) = \left\{p^2R_1\frac{E}{pZ_1\sin(\varphi_s-\varphi_2)}\right\}OG_1.\sin(\varphi-\varphi_2)$$

$$= \left\{pE\frac{\cos\varphi_s}{\sin(\varphi_s-\varphi_2)}\right\}OG_1\,.\sin(\varphi-\varphi_2),$$

$$= \left(pE\frac{\cos\varphi_s}{\sin(\varphi_s-\varphi_2)}\right)G_1C \text{ watts,}$$

where G_1C is the perpendicular distance of G_1 from the tangent OW.

Now $G_1C/\sin(\varphi_s - \varphi_2)$ is equal to ON', where N' is the point of intersection of the short-circuit line, OA_1, and a line, G_1M', drawn through G_1 parallel to the tangent OW. Also $ON'\cos\varphi_s$ is the ordinate at N'. Hence the I^2R loss is also given by

$$pE.NM,$$

where N is the projection of the point N' on the ordinate, G_1M, at G_1.

The power taken from the supply system is given by $EI\cos\varphi$, or by $pE.G_1M$, since $G_1M = (p.OG_1)\cos\varphi$.

Hence the power supplied to the load is given by G_1N, the difference between G_1M and NM, the scale being E times the current scale.

Now since the angle $MG_1M' = \varphi_2$: $G_1M = G_1M'\cos\varphi_2$; $G_1N = G_1N'\cos\varphi_2$; $NM = N'M'\cos\varphi_2$. Therefore, if the power scale is changed to ($E\cos\varphi_2$) times the current scale, the intercepts G_1M', G_1N', $N'M'$, on a line drawn through G_1 parallel to the tangent at the origin (i.e. this line is perpendicular to the diameter of the circle) give the power input, the power output, and the line losses ; the scale being 1 cm. $= pE\cos\varphi_2$ watts.

The efficiency [i.e. the ratio : (power taken from supply/power supplied to the load)] is given by A_1S/A_1K, where S is the point of intersection of OG_1 with the horizontal line, A_1K, drawn through A_1. Thus if from the

point S a line SU be drawn parallel to the tangent OW, and if T is the point of intersection of this line and the short-circuit line OA_1, then from the similar pairs of triangles OG_1M', OSU; OG_1N', OST; A_1KO, A_1ST we have

$$\frac{\text{Power supplied to load}}{\text{Power taken from supply}} = \frac{G_1N'}{G_1M'} = \frac{ST}{SU} = \frac{ST}{KO} = \frac{SA_1}{KA_1}$$

Construction of the Load Diagram from Test Data. The load diagrams hitherto considered have been deduced from a knowledge of the " constants " of the circuit, i.e. the resistance and reactances of " line " and the " load." But in the case of series circuits of the type shown in Fig. 64 it is apparent that the current circle could have been drawn if the magnitude and phase of the short-circuit current and the power factor of the load had been given. For example, the centre of the current circle is obtained quite easily by setting off the short-circuit line in its correct position with reference to the rectangular axes; bisecting this line, and drawing a perpendicular to intersect a line drawn through the origin and inclined at the angle φ_2 with respect to the horizontal axis, where $\cos \varphi_2$ is the power factor of the load and the angle φ_2 is measured in the positive direction when the power factor is leading, and in the negative direction when the power factor is lagging. The diagram is completed by drawing a horizontal line through the short-circuit point and a tangent from the origin to intersect the former. The distance between the short-circuit point and the point of intersection of the horizontal line and tangent is divided into 100 parts, with the zero at the short-circuit point, thereby giving the scale for efficiency. The scale for power follows directly from the scale for current and a knowledge of the line voltage and the power factor of the load.

In the case of the parallel circuit, the current circle can be drawn when the magnitudes and phase differences of the no-load and short-circuit currents are known.

Example. The following worked example is given to show the simplicity with which the performance of a simple transmission line can be obtained by means of the circle diagram.

A variable load of constant power factor is supplied from a generating station through a transmission line having an impedance of $1{\cdot}6 + j\,1{\cdot}95$ ohms. The voltage at the generating station is 2200 V.

Two cases will be considered: (1) power factor of load 0·9 lagging; (2) power factor of load 0·95 leading.

The current circles will be drawn by determining directly the vector for the short-circuit current. Thus, since $Z_1 = \sqrt{(1{\cdot}6^2 + 1{\cdot}95^2)} = 2{\cdot}52\ \Omega$,

$$I_s = E/Z_1 = 2200/2{\cdot}52 = 872\text{ A.} \quad \varphi_s = \cos^{-1}(1{\cdot}6/2{\cdot}52) = 50{\cdot}6^\circ.$$

Selecting a current scale of 1 cm. = 50 A., we draw the line OA_1 (Fig. 67), 17·44 cm. (= 872/50) in length and inclined at an angle of 50·6° to the vertical axis, to represent the short-circuit current.

Draw through the origin lines OH_1, OH_2, inclined at angles of $-25{\cdot}8°$ ($=\cos^{-1}0{\cdot}9$) and $+18{\cdot}2°$ ($=\cos^{-1}0{\cdot}95$), respectively, to the horizontal axis. Bisect OA_1 at F and draw the perpendicular FQ_2Q_1 to intersect the lines OH_2, OH_1, at Q_2 and Q_1 respectively. Then Q_1 is the centre of the current circle when the power factor of the load is 0·9 lagging, and Q_2 is the centre of the current circle when the power factor of the load is 0·95 leading

The diameters of the circles are

41·6 cm. ($= 17{\cdot}44/\sin(50{\cdot}6° - 25{\cdot}8°)$) for the load of power factor 0·9 (lagging).
18·7 cm. ($= 17{\cdot}44/\sin(50{\cdot}6° - 18{\cdot}2°)$) for the load of power factor 0·95 (leading).

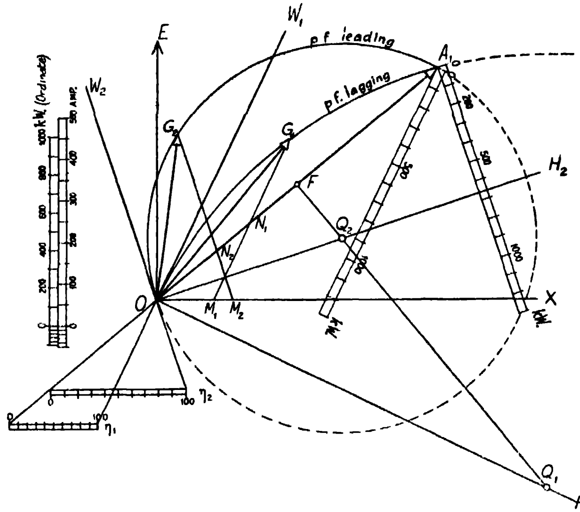


FIG. 67. LOAD DIAGRAM FOR SIMPLE TRANSMISSION LINE

The circle for the load of 0·95 power factor (leading) is shown complete in Fig. 67, but only a portion of the circle for the load of 0·9 power factor (lagging) is shown on account of space restrictions. Moreover, only the portions of both circles which are above the horizontal axis and between the no-load and short-circuit points are required for the present purpose.

The diagrams are completed by drawing tangents OW'_1, OW_2, at the origin and constructing the efficiency scales. To prevent confusion these scales are constructed below the horizontal axis. Thus the short-circuit line and the tangents are produced beyond the origin, and a horizontal line of *any* convenient length is drawn between the short-circuit line, produced, and each tangent. Each of these lines is divided into 100 equal parts, as shown in Fig. 67. The efficiency corresponding to a particular line current is obtained by producing the current vector backwards until it intersects the appropriate scale.

The scale for power may be obtained when the method of measuring this quantity is decided. Thus, if power is measured vertically, the scale is E times the current scale—i.e. 1 cm. = 2200 × 50 = 110,000 watts, or 1 cm. = 110 kW.—for both cases. But if power is measured in a direction parallel to the appropriate tangent (i.e. in a direction perpendicular to the diameter of the appropriate current circle, the scales for the two cases will not be the same, being 1 cm. = 2200 × 0·9 × 50 = 99,000 watts, or

1 cm. = 99 kW., for the 0·9 power-factor load; and 1 cm. = 2200 × 0·95 × 50 = 104,500 watts, or 1 cm. = 104·5 kW.. for the 0·95 power-factor load.

The power and losses are obtained by measuring these quantities in directions parallel to the appropriate tangents. For example, when the power factor is lagging the input, output, and losses corresponding to the current represented by the vector OG_1, Fig. 67, are given by G_1M_1, G_1N_1, N_1M_1, respectively, the scale being 1 cm. = 99 kW. Similarly, when the power factor is leading, the input, output, and losses corresponding to the current represented by the vector OG_2, are given by G_2M_2, G_2N_2, N_2M_2, respectively, the scale being 1 cm. = 104·5 kW.

The performance of the transmission line, as deduced from the load diagram of Fig. 67, is given in Table V, and the results are plotted in Fig. 68.

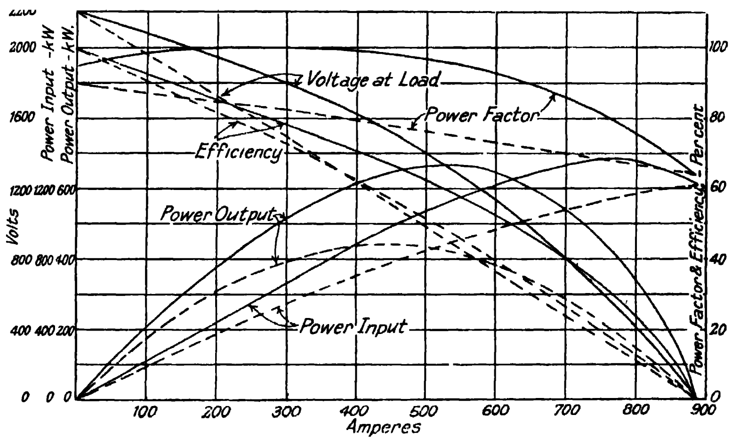


FIG. 68. PERFORMANCE CURVES OF TRANSMISSION LINE (DETERMINED FROM THE LOAD DIAGRAM OF FIG. 67)

The results show that a load of lagging power factor adversely affects the efficiency and voltage regulation of the line, and the power factor at which the generator operates.

SERIES-PARALLEL CIRCUITS

Load Diagram for a Series-parallel Circuit with Variable Impedance, of Constant Power Factor, in One Branch. The construction of the no-load and short-circuit diagram for this circuit involves the determination of (1) the diagrams for the joint admittance and joint impedance of the parallel branches, (2) the diagram for the joint impedance of the complete circuit, (3) the inversion of this diagram and the change of scale so that the vectors will represent currents.

The variable impedance will be denoted by Z_2, the fixed series impedance by Z_1, and the impedance of the non-variable branch by Z.

The no-load and short-circuit diagram is constructed as follows—Select a suitable scale for admittance and draw AB, Fig. 69, to

TABLE V

Measured and calculated quantities (from Fig. 67) for performance of transmission line.

Current (amp.)			100	200	300	400	445*	500	528†	600	7
Length OG_1 (cm.) / Length OG_2 (cm.)			2	4	6	8	8·9	10	0·56	12	14
Input	p.f. = 0·9 (lagging)	Length G_1M_1 (cm.)	1·94	3·8	5 5	7·1	7·76	8·52		9·8	10·87
		Power (kW)	192	376	544	703	768	844		970	1076
	p.f. = 0·95 (leading)	Length G_2M_2 (cm.)	2·08	4·2	6 32	8·4		10·23	10·68	11·75	12·76
		Power (kW)	217·5	439	660	877		1068	1116	1228	1332
Output	p.f. = 0·9 (lagging)	Length G_1N_1 (cm.)	1·73	3 1	3·97	4·44	4·46	4·42		3·9	2·87
		Power (kW)	172	307	393	440	441	438		386	284
	p.f. = 0·95 (leading)	Length G_2N_2 (cm.)	1·93	3·55	4·92	5 9		6·36	6·4	6·2	5·16
		Power (kW)	201·8	371	514	616		664	669	648	540
Losses	p.f. = 0·9 (lagging)	Length N_1M_1 (cm.)	0·21	0 7	1·53	2 66	3·3	4·1		5·9	8·0
		Power (kW)	19·8	69	151	263	327	406		584	792
	p.f. = 0·95 (leading)	Length N_2M_2 (cm.)	0·15	0·65	1·4	2·5		3·87	4·28	5·55	7·6
		Power (kW)	15 7	68	146	261		404	447	580	792
Efficiency (per cent)		p.f. = 0 9 (lagging)	89 6	81·7	72·3	62·6	57·4	51·9		39·8	26·6
		p.f. = 0 95 (leading)	92·7	84 6	77·9	70 2		62·2	60	52·8	40·5
Voltage at load	p.f. = 0 9 (lagging)	Length G_1A_1 (cm.)	15 6	13·7	11·8	9 8	8·9	7·8		5·76	3·7
		Volts	1965	1726	1487	1235	1120	982		726	466
	p.f. = 0 95 (leading)	Length G_2A_1 (cm.)	16·6	15·55	14·3	12 82		11·16	10·56	8·97	6·42
		Volts	2090	1960	1802	1615		1406	1330	1130	809
Voltage drop in line	p.f. = 0 9 (lagging)	Length OG_1 (cm.)	2	4	6	8	8·9	10		12	14
		Volts	252	504	756	1080	1120	1260		1512	1764
	p.f. = 0·95 (leading)	Length OG_2 (cm.)	2	4	6	8		10	10·56	12	14
		Volts	252	504	756	1080		1260	1330	1512	1764
Power factor at generator (per cent)	load p.f. 0 9 (lagging)		87·3	85 5	82·5	79·9	78·6	76·8		73·5	70
			←				lag				→
	load p.f. 0·95 (leading		98·8 (lead)	99 8 (lead)	1·0	99 6		97 2	96·1	93	86·5
					←			lag			→

SCALES.—Current : 1 cm. = 50 A.
Power : 1 cm. = 99 kW., for p.f. = 0 9 (lagging) ; 1 cm. = 104·5 kW., for = 0·95 for p.f. (leading).
Voltage : 1 cm. = 2·52 × 50 = 126 V.

* Maximum load when power factor of load is 0·9 (lagging). † Maximum load when power factor of load is 0·95 (leading).

represent the admittance, Y_3, of the branch of fixed impedance. It is preferable for reasons which will be explained later, to draw AB with respect to temporary axes, AX', AY', instead of the permanent axes of reference. From B draw the line $BC\infty$ to represent the admittance of the variable branch, the line being inclined at the angle φ_2 with respect to the temporary vertical axis, where $\cos\varphi_2$ is the power factor of the variable branch. Vectors drawn from A to the line $BC\infty$ therefore represent the joint admittance of the parallel branches. Hence the line $BC\infty$ is the locus of

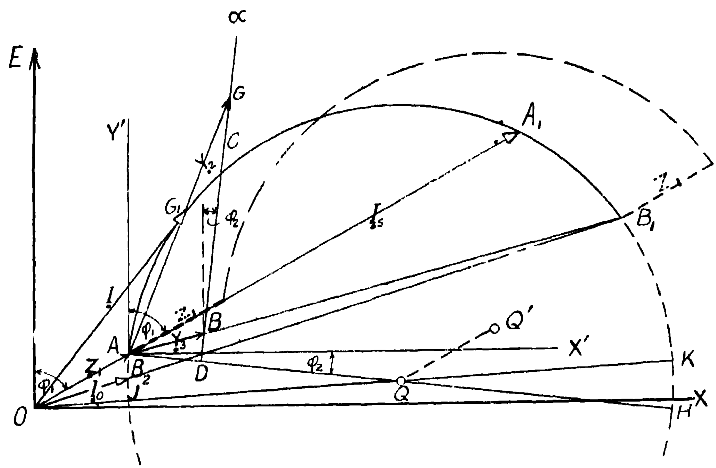


FIG. 69. NO-LOAD AND SHORT-CIRCUIT DIAGRAM FOR SIMPLE SERIES-PARALLEL CIRCUIT

the joint admittance vector when the impedance of the variable branch is positive and is varied between zero and infinity.

The joint impedance of the parallel branches is obtained by the inversion of the locus $BC\infty$ with respect to the pole A. This inversion gives the arc AG_1B_1, the centre of which lies at Q in the line AH inclined at φ_2 with respect to the temporary horizontal axis. Vectors drawn from A to the arc AG_1B_1 therefore represent the joint impedance of the parallel branches when the impedance of the variable branch is positive and is varied from zero to infinity. Similarly, vectors drawn from A to the lower arc B_1HA represent the joint impedance of the branch circuits when the impedance of the variable branch is negative and is varied from zero to infinity (i.e. when the circuit Z_2 is acting as a generator at a constant power factor, $\cos\varphi_2$).

To obtain the joint impedance of the whole circuit the series impedance, Z_1, must be added vectorially to the joint impedance vectors for the branch circuits. If the addition is carried out in this manner, another circle, which is shown dotted in Fig. 69, will be obtained. The diameter of this new circle is the same as that of the original circle, but its centre is at Q'; the distance QQ' being equal, on the impedance scale, to Z_1, and the inclination of QQ' to the vertical axis being equal to φ_1, where $\tan \varphi_1 = X_1/R_1$. This circle is therefore the locus of the joint impedance vectors for the complete circuit when the axes AX', AY', are the axes of reference.

Instead of constructing the second circle to obtain the joint impedance of the complete circuit, we may make the original circle represent this quantity by shifting the temporary origin (A) to O, where the distance OA represents the series impedance Z_1, and the inclination of OA to the vertical is equal to φ_1. Thus the single circle AG_1B_1H may represent either (1) the locus of the vector of the joint impedance of the parallel branches of the circuit—in which case the origin is at A and the axes of reference are AX', AY'—or (2) the locus of the vector of the joint impedance of the complete circuit—in which case the origin is at O and the axes of reference are OX, OY.

The joint admittance of the complete circuit is obtained by the inversion of the circle AG_1B_1H with respect to the pole O. This inversion, as explained on p. 105, gives a circle the centre of which lies in the line joining O and Q. But by a suitable choice of the new admittance scale the original circle may be made to represent its inverse circle. For example, if J, K, are the points of intersection of the line OQ, produced, and the circumference of the circle AG_1B_1H, then if this circle is to represent the locus of both impedance and admittance vectors, we must have $m.OJ = 1/n.OK$, or $n = 1/(m \times OJ.OK)$, where m, n, are the scales for impedance and admittance respectively.

Now in the original diagram for the admittance of the parallel branch circuits the points B and ∞ correspond to zero and infinite admittance, respectively, of the variable branch. The inversion of these points with respect to A gives the points B_1 and A respectively, on the impedance circle, and the inversion of points B_1 and A with respect to O gives the points B_2 and A_1 on the admittance circle. Hence B_2 is the "no-load" point, and A_1 is the "short-circuit" point in the final admittance diagram. By changing the scale of this diagram to E times the admittance scale, where E is the supply voltage, we obtain the current diagram for the circuit, B_2 being the "no-load" point and A_1 the "short-circuit" point. Hence the

no-load current is given by the vector OB_2 and the short-circuit current is given by the vector OA_1.

The *load diagram*, Fig. 70, is obtained from the current diagram by determining the datum lines for (1) the power taken from the supply system, (2) the power expended in the series impedance, (3) the power expended in the branch of fixed impedance, and (4) the power supplied to the variable branch. The additional construction involved after the no-load and short-circuit points have been determined and the circle has been drawn, is as follows—

From the origin draw a tangent OF to the circle, bisect it at D and draw a perpendicular $DUHV$ to the line OQ joining the origin and the centre of the circle. This perpendicular is called the "semi-polar" of the circle with respect to the origin.

From the short-circuit point, A_1, draw the tangent A_1V.

Join the no-load and short-circuit points, and produce the line (A_1B_2) to cut the horizontal axis at T. Join T and V (the intersection of the semi-polar and the tangent at the short-circuit point).

Join the short-circuit point and the point, U, at which the semi-polar intersects the horizontal axis.

The *datum line for obtaining the power taken from the supply system* is the horizontal, or abscissæ, axis, and is called the "input" datum line. The power taken from the supply system, when the line current is represented by the vector OG_1, is proportional to the ordinate, G_1M, drawn from the point G_1.

The *datum line for obtaining the I^2R loss in the series impedance*, Z_1, is the "semi-polar" of the current circle with respect to the origin O. This line, UHV, is called the "primary loss" datum line. The I^2R loss in the series impedance, corresponding to a line current represented by the vector OG_1, is proportional to the perpendicular distance, G_1H, of the point G_1 from the semi-polar.

The loss is also represented by LS, which is the intercept on the line G_1S, drawn through G_1 parallel to the semi-polar, made by the horizontal axis and the line UA_1 (which joins the short-circuit point and the point at which the semi-polar intersects the horizontal axis).

The *datum line for obtaining the power expended in the branch of fixed impedance* is the tangent drawn from the short-circuit point. The power expended in this branch, when the line current is represented by the vector OG_1, is proportional to the perpendicular distance, G_1K. of the point G_1 from this tangent.

The *datum line for obtaining the power supplied to the branch of variable impedance* is the line joining the no-load and short-circuit points, and is called the "output" datum line. The power supplied

to this branch, when the line current is represented by the vector OG_1, is proportional to G_1N, which is the intercept, made by the circumference of the circle and the "output" line, on the line G_1W, drawn from the point G_1 parallel to the line VT.

Hence NW is proportional to (I^2R loss in the series impedance + power expended in the branch of fixed impedance).

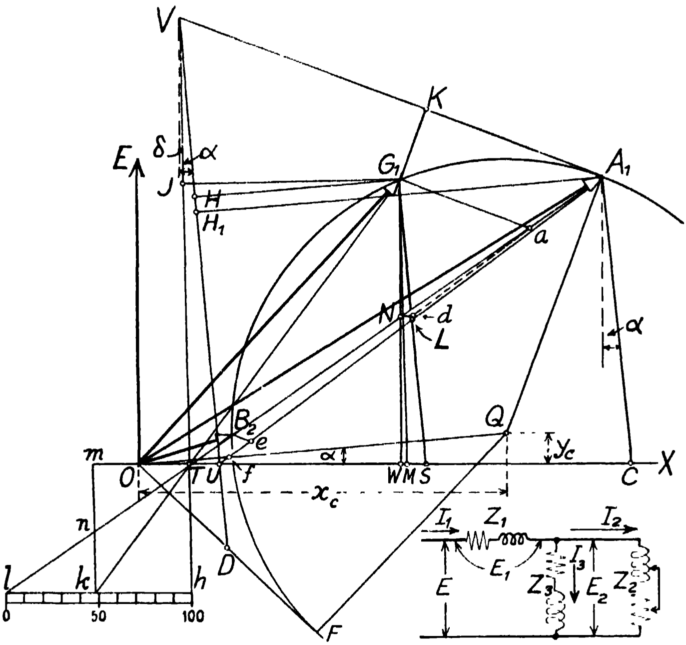


FIG. 70. LOAD DIAGRAM FOR SIMPLE SERIES-PARALLEL CIRCUIT

The scales for the quantities obtained by measurement from the various datum lines are deduced on pp. 137 and 138.

The scale for *efficiency* is constructed on a horizontal line drawn in any convenient position between boundary lines formed by the "output" datum line, A_1B_2T, or its extension, and a line, VT, passing through the points of intersection of (i) the "output" datum line and the horizontal axis, (ii) the "primary loss" datum line (i.e. the semi-polar) and the tangent drawn from the short-circuit point. The efficiency, i.e. the ratio

$$\frac{\text{power supplied to the branch of variable impedance}}{\text{power taken from the supply system}}$$

is given directly on this scale by the point of intersection made by

a line drawn through T and the appropriate point on the current circle.

The triangle OG_1A_1 is the voltage triangle for the circuit; the side OA_1 representing the supply voltage, OG_1 the voltage drop in the series impedance, and G_1A_1 the voltage at the terminals of the parallel branch circuits.

The sides G_1A_1, G_1B_2, of the triangle $A_1G_1B_2$ represent, to dissimilar scales, the magnitudes of the currents in the fixed and variable branches respectively

Proof. **Line Current and Branch-circuit Currents.** The line current corresponding to any particular value, Z_2, of the variable impedance is given by

$$I_1 = \frac{E}{Z_1 + [1/(Y_2 + Y_3)]}$$

where Y_2 is the value of admittance corresponding to the impedance Z_2, and Y_3 is the admittance of the non-variable branch.

The corresponding currents in the branch circuits are: $I_2 = E_2Y_2$ in the variable branch, and $I_3 = E_2Y_3$ in the non-variable branch, where E_2 is the voltage at the terminals of these branches and is given by $E_2 = E - I_1Z_1$.

$$\text{Now } I_1 = I_2 + I_3 = I_2 + Y_3(E - I_1Z_1),$$

and when the current in the variable branch is zero, the line current is

$$I_O = Y_3(E - I_OZ_1).$$

Whence, by subtraction,

$$\begin{aligned} I_1 - I_O &= I_2 + Y_3(E - I_1Z_1) - Y_3(E - I_OZ_1) \\ &= I_2 - Y_3Z_1(I_1 - I_O) \end{aligned}$$

or

$$I_2 = (I_1 - I_O)\,(1 + Z_1/Z_3).$$

Thus the current in the variable branch is equal to the vectorial difference of the line and no-load currents multiplied by the complex number $(1 + Z_1/Z_3)$, which is a constant quantity for a given circuit.

Now B_2G_1, Fig. 70, is the difference of the line current and no-load current vectors. Hence, B_2G_1 represents the magnitude of the current in the variable branch to a scale $\sqrt{[1 + 2\,(Z_1/Z_3)\cos(\varphi_1 - \varphi_3) + (Z_1/Z_3)^2]}$ times the scale of the line current.

The current I_3 in the branch of fixed impedance is proportional to the voltage, E_2, at the terminals of the parallel branches. Now the triangle OG_1A_1 is the voltage triangle for the circuit (see p. 126), and the voltage at the terminals of the parallel branches is represented by G_1A_1, the scale being Z_1 times the scale for the line current. Hence the magnitude of the current in the branch of fixed impedance is given by $I_3 = E_2/Z_3 = (pZ_1/Z_3)G_1A_1$: i.e. the magnitude of this current is represented by G_1A_1, the scale being 1 cm. $= (pZ_1/Z_3)$ amp. But the vector G_1A_1 only gives the phase of this current in the special case when the reactance of this branch is zero.

Power Taken from the Supply System. At no load the power taken from the supply system is given by: $P_o = EI_o \cos\varphi_o$, where φ_o is the phase difference between the no-load current and the supply voltage. The power taken from the supply system at short circuit is given by $P_s = EI_s \cos\varphi_s$.

Now if the diagram of Fig. 70 is drawn to a scale of 1 cm. $= p$ amp., the power taken at no load is represented by $E.p.OB_2.\cos\varphi_o$, or by pEy_o, where y_o is the ordinate at B_2. Similarly the power taken at short circuit is represented by $E.p.OA_1 \cos\varphi_s$, or by pEy_s, where y_s is the ordinate at A_1, and generally, if any particular line current, I, is represented by the vector OG_1,

the power taken from the supply system is represented by pEy, where y is the ordinate at G_1. Therefore the power taken from the supply system is represented by the ordinate of the extremity of the current vector, the scale being 1 cm. = pE watts.

Power Expended in Series Impedance. The equation representing the I^2R loss in the series impedance, Z_1, is deduced as follows—

Let the line current I be represented by the vector OG_1, the co-ordinates of the point G_1 being x, y.

Then, $\quad I^2R_1 = (p.OG_1)^2 R_1 = p^2(x^2 + y^2)R_1.$

If r is the radius of the current circle and x_c, y_c, are the co-ordinates of its centre, the equation to the circle is

$$(x - x_c)^2 + (y - y_c)^2 = r^2,$$

from which we obtain

$$x^2 + y^2 = 2xx_c + 2yy_c - (x_c^2 + y_c^2 - r^2) \qquad . \quad . \quad . \quad (47)$$

Hence the I^2R loss in the series impedance is represented by

$$\begin{aligned} I^2R_1 &= p^2(x^2 + y^2)R_1 \\ &= p^2R_1[2xx_c + 2yy_c - (x_c^2 + y_c^2 - r^2)] \\ &= 2p^2R_1[xx_c + yy_c - \tfrac{1}{2}(x_c^2 + y_c^2 - r^2)] \\ &= [2p^2R_1\sqrt{(x_c^2 + y_c^2)}]\frac{xx_c + yy_c - \frac{1}{2}(x_c^2 + y_c^2 - r^2)}{\sqrt{(x_c^2 + y_c^2)}} \\ &= [2p^2R_1\sqrt{(x_c^2 + y_c^2)}]G_1H, \end{aligned}$$

since the expression $[xx_c + yy_c - \frac{1}{2}(x_c^2 + y_c^2 - r^2)]/\sqrt{(x_c^2 + y_c^2)}$ represents the perpendicular distance G_1H of the point x, y (i.e. the point G_1), from the straight line which is represented by the equation

$$xx_c + yy_c - \tfrac{1}{2}(x_c^2 + y_c^2 - r^2) = 0,$$

this line being the "semi-polar"* of the origin with respect to the circle. Thus the semi-polar of the origin with respect to the current circle is the datum line from which the I^2R loss in the series portion of the circuit is measured, the loss being given by the perpendicular distance, from the semi-polar of the appropriate point on the current circle, and the scale being 1 cm. = $2p^2R_1\sqrt{(x_c^2 + y_c^2)}$ watts.

The proof for the alternative method of obtaining the I^2R loss in the ***series*** *impedance* (i.e. from the intercept, made by the line A_1U and the horizontal axis, on a line drawn from G_1 parallel to the semi-polar) is as follows—

If from the points G_1 and A_1 the lines G_1S, A_1C, and G_1H, A_1H_1, Fig. 70, are drawn parallel to, and perpendicular to, respectively, the semi-polar VU, then from the similar triangles USL, UCA_1, we have

$$\frac{LS}{A_1C} = \frac{US}{UC} = \frac{G_1H}{A_1H_1}$$

i.e.
$$\frac{LS}{G_1H} = \frac{A_1C}{A_1H_1}.$$

Now G_1H, as already shown, represents the I^2R loss in the series impedance when the current is represented by OG_1, the scale being 1 cm. = $2p^2R_1\sqrt{(x_c^2 + y_c^2)}$ watts. Also A_1H_1 and A_1C both represent the I^2R loss at short circuit, the scale for the former being 1 cm. = $2p^2R_1\sqrt{(x_c^2 + y_c^2)}$, and that for the latter being 1 cm. = $pE \cos a$ watts, where a is the angle which the semi-polar makes with the vertical axis.

* The semi-polar is the line which is parallel to the polar and is mid-way between the polar and the origin. Both the semi-polar and polar, in the present case, are perpendicular to the line joining the origin and the centre of the circle, the equation to this line being $xy_c - yx_c = 0$.

Hence,

$$A_1C(pE \cos a) = A_1H_1[2p^2R_1\sqrt{(x_c^2 + y_c^2)}]$$

i.e.

$$\frac{A_1C}{A_1H_1} = \frac{2p^2R_1\sqrt{(x_c^2 + y_c^2)}}{pE \cos a}$$

Substituting this value for A_1C/A_1H_1 in the above equation, we have

$$\frac{LS}{G_1H} = \frac{2p^2R_1\sqrt{(x_c^2 + y_c^2)}}{pE \cos a}$$

Whence,

$$LS(pE \cos a) = G_1H[2p^2R_1\sqrt{(x_c^2 + y_c^2)}].$$

Therefore, LS represents the I^2R loss in the series impedance, to the scale 1 cm. $= pE \cos a$ watts, when the current is represented by OG_1.

Construction for the Polar and Semi-polar of a Given Point with Respect to a Circle. Let Q be the centre of the circle, C, Fig. 71, and P the point from which the polar with respect to the circle is required. From P draw tangents to the circle, and join the points of contact by the line ADB. This line is the *polar* of the point P with respect to the circle, and is perpendicular to the line PQ, joining the point P and the centre Q of the circle.

To obtain the semi-polar, bisect DP at E and draw through E a line FEG, parallel to the polar. Then the line FEG is called the *semi-polar* of the point P with respect to the circle.

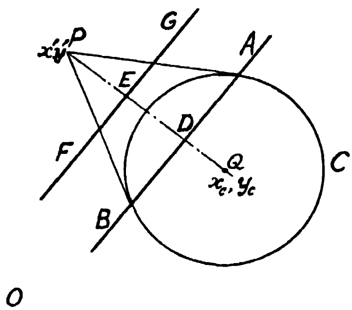


FIG. 71. PERTAINING TO POLAR AND SEMI-POLAR

In cases where the determination of the points of contact of the tangents is difficult, or may lead to an inaccurate construction for the polar, and semi-polar, the following construction may be adopted: Divide the line PQ, joining the centre of the circle and the point P, at D and E such that $PQ.QD = r^2$, or $PD = PQ[1 - (r/PQ)^2]$; and $PE = \frac{1}{2}PQ[1 - (r/PQ)^2]$. Draw through the points D, E, lines perpendicular to the line PQ. Then the line through D is the polar, and that through E is the semi-polar

The proof for this construction is as follows: Let the equation of the circle be $(x - x_c)^2 + (y - y_c)^2 = r^2$, and let x', y', be the co-ordinates of the point, P, for which the polar is required. Then the equation of the polar is $(x - x_c)(x' - x_c) + (y - y_c)(y' - y_c) = r^2$, or

$$x(x' - x_c) + y(y' - y_c) - x_c(x' - x_c) - y_c(y' - y_c) - r^2 = 0,$$

which is the equation to a straight line. Moreover, since the equation to the line joining the point P and the centre, Q, of the circle is

$$\frac{x - x'}{x_c - x'} - \frac{y - y'}{y_c - y'} = 0,$$

or

$$x(y_c - y') - y(x_c - x') - x'y_c + x_cy' = 0,$$

this line is perpendicular to the polar

Hence the distance between the centre, Q, of the circle and the point of intersection, D, of the line PQ and the polar (i.e. the perpendicular distance of the centre of the circle from the polar) is

$$DQ = \frac{x_c(x_c - x') + y_c(y_c - y') - x_c(x_c - x') - y_c(y_c - y') + r^2}{\sqrt{[(x_c - x')^2 + (y_c - y')^2]}}$$

$$= \frac{r^2}{\sqrt{[(x_c - x')^2 + (y_c - y')^2]}}$$

Also the distance between the points P and Q is

$$PQ = \sqrt{[(x_c - x')^2 + (y_c - y')^2]}.$$

Therefore $$PQ.DQ = r^2.$$

Power Expended in the Branch Circuit of Fixed Impedance. The power expended in this portion of the circuit is given by $P_3 = (E_2/Z_3)^2 R_3$.

Now the triangle OA_1G_1, Fig. 70, is the voltage triangle for the system, the side OA_1 representing the supply voltage, OG_1 the voltage drop in the series impedance, and G_1A_1 the voltage at the terminals of the parallel branches.

Since the vector OA_1 represents the short-circuit current to a scale of 1 cm. $= p$ amp., and this current is given by E/Z_1, the same vector, OA_1, will also represent the supply voltage to a scale Z_1 times the current scale, i.e. the scale for voltage is 1 cm. $= pZ_1$ volts. Therefore

$$P_3 = (E_2/Z_3)^2 R_3 = R_3(pZ_1 . G_1A_1/Z_3)^2 = (p^2 R_3 Z_1^2/Z_3^2)[(x_s - x)^2 + (y_s - y)^2],$$

since $G_1A_1^2 = (x_s - x)^2 + (y_s - y)^2$, where x_s, y_s, are the co-ordinates of the short-circuit point A_1.

Expanding this expression and substituting for $x^2 + y^2$ from equation (47), we have

$$P_3 = (p^2 R_3 Z_1^2/Z_3^2)\, 2\{x(x_c - x_s) + y(y_c - y_s) - \tfrac{1}{2}[(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]\}.$$

Whence,

$$P_3 = \frac{2p^2 Z_1^2 R_3}{Z_3^2} \sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}$$
$$\frac{x(x_c - x_s) + y(y_c - y_s) - \tfrac{1}{2}[(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]}{\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}}$$
$$= \frac{2p^2 Z_1^2 R_3}{Z_3^2} \sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]} . G_1K,$$

since the expression

$$\frac{x(x_c - x_s) + y(y_c - y_s) - \tfrac{1}{2}[(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]}{\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2}}$$

represents the perpendicular distance, G_1K of the point x, y (i.e. the point G_1) from the tangent at A_1. Therefore the tangent at the short-circuit point is the datum line from which the power expended in the branch of constant impedance is measured, the scale being 1 cm.

$$= (2p^2 R_3 Z_1^2/Z_3^2)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]} \text{ watts.}$$

Power Expended in the Branch Circuit of Variable Impedance. To show that this quantity is represented by G_1N—Draw from G_1 a line, G_1a, parallel to VA, to intersect A_1B_2 (the line joining the no-load and short-circuit points) at a, and from a draw a line parallel to A_1U to intersect at G_1S (which is parallel to the semi-polar) at d. From d draw a horizontal line to intersect G_1W (which is parallel to VT) at N. Then this point, N, coincides with the intersection of A_1B_2 and G_1S, since the triangles aNb, A_1TU are similar.

Now the portion dS, of G_1S, is made up of two parts, dL and LS; the former representing the power expended in the branch of fixed impedance, and the latter representing the I^2R loss in the series impedance, the scale in each case being 1 cm. $= pE \cos a$ watts. Therefore the ordinate at d, or, alternatively, the ordinate at N, represents the power expended in the fixed portion of the circuit, the scale being 1 cm. $= pE$ watts. Hence, NW represents this power to the scale 1 cm. $= pE \cos \delta$ watts, where δ is the inclination of VT to the vertical axis.*

* The line VT may be called the "total loss" datum line, since the length of the perpendicular, G_1J, drawn from G_1, is proportional to the intercept NW.

The remaining portion, G_1N, of G_1W must therefore represent the power supplied to the variable branch to the scale 1 cm. $= pE \cos \delta$ watts. Since δ is usually a very small angle, the power scale when measuring parallel to VT is practically equal to the power scale when measuring along the ordinate.

Therefore in the load diagram the power expended in, or supplied to, all parts of the circuit is obtained by drawing from G_1 (i) the ordinate G_1M, (ii) a parallel, G_1W, to VT, (iii) a parallel, G_1S, to the semi-polar VU. The power taken from the supply system is then given by the ordinate, G_1M, the scale being 1 cm. $= pE$ watts. The power expended in the series impedance is given by the intercept LS, or, alternatively, by the ordinate at L, the scales being 1 cm. $= pE \cos \alpha$ watts in the former case, and 1 cm. $= pE$ watts in the latter case. The power expended in the fixed portions of the circuit (i.e. the series impedance and the fixed branch of the parallel portion of the circuit) is given by NW, or, alternatively, by the ordinate at N, the scales being 1 cm. $= pE \cos \delta$ watts in the former case, and 1 cm. $= pE$ watts in the latter case. The power supplied to the variable branch of the circuit is given by G_1N, or, alternatively, by the difference in the ordinates at G_1 and N, the scales being 1 cm. $= pE \cos \delta$ watts in the former case, and 1 cm. $= pE$ watts in the latter case.

Efficiency. The ratio (power supplied to branch of variable impedance/power taken from supply system) is given by G_1N/G_1W.

Now, if the lines A_1B_2T and VT are produced beyond T, and any horizontal line be drawn to intersect their extensions at l and h, respectively; and if G_1 be joined to T and produced so as to cut this horizontal line at k, the efficiency is given by lk/lh. Thus, if from k a line, km, is drawn parallel to VT to intersect Tl and the horizontal axis at n and m respectively, we have, from the similar triangles Thl, nkl

$$\frac{nk}{Th} = \frac{lk}{lh}$$

But
$$\frac{nk}{mk} = \frac{nk}{Th} = \frac{G_1N}{G_1W}$$

Therefore,
$$\frac{lk}{lh} = \frac{G_1N}{G_1W}$$

Hence if lh be divided into 100 equal parts, with the zero point at l and the 100 point at h, the percentage efficiency is given directly by the scale reading at k (i.e. the point at which the line joining G_1 and T intersects the scale).

Construction of the Load Diagram from Test Data. For the case, discussed above, where the power factor of the variable branch circuit is constant and the impedance of this portion is variable between zero and infinity, the following construction for the no-load and short-circuit diagram may be adopted when the magnitude and phases of the no-load and short-circuit currents are known.

Draw rectangular axes OX, OY, and from the origin O draw the vectors OB_2, OA_1, Fig. 72, to represent the no-load and short-circuit currents, respectively, to a convenient scale, the inclination of these vectors to the vertical axis being φ_o and φ_s respectively. Join the points B_2 and A_1; bisect this line at F, and draw the perpendicular at FQ. From B_2 draw the line B_2Q to intersect this perpendicular at Q, which is the centre of the current circle. The angle which B_2Q makes with the horizontal axis. B_2X'', at B_2 is

equal to $\pm\varphi_2 + 2\psi$,*where $\cos\varphi_2$ is the power factor of the variable branch (φ_2 being positive for a leading power factor and negative for a lagging power factor), and ψ is the angle OA_1B_2. Hence when the power factor, $\cos\varphi_2$, of the variable branch is *lagging*, $B_2X''Q$ is drawn *below* the horizontal axis, B_2X'', the angle $X''B_2Q$ being

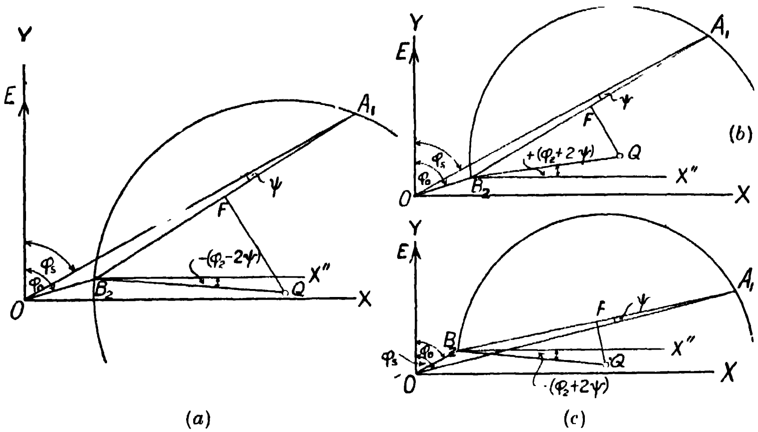


FIG. 72. CONSTRUCTION OF NO-LOAD AND SHORT-CIRCUIT DIAGRAM FROM DATA OF NO-LOAD AND SHORT-CIRCUIT TESTS

equal, numerically, to $(\varphi_2 - 2\psi)$* ; but if the power factor is *leading*, B_2Q is drawn *above* the horizontal axis, B_2X'', and the angle $X''B_2Q$ is equal, numerically, to $(\varphi_2 + 2\psi)$.*

Proof. A reference to Fig. 69, which refers to the case when the power factor of the variable branch is lagging, and the no-load point lies below the short-circuit line, will show that the line AQ containing the centre of the current circle is inclined at the angle φ_2 with respect to the horizontal axis AX', this angle being below the horizontal axis because the power factor is lagging. Now since the angles AQB_2, AA_1B_2, both subtend the same arc, AB_2, of the circle B_2AA_1H, but the former angle is at the centre, and the latter is at the circumference, of the circle, the angle AQB_2 is double the angle AA_1B_2. Hence the angle which the line B_2Q makes with a horizontal axis drawn through B_2 is equal, numerically, to $\varphi_2 - 2\psi$. Therefore the line B_2Q, Fig. 72(a), which is inclined at an angle equal to $-\varphi_2 + 2\psi = -(\varphi_2 - 2\psi)$, with respect to the horizontal axis B_2X'', passes through the centre of the circle.

In a similar manner it may be shown that when the power factor of the variable branch is leading, the line B_2Q, Fig. 72(b), which is inclined at an angle equal to $+\varphi_2 + 2\psi$ with respect to the horizontal axis B_2X'', passes through the centre of the circle.

The diagram is completed by constructing the efficiency scale and drawing the datum lines for input, output, etc., in the manner already described.

* If the no-load point lies above the short-circuit line, i.e. if $\varphi_s > \varphi_0$, as in Fig. 72 (c), the negative sign must be given to 2ψ.

CHAPTER VIII

THE SINGLE-PHASE TRANSFORMER

One of the greatest advantages of alternating currents over direct currents is the facility with which the voltage can be changed from a low to a high value, or *vice versa*, to meet the requirements of, say, long distance transmission or distribution for industrial power and lighting supplies. In all large A.C. distribution systems the

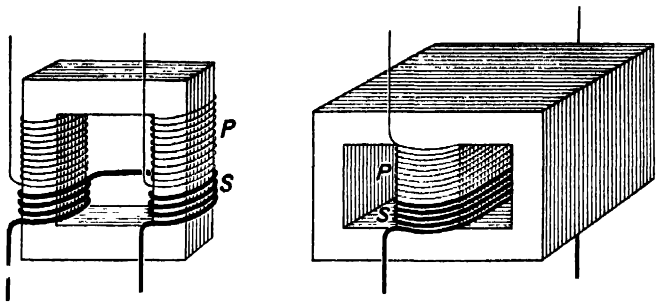
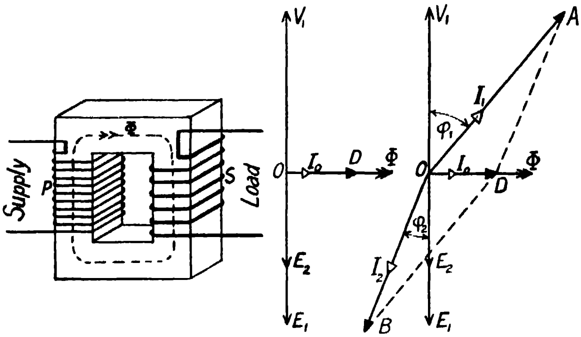


Fig. 73. Elementary Diagrams Showing Arrangements of Core and Windings in Core-type (left) and Shell-type (right) Transformers

voltage at a consumer's premises may have a value quite different from that at the generator, and several changes of voltage may occur between the generator and the consumer. These changes of voltage are effected, with almost negligible loss of energy, by means of stationary transformers working on the principle of mutual induction.

Principle of Transformer. A transformer consists essentially of two electric circuits, tightly coupled magnetically. One circuit (called the *primary*) is designed to receive energy from an A.C. supply system; and the other circuit (called the *secondary*) is designed to deliver energy of the same form and frequency, but usually of a different voltage, to a load. In practice, the tight magnetic coupling between the circuits is obtained by arranging them in the form of either concentric or sandwiched coils, spaced as closely together as the conditions of construction and operation will permit, and providing a magnetic circuit of low reluctance. For low-frequency systems the magnetic circuit is built of alloyed steel laminations with interleaved joints, so as to reduce the reluctance. The sketches in Fig. 73 show two arrangements for the coils and

laminated core. Hence when the primary is connected to an A.C. supply system, an alternating flux is produced in the core, and this flux links with the turns of both primary and secondary windings, thereby inducing E.M.Fs. in these windings. The E.M.F. induced in the primary is in the nature of a back-E.M.F. and opposes the supply voltage; that induced in the secondary is expended in producing current in the load. The transference of energy from the primary to the secondary, therefore, takes place electromagnetically



(a) (b) (c)

FIG. 74. MAGNETIC CIRCUIT AND VECTOR DIAGRAMS OF IDEAL TRANSFORMER

in virtue of the magnetic coupling between the circuits. Thus the magnetic coupling plays an important part in the action of the transformer.

Elementary Theory of Ideal Transformer. In an ideal transformer (i.e. one without losses or magnetic leakage) an alternating flux Φ, produced by the magnetizing effect of the primary, links with the whole of the turns of both primary and secondary windings as indicated in Fig. 74 (*a*). This diagram shows the primary (P) and secondary (S) windings spaced apart for clearness in representation, but in actual transformers these windings would be in the form of concentric coils.

If N_1, N_2 denote the number of turns in primary and secondary windings respectively, the E.M.Fs. induced are $e_1 = -(N_1/10^8)d\Phi/dt$ and $e_2 = -(N_2/10^8)d\Phi/dt$. Hence $e_1/e_2 = N_1/N_2$, i.e. the ratio of the induced E.M.Fs. is equal to the ratio of the numbers of turns in the respective windings. This ratio is called the "turn ratio" or the "ratio of transformation." It is denoted by K.

If the flux follows a sine law, i.e. $\Phi = \Phi_m \sin \omega t$, then $e_1 = -\omega N_1 \times \Phi_m \times 10^{-8} \cos \omega t = \omega N_1 \Phi_m \times 10^{-8} \sin(\omega t - \frac{1}{2}\pi)$, and $e_2 = -\omega N_2 \Phi_m$

$\times\ 10^{-8}\cos\omega t = \omega N_2\Phi_m \times 10^{-8}\sin(\omega t - \frac{1}{2}\pi)$. Whence the R.M.S. values of the induced E.M.Fs. are

$$E_1 = (\omega N_1\Phi_m \times 10^{-8})/\sqrt{2} = (2\pi/\sqrt{2})\Phi_m N_1 f \times 10^{-8}$$
$$= 4{\cdot}44\Phi_m N_1 f \times 10^{-8} \quad . \quad (48)$$
$$E_2 = (\omega N_2\Phi_m \times 10^{-8})/\sqrt{2} = (2\pi/\sqrt{2})\Phi_m N_2 f \times 10^{-8}$$
$$= 4{\cdot}44\Phi_m N_2 f \times 10^{-8} \quad . \quad (49)$$

where f is the frequency of the alternating flux.

Now, in the ideal case, E_1 must balance the voltage applied to the primary. Hence if the R.M.S. value of this voltage is constant, and the frequency is also constant, the maximum value, Φ_m, of the flux must be maintained constant. Therefore a definite number of ampere-turns must be supplied by the primary to maintain the flux, and these ampere-turns must remain constant, irrespective of the load current in the secondary. Obviously this result is only possible when the ampere-turns produced by the current in the secondary winding are neutralized by an equivalent number of ampere-turns in the primary winding. Thus the effect of loading, or taking current from, the secondary is to cause a corresponding increase and alteration in phase of the current in the primary, so that the resultant ampere-turns of the two windings is constant and equal to the ampere-turns necessary to produce the flux.

If $\underset{\cdot}{I}_1$, $\underset{\cdot}{I}_2$ are the primary and secondary currents for a given load, and $\underset{\cdot}{I}_0$ is the primary current when $\underset{\cdot}{I}_2$ is zero, then we must have $\underset{\cdot}{I}_1 N_1 + \underset{\cdot}{I}_2 N_2 = \underset{\cdot}{I}_0 N_1$, or $\underset{\cdot}{I}_1 N_1 = -\underset{\cdot}{I}_2 N_2 + \underset{\cdot}{I}_0 N_1$.

Vector diagrams representing this, and the no-load, condition are shown in the diagrams (*b*), (*c*) of Fig. 74. In these diagrams the vector *OD* represents the magnetizing ampere-turns required to maintain the flux; *OB*, the ampere-turns produced by the current in the secondary; and *OA*, the ampere-turns actually produced by the primary. The induced E.M.Fs. are represented by OE_1 and OE_2, and the supply voltage by OV_1.

Observe that in the special case when the magnetizing ampere-turns are very small in comparison with the secondary ampere-turns, the primary and secondary ampere-turns will be practically equal and in opposition. Hence under these conditions we have, to a very close approximation: $\underset{\cdot}{I}_1 N_1 = -\underset{\cdot}{I}_2 N_2$, or $I_1/I_2 = N_2/N_1$, i.e. the ratio of the primary and secondary currents is constant. This important relationship is the basis of the current transformer—an instrument transformer used with a low-range ammeter for the measurement of currents in circuits where the direct connection of the ammeter would be impracticable (see Chapter XVII).

In diagram (*c*), which represents loaded conditions, it will be

observed that the phase difference, φ_1, between the supply voltage and primary current is, with lagging secondary currents, always greater than that, φ_2, between the secondary voltage and current. Moreover, if φ_2 is constant, φ_1 will depend upon the magnitude of the load current, decreasing as the load current increases.

Differences Between Ideal and Actual Transformer. In an actual transformer the primary and secondary windings each have resistance; magnetic leakage takes place between these windings; and hysteresis and eddy-current losses occur in the magnetic core.

The general effect of the resistance of the windings and the losses in the core is to cause loss of energy and heating. The resistance of the windings also causes voltage drops in both primary and secondary windings, and a diminution of the flux with increase of load current. The core losses also cause the exciting ampere-turns for the magnetic circuit to be slightly greater than the magnetizing ampere-turns and out of phase with the latter.

Magnetic leakage affects not only the flux linked with the secondary winding (which is now less than the corresponding flux linked with the primary winding), but also the phase differences between the induced E.M.Fs. and the currents.

Transformer with Losses, but without Magnetic Leakage. This case represents an approximation to practical conditions. The primary and secondary windings have resistance, and the magnetic core has hysteresis and eddy current losses such as would occur in a normal transformer.

Consider, first, the effect of the resistance of the windings on the operation of the transformer. The resistance of the primary winding causes a voltage drop, and therefore the E.M.F. induced in this winding has now to balance the vector difference of the supply voltage and the voltage drop in the primary winding. Thus $\underset{\cdot}{E}_1 = \underset{\cdot}{V}_1 - \underset{\cdot}{I}_1 R_1$, where $\underset{\cdot}{V}_1$ is the supply voltage, $\underset{\cdot}{I}_1$, the current in, and R_1 the resistance of, the primary winding.

Hence the induced E.M.F. E_1 decreases as I_1 increases, and therefore, with constant supply voltage and frequency, the flux decreases with an increase of primary current.

The decrease in the flux causes a corresponding decrease in the E.M.F. (E_2) induced in the secondary winding, but as we are assuming no magnetic leakage the ratio E_1/E_2 will be constant.

The secondary terminal voltage, V_2, is equal to the vector difference between the induced E.M.F., E_2, and the voltage drop, I_2R_2, in the secondary winding. Thus $\underset{\cdot}{V}_2 = \underset{\cdot}{E}_2 - \underset{\cdot}{I}_2 R_2 = \underset{\cdot}{E}_1/K - \underset{\cdot}{I}_2 R^2 = [(\underset{\cdot}{V}_1 - \underset{\cdot}{I}_1 R_1)/K] - \underset{\cdot}{I}_2 R_2$, where $K = E_1/E_2$, and is called the *ratio of transformation* or, shortly, the *ratio*.

Therefore the secondary terminal voltage will change with a change of load (magnitude as well as power factor). But as, in commercial transformers, the voltage drop in each winding due to resistance rarely exceeds 1 per cent of the corresponding terminal voltage even in transformers of small output, and may not exceed one-quarter of 1 per cent in a large transformer, the effect of

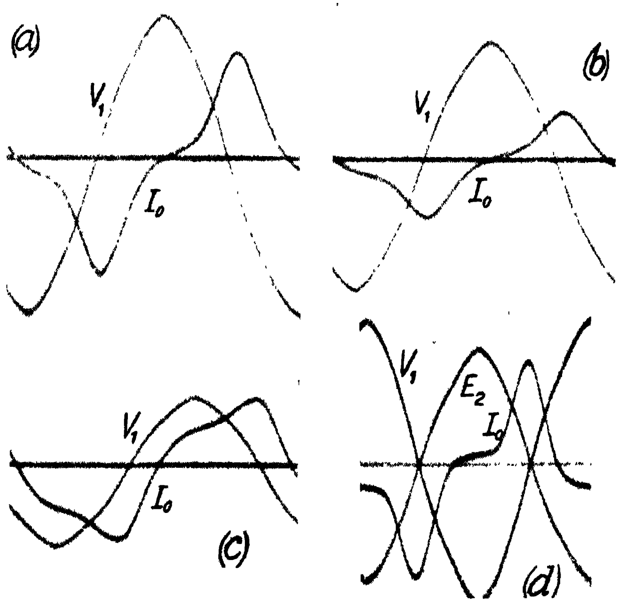


FIG. 75. OSCILLOGRAMS OF EXCITING CURRENT (I_0), PRIMARY APPLIED VOLTAGE (V_1), AND SECONDARY INDUCED VOLTAGE (E_2) OF TRANSFORMERS

(*a*), (*b*), (*c*) refer to a transformer supplied at three different voltages; (*d*) refers to another transformer, supplied at 10 per cent above normal voltage

R.M.S. Values of Voltages and Currents—(*a*) $V_1 = 120$ V., $I_0 = 2{\cdot}16$ A.; (*b*) $V_1 = 100$ V. (normal), $I_0 = 1{\cdot}1$ A.; (*c*) $V_1 = 60$ V., $I_0 = 0{\cdot}25$ A.; (*d*) $V_1 = 110$ V., $I_0 = 4{\cdot}6$ A., $E_2 = 225$ V.

resistance will cause only a relatively small change in the secondary terminal voltage with change of load.

The manner in which this change of secondary terminal voltage is influenced by the power factor of the load is shown in the vector diagrams of Fig. 76, one diagram (*a*) being drawn for a load of lagging power factor, and the other diagram (*b*) being drawn for a load of leading power factor.

If the power factor of the secondary side is $\cos \varphi_2$ and that of

the primary (input) side is $\cos \varphi_1$, the secondary terminal voltage is given approximately by

$$V_2 = [(V_1 - I_1R_1 \cos \varphi_1)/K] - I_2R_2 \cos \varphi_2.$$

The effect of the core losses (i.e. hysteresis and eddy currents) and magnetic saturation is to cause the exciting current for the magnetic circuit to lead the flux and its wave-form to be distorted in the manner explained in Chapter XIV, and shown graphically

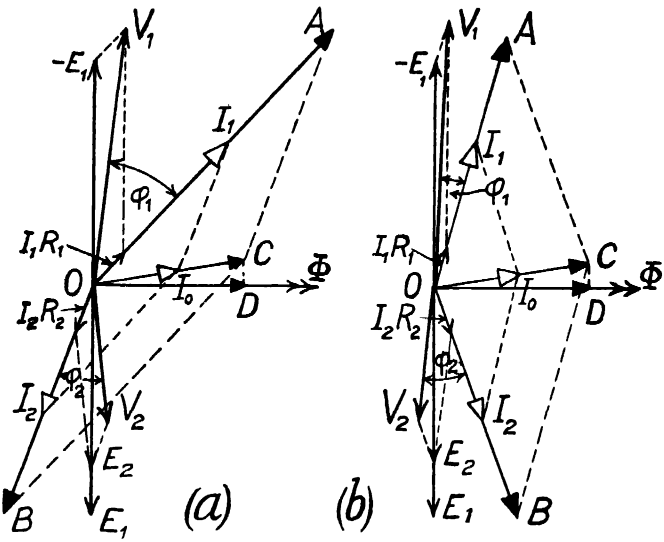


FIG. 76. VECTOR DIAGRAMS FOR A TRANSFORMER HAVING IRON AND COPPER LOSSES, BUT NO MAGNETIC LEAKAGE
(*a*) Secondary current lagging. (*b*) Secondary current leading

in the oscillograms of Fig. 75, which refer to a small transformer excited at various voltages.

With a sine-wave supply voltage, the flux will be sinusoidal, but the exciting current will be distorted. Therefore, in drawing the vector diagram for these conditions, we have to show the *equivalent* (sine wave) exciting current (see p. 329).

Vector Diagram for Transformer with Losses, but without Magnetic Leakage. In Fig. 76 are given two vector diagrams for a transformer with losses, but without magnetic leakage. One diagram (*a*) is drawn for a load having a lagging power factor, and the other (*b*) is drawn for a load having a leading power factor. In both diagrams, representation is as follows: flux, $O\Phi$; induced E.M.Fs.,

OE_1, OE_2; exciting (no-load) current, OI_0; primary current, OI_1; secondary current, OI_2; primary ampere-turns, OA; secondary ampere-turns, OB; exciting ampere-turns (resultant of OA and OB), OC; magnetizing ampere-turns, OD; actual voltage drop in primary, I_1R_1; actual voltage drop in secondary, I_2R_2; supply voltage, OV_1; secondary terminal voltage, OV_2. The vectors representing the voltage drops have been drawn very much exaggerated in order to render them clear in the diagram.

Magnetic Leakage. The arrangement of the primary and secondary windings on separate limbs of the core, as shown in diagram

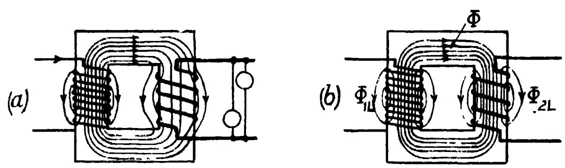


FIG. 77. MAGNETIC LEAKAGE IN TRANSFORMER
(*a*) Flux distribution in core-type transformer having windings on separate limbs.
(*b*) Hypothetical component fluxes

(*a*), Fig. 74, results in considerable magnetic leakage, especially when the secondary is loaded, as the ampere-turns produced by the secondary winding, being practically in opposition to the primary ampere-turns, tend to force the flux (which at no load links with the secondary winding) into the leakage paths as represented in diagram (*a*), Fig. 77. The leakage obviously increases with an increase of load current, because this results in increased ampere-turns of both windings and the production of a high magnetomotive force across the leakage paths. Hence a transformer having the windings arranged as shown in Fig. 74 (*a*) would have a heavy drooping characteristic for the secondary voltage.

The theory of the transformer with magnetic leakage may be developed by considering fictitious leakage fluxes to be superimposed upon the ideal flux, Φ, which links both windings and remains constant. These conditions are represented in diagram (*b*), Fig. 77.

The leakage fluxes take paths which comprise air and iron, the reluctance of the air portions of the paths being many times that of the iron portions. Hence the leakage fluxes will be directly proportional to, and in phase with, the ampere-turns expended across the leakage paths: i.e. the primary leakage flux, Φ_{L1}, will be proprotional to, and in phase with, the primary ampere-turns or the primary current; the secondary leakage flux, Φ_{L2}, will be

proportional to, and in phase with, the secondary ampere-turns or the secondary current.

The flux linking the primary winding will therefore be $\Phi + \Phi_{L1}$, vectorially, and that linking the secondary winding will be $\Phi + \Phi_{L2}$, vectorially.

The E.M.F. induced in the primary winding is then due to the flux $(\Phi + \Phi_{L1})$, and that induced in the secondary winding is due to the flux $(\Phi + \Phi_{L2})$.

The vector diagram representing these conditions is shown in Fig. 78, in which $O\Phi$ represents the ideal flux; OC, the no-load or exciting ampere-turns producing this flux; OA, OB, the primary and secondary ampere-turns respectively; $O\Phi_{L1}$, $O\Phi_{L2}$, the leakage fluxes; $O\Phi_1$ ($= \Phi + \Phi_{L1}$, vectorially), the flux linked with the primary winding; $O\Phi_2$ ($= \Phi + \Phi_{L2}$, vectorially), the flux linked with the secondary winding; OE_1', OE_2', the induced E.M.Fs. due to $O\Phi_1$ and $O\Phi_2$ respectively.

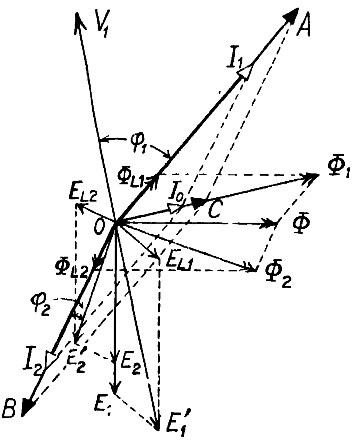


FIG. 78. VECTOR DIAGRAMS FOR A TRANSFORMER HAVING MAGNETIC LEAKAGE AND IRON LOSSES, BUT NO COPPER LOSSES

Obviously each of the induced E.M.Fs., OE_1', OE_2', may be considered to have a component due to the ideal flux, Φ, and a component due to the appropriate leakage flux. The component E.M.Fs. are shown by OE_1, OE_2, OE_{L1}, OE_{L2}, of which OE_1, OE_2 are the components due to the ideal flux, Φ, and OE_{L1}, OE_{L2} are those due to the leakage fluxes Φ_{L1}, Φ_{L2} respectively.

Leakage Reactance. The component E.M.F.s OE_{L1}, OE_{L2}, due to the leakage fluxes are proportional to, and have a phase difference of 90 degrees (lagging) from, the primary and secondary currents respectively. Hence so far as phase difference from, and proportionality to, the current are concerned, the E.M.Fs. are similar in nature to the internal E.M.F. in a reactance coil. We may therefore conveniently consider that a transformer with magnetic leakage is equivalent to an ideal transformer with reactance coils connected in both primary and secondary circuits, as shown in Fig. 79, such that the internal E.M.F. in each reactance coil is equal to that due to the corresponding leakage flux in the actual transformer.

Hence if X_1, X_2 are the reactances of the coils in the primary and secondary circuits respectively, we have: $X_1 = E_{L1}/I_1$; $X_2 = E_{L2}/I_2$; where E_{L1}, E_{L2} are the component E.M.Fs. due to the leakage fluxes in the actual transformer, and I_1, I_2 are the primary and secondary currents respectively. Since the reactances X_1, X_2 are obtained by dividing the E.M.F. due to a leakage flux by the current producing it, they are called *leakage reactances*, X_1 being called the *primary leakage reactance* and X_2 the *secondary leakage reactance*.

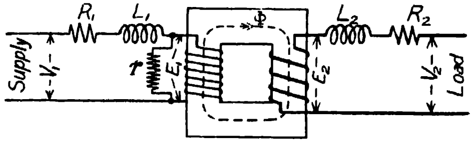


FIG. 79. MODIFICATION OF CIRCUIT DIAGRAM OF IDEAL TRANSFORMER TO REPRESENT THE ACTUAL CASE OF A TRANSFORMER WITH LOSSES AND MAGNETIC LEAKAGE

Similarly, the E.M.Fs. E_{L1} and E_{L2} are called the leakage-reactance E.M.Fs. of the primary and secondary windings respectively.

Vector Diagram of Transformer with Losses and Magnetic Leakage. Fig. 80 shows the vector diagram for a practical transformer with losses and magnetic leakage. The vector $O\Phi$ represents the ideal or no-load flux (Φ), which is considered to remain constant at all loads; OC, the exciting, or no-load, ampere-turns; OD, the magnetizing ampere-turns; OE_1, OE_2, the E.M.Fs. induced in the primary and secondary windings respectively by the flux Φ; OA, OB, the primary and secondary ampere-turns due to the currents OI_1 and OI_2 respectively; OE_{R1}, OE_{R2}, the internal E.M.Fs. due to the resistances of the primary and secondary windings; OE_{L1}, OE_{L2}, the leakage reactance (internal) E.M.Fs. of the primary and secondary windings respectively.

The secondary terminal voltage is represented by OV_2, and is equal to the resultant E.M.F. in the secondary, which is the vector sum of OE_2, OE_{R2}, and OE_{L2}. These E.M.Fs. are shown in the vector polygon OE_2aV_2 by the sides OE_2, E_2a, aV_2 taken in order.

The primary terminal voltage (i.e. the supply voltage) is represented by OV_1 and *balances* the resultant E.M.F. in the primary, which is equal to the vector sum of OE_1, OE_{R1}, OE_{L1}, and is shown by the vector OV_1'. The vector polygon for these E.M.Fs. is OE_1bV_1', of which the sides OE_1, E_1b, bV_1' represent the E.M.Fs. E_1, E_{R1}, E_{L1} taken in order.

Observe that for both primary and secondary circuits the terminal voltage was determined from the *internal* E.M.Fs. on the same principles as were employed in Chapter III.

Alternatively, the terminal voltages could have been obtained by considering the actual voltage drops due to resistance and leakage reactance instead of the corresponding internal E.M.Fs. In this case the secondary terminal voltage would have been obtained from the induced E.M.F. (OE_2) *minus* the voltage drop due to secondary resistance, *minus* the voltage drop due to secondary

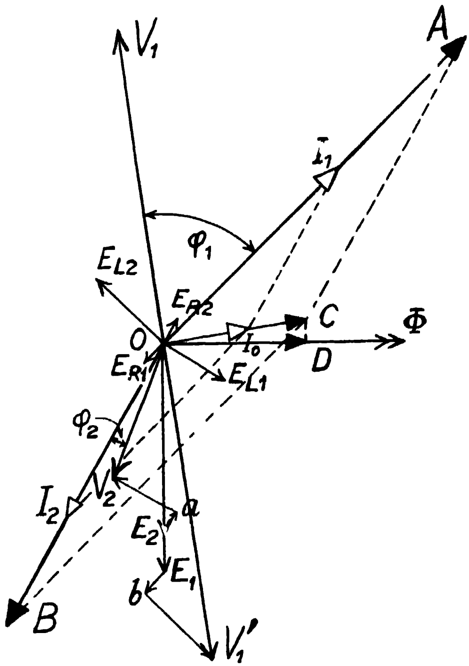


FIG. 80. VECTOR DIAGRAM FOR A TRANSFORMER HAVING LOSSES AND MAGNETIC LEAKAGE

leakage reactance. The primary terminal voltage (i.e. the supply voltage) would have been obtained from the *reversed* induced E.M.F., *plus* the voltage drop due to primary resistance, *plus* the voltage drop due to primary leakage reactance.

Simplified or Approximate Vector Diagram. A considerable simplification results if the vector diagram is drawn for a 1 : 1 ratio of transformation (i.e. the induced E.M.Fs. OE_1, OE_2, are equal) and for zero exciting ampere-turns. The primary and secondary ampere-turns and currents are then equal and in opposition; the vectors representing the internal E.M.Fs. due to resistance are in opposition, and likewise also those representing the internal E.M.Fs. due

to leakage reactance. The resulting vector diagram is shown in Fig. 81.

This diagram may be further simplified as shown in Fig. 82, in which the internal E.M.Fs. due to resistance and leakage reactance are compounded to form a single impedance-E.M.F., which is represented by the vector V_2V_1'. Then if the secondary current,

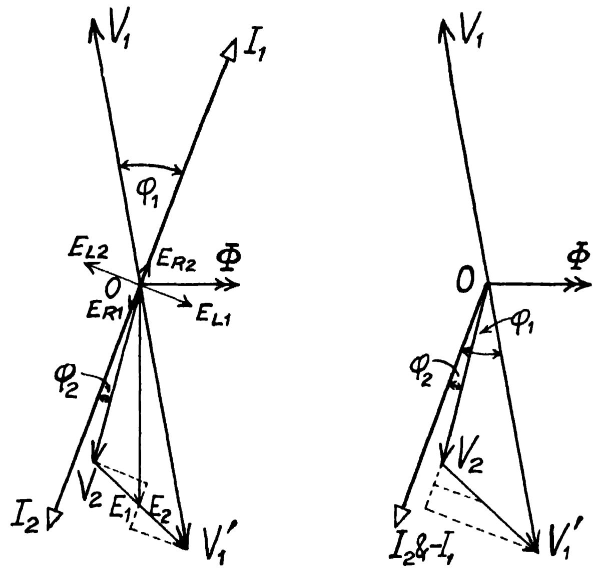


FIG. 81 FIG. 82

SIMPLIFIED VECTOR DIAGRAMS FOR A TRANSFORMER HAVING A RATIO OF TRANSFORMATION OF UNITY

secondary terminal voltage, and reversed primary terminal voltage are the quantities only considered, the diagram reduces to six vectors, of which three are inter-related in the form of a right-angled (impedance-voltage) triangle.

Equivalent Resistance and Leakage Reactance. The diagram of Fig. 82 would be suitable for a transformer in which the ratio of transformation was K, instead of unity, if different scales were adopted for primary and secondary quantities. For example, the scale for secondary E.M.Fs. and voltages would be K times that for the primary E.M.Fs. and voltages; the scale for secondary current and ampere-turns would be $1/K$ that for the corresponding primary quantities.

The same result, however, would be obtained without changing

the scale of the diagram if all the actual secondary E.M.Fs. were multiplied by $K(=N_1/N_2)$ before representation in the diagram, and the actual secondary current and ampere-turns were multiplied by $1/K$. Thus the actual secondary internal E.M.Fs. due to resistance and reactance would appear in the diagram as quantities K times their actual value. These quantities therefore represent E.M.Fs. in the primary which are equivalent to the secondary internal E.M.Fs. due to resistance and reactance. Hence, if we divide each quantity by the primary current, we obtain a resistance and a reactance respectively, which, when carrying the primary current, will produce in the primary, internal E.M.Fs. equivalent to the actual internal E.M.Fs. in the secondary. The value of resistance so obtained is called the *equivalent secondary resistance referred to the primary*, and is denoted by R_2'. Similarly, the value of reactance so obtained is called the *secondary leakage reactance referred to the primary*, and is denoted by X_2'.

Hence, on the assumption that $I_2/I_1 = K$, we have

$$R_2' = KI_2R_2/I_1 = K^2R_2,$$
$$X_2' = KI_2X_2/I_1 = K^2X_2.$$

Thus the equivalent primary resistance (R_1') referred to the secondary $= (I_1R_1/K)/I_2 = R_1/K^2$. Similarly, the equivalent primary leakage reactance (X_1') referred to the secondary $= (I_1X_1/K)/I_2 = X_1/K^2$.

The comparison between equivalent quantities for primary and secondary circuits may be shown thus—

	Secondary Circuit: Actual Quantities	Primary Circuit: Equivalent Quantities	Primary Circuit: Actual Quantities	Secondary Circuit: Equivalent Quantities
E.M.F.	E_2	KE_2	E_1	E_1/K
Current	I_2	I_2/K	I_1	KI_1
Resistance . . .	R_2	K^2R_2	R_1	R_1/K^2
Leakage reactance . .	X_2	K^2X_2	X_1	X_1/K^2

Conversely, primary quantities may be referred to the secondary.

Approximate Relative Magnitudes of Primary and Secondary Resistances and Leakage Reactances. In a commercial transformer the I^2R losses at full load are divided approximately equally between the primary and secondary windings. Thus, approximately, $I_1^2R_1 = I_2^2R_2$. Therefore $R_1/R_2 = (I_2/I_1)^2 = K^2$, or $R_1 = K^2R_2 = R_2'$.

Also since at full load the primary and secondary ampere-turns are approximately equal, and the leakage paths have approximately

equal reluctances, we have, approximately, $X_1/X_2 = (N_1/N_2)^2 = K^2$ i.e. $X_1 = K^2X_2 = X_2'$.

Determination of Impedance and Leakage Reactance of Transformer. These quantities are determined experimentally by an *impedance* or *short-circuit test*. One winding, usually the secondary, is short-circuited, and a *low voltage* (about 5 to 10 per cent of normal voltage) at normal frequency is applied to the other winding, so as to cause approximately full-load current to circulate. The input voltage, current, and power are measured.

Let these quantities be denoted by V, I, P respectively. Then V represents the impedance voltage of the complete transformer and P represents the total I^2R loss in the windings, because, owing to the low impressed voltage and the short-circuited secondary, the flux will have a very low value (only a few per cent of normal) and therefore the iron losses can be completely ignored in comparison with the I^2R losses.

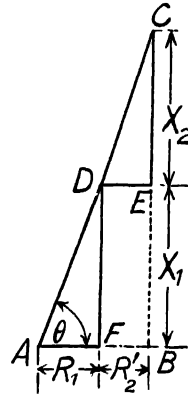


FIG. 83. IMPEDANCE TRIANGLES OF TRANSFORMER

The impedance of the transformer is then given by V/I, the resistance by P/I^2, and the leakage reactance by $\sqrt{[(V/I)^2 - (P/I^2)^2]}$. Observe that $P/I^2 = R_1 + R_2'$, so that if R_1 is known, R_2' can be calculated. The impedance triangle can then be divided into the appropriate equivalent triangles for primary and secondary, as shown in Fig. 83. Thus, if ABC is the equivalent impedance for the complete transformer, referred to the primary, AFD is the impedance triangle for the primary alone, and DEC is the equivalent impedance triangle for the secondary alone.

In practice, the value of resistance calculated from P/I^2 will usually be higher than that calculated from the measured resistances of the windings, the difference being attributable to eddy current losses in the conductors and their supports.

Example. The low-voltage winding of a 300-kVA., 11,000/2200-V., 50-cycle transformer has 190 turns and a resistance of 0·06 Ω. The high-voltage winding has 910 turns and a resistance of 1·6 Ω. When the low-voltage winding is short-circuited, the full-load current is obtained with 550 volts applied to the high-voltage winding. Calculate (1) the equivalent resistance and leakage reactance referred to the high-voltage side, (2) the leakage reactance of each winding.

Taking the full-load efficiency as 98·5 per cent, the full-load primary current = 300,000/11,000 × 0·985 = 27·7 A. Hence,

Equivalent impedance (Z)

$$= 550/27{\cdot}7 = 19{\cdot}8\ \Omega.$$

Equivalent resistance of secondary referred to primary

$= R_2' = 0{\cdot}06(910/190)^2 = 1{\cdot}38\ \Omega.$

Equivalent resistance of transformer (referred to primary)

$= R_1 + R_2' = 1{\cdot}6 + 1{\cdot}38 = 2{\cdot}98\ \Omega.$

Equivalent leakage reactance of transformer (referred to primary)

$= X = \sqrt{[Z^2 - (R_1 + R_2')^2]} = 19{\cdot}5\ \Omega.$

On the assumption that the ratio (leakage reactance/resistance) is the same for each winding, we have

Leakage reactance of primary

$= X_1 = 19{\cdot}5 \times 1{\cdot}6/2{\cdot}98 = 10{\cdot}5\ \Omega.$

Equivalent leakage reactance of secondary referred to primary

$= X_2' = 19{\cdot}5 \times 1{\cdot}38/2{\cdot}98 = 9{\cdot}0\ \Omega.$

Leakage reactance of secondary

$= 9(190/910)^2 = 0{\cdot}39\ \Omega.$

Voltage Regulation and Its Predetermination. The voltage regulation (called also *inherent regulation*) of a transformer is defined as

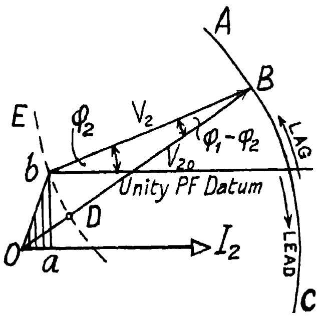


FIG. 84

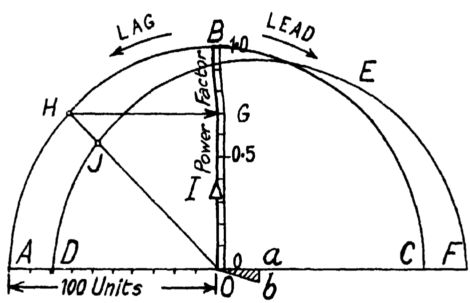


FIG. 85

GRAPHIC METHODS OF DETERMINING VOLTAGE REGULATION OF TRANSFORMER

the difference between the no-load and full-load secondary terminal voltages, expressed as a percentage of the no-load voltage, for constant supply voltage and frequency. Thus, if V_{20} denotes the terminal voltage of the secondary at no load, and V_2 denotes the voltage at full load, then the voltage regulation $= 100\,(V_{20} - V_2)/V_{20}$.

The voltage regulation for a given load and power factor can be readily determined by the aid of Fig. 82 when the impedance voltage triangle is known. It is more convenient, however, to draw the diagram in the form shown in Fig. 84, in which the impedance triangle is in a fixed position for all power factors.

Thus draw the vector OI_2, representing the secondary current, horizontal and on it construct the impedance voltage triangle Oab

corresponding to this current. From O draw an arc AC of radius equal to the no-load secondary voltage, and from the apex, b, of the impedance triangle draw bB, making the angle φ_2 with the horizontal at b, $\cos \varphi_2$ being the power factor of the load. Join OB.

Then ObB is the voltage triangle for the transformer, and the secondary terminal voltage corresponding to a load current I_2 is given by bB. The angle BOI_2 is the approximate phase difference between the primary voltage and primary current.

If with centre B and radius Bb we draw the arc EbD, then the voltage regulation is given by 100 OD/OB.

The diagram is readily adapted for other load conditions by constructing proportional impedance triangles, as indicated in Fig. 84, and drawing the vector for the secondary terminal voltage from the appropriate apex.

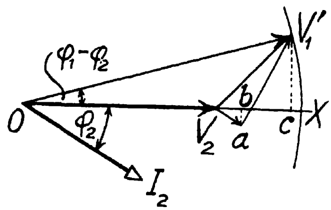


FIG. 86. VECTOR DIAGRAM ILLUSTRATING DERIVATION OF APPROXIMATE FORMULA FOR VOLTAGE REGULATION

If the voltage regulation is required for a constant load current at variable power factor, the construction shown in Fig. 85 (called the Kapp diagram) is more convenient. Thus with O as centre in a horizontal axis AOC, draw a semicircle ABC having a radius equal to 100 units of length (e.g. 100 mm.). Let this length represent the no-load secondary voltage [= primary (supply) voltage/K]. Draw the perpendicular OB and let this represent the vector of the secondary current. Construct the impedance voltage triangle Oab as shown, and with b as centre, draw the arc DEF. Mark on OB a scale of power factor with 0 at O and 1·0 at B. Then the percentage voltage regulation for any power factor is obtained as follows: (1) project the appropriate point on the power-factor scale, e.g. G, to the circumference of the semicircle ABC, using the quadrant AB for a lagging power factor and the quadrant BC for a leading power factor; (2) join the point of intersection, e.g. H to O; (3) measure the intercept, HJ, which gives to a close approximation the voltage regulation directly as a percentage.

Calculation of Voltage Regulation. In cases where a graphical determination will not give accurate results, the voltage regulation may be calculated. Thus, if the vector diagram of Fig. 82 is redrawn, as in Fig. 86, with the vector OV_2 in the horizontal axis OX, then, since in practice $(\varphi_1 - \varphi_2)$ is a very small angle, the projection,

Oc, of OV_1' on OX, may be considered equal to OV_1', which represents the no-load secondary voltage, V_{20}. Now $Oc = OV_2 + V_2b + bc$, where b is the projection of the right-angled corner a of the impedance triangle on OX. But the side V_2a of this triangle $= (R_2 + R_1')I_2$, so that $V_2b = (R_2 + R_1')I_2 \cos \varphi_2$. Similarly, the side $aV_1' = (X_2 + X_1')I_2$, so that $bc = (X_2 + X_1')I_2 \sin \varphi_2$.

Hence, the percentage voltage regulation

$$= 100(Oc - OV_2)/Oc = 100(V_2b + bc)/Oc$$
$$= 100[(R_2 + R_1')I_2 \cos \varphi_2 + (X_2 + X_1')I_2 \sin \varphi_2]/V_{20} \quad . \quad (50)$$

Example. Calculate the voltage regulation at a power factor of 0·8 (lag) for the transformer, of which data are given in the example on p. 155.

We have, from p. 156,

$$R_2 + R_1' = 0{\cdot}06 + 1{\cdot}6(190/910)^2 = 0{\cdot}129\ \Omega.$$
$$X_2 + X_1' = 0{\cdot}39 + 10{\cdot}5(190/910)^2 = 0{\cdot}897\ \Omega.$$

The full-load current $I_2 = 300{,}000/2200 = 136$ A.

Whence, since $\cos \varphi_2 = 0{\cdot}8$,

and $\sin \varphi_2 = \sqrt{(1 - 0{\cdot}8^2)} = 0{\cdot}6$,

$$\%\ \text{regulation} = \frac{100(0{\cdot}129 \times 136 \times 0{\cdot}8 + 0{\cdot}897 \times 136 \times 0{\cdot}6)}{11{,}000 \times 190/910}$$
$$= 3{\cdot}8 \text{ per cent}$$

Equivalent Circuit. When developing the theory of the transformer with magnetic leakage and losses, it was shown (p. 150) that such a transformer is equivalent to an ideal transformer (i.e. one without magnetic leakage or losses, but with a magnetic circuit requiring a definite number of ampere-turns), together with external (series) resistances and reactance coils in the primary and secondary circuits, and a resistance shunted across the primary to represent the iron losses. The circuit diagram for these conditions is shown in Fig. 79.

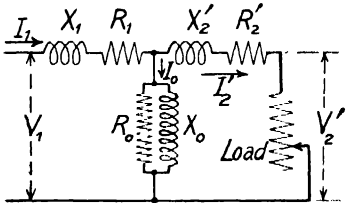
To obtain the equivalent electric circuit, the magnetic coupling between the primary and secondary circuits in the ideal transformer must be replaced by an electric coupling. In order to effect this result, we make the ratio of transformation unity, adjusting the series resistances and reactances to suit, and connect across the primary circuit a reactance coil which will take a current equal to the magnetizing current. Thus, so far as voltages and currents are concerned, the series-parallel circuit shown in Fig. 87 will give conditions equivalent to those in the transformer. This electric circuit is therefore called the *equivalent circuit* of the transformer. It may be reduced to a simpler, i.e. a parallel, circuit shown in Fig. 88, but this gives only a rough approximation to the actual conditions. The general vector diagram for such a parallel circuit has already

been deduced (p. 104), and therefore the approximate performance of the transformer may be predetermined by the method described on pp. 120, 121.

The general vector diagram corresponding to the more accurate equivalent circuit (Fig. 87) is deduced in Chapter XXII, together with the method of predetermining its performance.

By ignoring the magnetizing current and iron losses, the equivalent circuit diagrams of Figs. 87 and 88 reduce to a series circuit similar to that shown in Fig. 64 (p. 122) and discussed on pp. 122–129.

Losses and Efficiency. The losses occurring in a loaded transformer are: (1) hysteresis and eddy-current losses in the iron core;



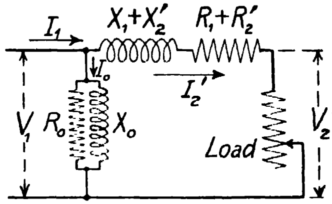


FIG. 87 FIG. 88

EQUIVALENT CIRCUIT DIAGRAMS FOR TRANSFORMER

and (2) losses in the windings due to resistance and also to eddy currents in the conductors. The losses occurring under item (1) are called *iron losses*; those occurring under item (2) are called *copper losses*.

Since for all ordinary loads under normal conditions of operation the main flux remains almost constant at its no-load value, the iron losses may be considered as a constant loss.

The copper losses may be considered as approximately proportional to the square of the current (primary or secondary).

Now, efficiency = power output/power input = (power input − losses)/power input = 1 − (losses/power input).

Hence, the maximum efficiency will occur at the load for which the ratio (losses/power input) is a minimum. At unity power factor this load corresponds to equality of copper and iron losses.* Thus

* At unity power factor the power input = $V_1 I_1$ and the total losses = $P_i + I_1^2 R$. The minimum value of the ratio (total losses/input) is obtained by equating the first derivative of this expression to zero and solving the equation.

Thus, $$\frac{d}{dI_1}\left(\frac{P_i}{V_1 I_1} + \frac{I_1^2 R}{V_1 I_1}\right) = 0$$

i.e. $$-(P_i/V_1)\,I_1^{-2} + R/V_1 = 0$$

Whence $$I_1^2 R = P_i$$

if P_i denotes the iron losses, I_1 the primary current, and R the equivalent resistance of the transformer referred to the primary, then for maximum efficiency, $I_1^2R = P_i$, or $I_1 = \sqrt{(P_i/R)}$.

If the iron and copper losses are plotted as a percentage of the input as shown in Fig. 89, the percentage efficiency curve is readily deduced, and it will be apparent that the point of intersection of the copper and iron loss curves corresponds to the maximum efficiency.

By suitable design, therefore, it is possible to make the maximum efficiency occur at any desired load. For example, if the full-load copper loss is equal to the iron loss, the maximum efficiency will

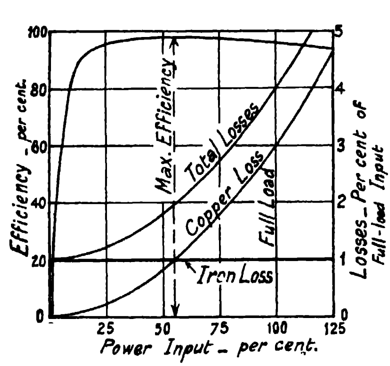


FIG. 89. CURVES OF EFFICIENCY AND LOSSES

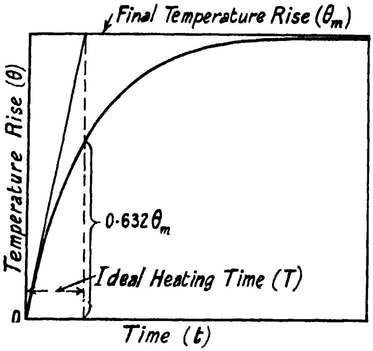


FIG. 90. THEORETICAL HEATING CURVE

occur at full load; but if the full-load copper loss is three times the iron loss, the maximum efficiency will occur at 57·7 per cent [$= \sqrt{(1/3)}$] of full load, assuming unity power factor.

All-day Efficiency. Since transformers supplying lighting and general networks have usually to remain in circuit for long periods, the cost of the energy expended in the losses is an important item in the operation of the transformer, particularly when the transformer is lightly loaded for long periods. The selection of a transformer for a given service cannot therefore be made on the basis of a maximum efficiency at full-load, unless the transformer is to be fully loaded during the greater portion of the time it is in service. Instead, the basis should be a maximum *all-day efficiency*, a quantity which takes into account the *energy* output and input.

Thus,

$$\text{all-day efficiency} = \frac{\text{energy output for a given period}}{\text{energy input during the same period}} \text{ (e.g. 24 hours)}$$

Example. A 300 kVA. transformer (A) has an iron loss of 1·25 kW. and a full-load copper loss of 3·75 kW. Another 300 kVA. transformer (B) has an iron loss of 2·5 kW. and a full-load copper loss of 2·5 kW. If each transformer is connected separately to a load which varies over a period of 24 hours in the following manner—

Periods— 1 a.m.—6 a.m.	—7 a.m.	—8 a.m.	—9 a.m.	—12 noon	—2 p.m.	—6 p.m.	—8 p.m.	— 10 p.m.	—1 a.m.
Load (kW.)— 100	200	300	360	300	280	300	360	280	200

calculate for each transformer—(1) the all-day efficiency, (2) the cost of the losses over the above period on the basis of energy at a flat rate of 0·8d. per kWh. Calculate also the maximum efficiency of each transformer and the load at which this maximum efficiency occurs.

The first step is to calculate the losses corresponding to the loadings given in the load curve. The iron losses are, of course, constant and the copper losses at any load, say P kW., are obtained by multiplying the full-load copper loss by $(P/300)^2$.

Next, the kWh. expended in supplying the losses is calculated, and then the kWh. input and output during the loading periods are determined. The results are shown in tabular form.

		100	200	280	300	360
Load (kW.)		100	200	280	300	360
Loading period (hours)		5	4	4	8	3
Copper losses (kW.)	Transformer A	0·42	1·67	3·26	3·75	5·4
	Transformer B	0·28	1·1	2·18	2·5	3·6
Total losses (kW.)	Transformer A	1·67	2·95	4·51	5·0	6·65
	Transformer B	2·78	3·6	4·68	5·0	6·1
kWh. expended in supplying losses	Transformer A	8·35	11·8	18·1	40	19·95
	Transformer B	13·9	14·4	18·7	40	18·3
kWh. output during loading period		500	800	1120	2400	1080

Total kWh. output during 24-hour period = 5900.

Total kWh. expended in supplying losses during 24-hour period, transformer A = 98·3; transformer B = 105·3.

Whence,

All-day efficiency, transformer A = 5900/(5900 + 98·3) = 0·984

All-day efficiency, transformer B = 5900/(5900 + 105·3) = 0·975

Cost of losses per diem, transformer A = 98·3 × 0·8 = 78·6 pence

Cost of losses per diem, transformer B = 105·3 × 0·8 = 84·3 pence.

The maximum efficiency of transformer B obviously occurs at full load (since the iron and copper losses are then equal) and its value = 300/305 = 0·984.

In the case of transformer A, we proceed as follows: Let x = load (kW.) at which maximum efficiency occurs. Then $(x/300)^2 \times 3{\cdot}75 = 1{\cdot}25$, whence x = 173·2 kW.

At this load, the total losses = 2·5 kW. and the efficiency = 173·2/175·7 = 0·986.

Heating and Temperature Rise. The iron and copper losses produce heating of the core and windings, and result in an increase of temperature of these parts with respect to the surrounding air or cooling medium. The rise of temperature will continue until the rate at which heat is dissipated by the cooling system is equal to rate at which heat is produced by the losses.

If the rate of production of heat (i.e. the losses) is assumed to be constant, the transformer is assumed to be a homogeneous body, and the heat is dissipated naturally by convection, radiation, etc., the temperature rise will be a logarithmic function of the loading time (i.e. the duration of the loading measured from the time of application of load to the transformer initially at air temperature). The temperature-time curve is of exponential form, and the temperature rise, θ, at any instant, t, measured from the start, is given by

$$\theta = \theta_m(1 - \varepsilon^{-t/T}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where θ_m is the maximum temperature rise, and T is the *heating time-constant*, or *ideal heating time*, of the transformer.

The heating time-constant is defined as the time, measured from the start, which would be required to reach the maximum temperature rise (θ_m) if none of the heat were dissipated (i.e. if all the heat produced by the losses were stored in the transformer and the initial rate of temperature rise were maintained).

The value of the heating time-constant may be obtained either by drawing a tangent to the initial portion of the temperature-time curve and determining the time corresponding to the intersection of this line and the temperature line θ_m, as shown in Fig. 90, or by calculating the ratio—(thermal capacity of the transformer/specific cooling factor), the specific cooling factor being the heat dissipated per second per 1° C. temperature rise.

If, in equation (51) we make the substitution $t = T$, we obtain $\theta = \theta_m(1 - \varepsilon^{-1}) = 0{\cdot}632\theta_m$. Hence, instead of determining T by drawing a tangent to the temperature-time curve at the origin, we can determine from this curve the time required to attain a temperature rise of 63·2 per cent of the final temperature rise.

The heating time-constant of actual transformers varies from about 1 hour to 5 hours, depending upon the kVA. rating, voltage, and the method of cooling.

Although the theoretical equation, $\theta = \theta_m(1 - \varepsilon^{-t/T})$, indicates that the final temperature is attained when the time is infinite, in practice this temperature is attained after a few hours of loading

in the case of small transformers and after several hours in the case of large transformers.

Moreover, in practice, the actual heating and cooling conditions may contain variable factors, and also the actual transformer is not a homogeneous mass. These conditions will, of course, have some effect on the shape of the temperature-time curve.

The permissible values of the final temperature rise of the windings (as determined by resistance measurements) for constant loading at the rated load are: 55° C. for air-cooled transformers (75° C. is permissible when mica or asbestos insulation is employed); 60° C. for oil-immersed transformers with natural circulation; 65° C. to 70° C. for oil-immersed transformers with forced circulation; the ambient air temperature not exceeding 30° C. and the inlet temperature of the cooling water (when employed) not exceeding 20° C.

Methods of Cooling. The natural cooling surface of the core and windings is, except in the smallest sizes of transformers, insufficient to dissipate the heat produced by the losses (even when the efficiency at full load is of the order of 98 per cent), and therefore artificial cooling is necessary for transformers of medium and large sizes.

Artificial cooling may take the form of either an air blast or oil immersion. In the latter case, the transformer is immersed in oil contained in a steel tank, the external surface of which, augmented if necessary by corrugations, fins or radiator tubes, provides the cooling surface for heat dissipation. A typical arrangement is shown in Fig. 178 (p. 264). The oil performs the double function of transferring the heat from the core and windings to the cooling surface, and of providing additional insulation for the windings. In cases where, with natural circulation of the oil and external air, the augmented cooling surface of the tank is insufficient for cooling purposes, the circulation is augmented either by an external air blast or an internal cooler with water circulation. If these devices are insufficient to produce the requisite cooling, the oil is circulated by means of a pump, and is cooled by an external cooler with either water or an air blast as the cooling medium.

Auto-transformer. An auto-transformer is a transformer with a *single* winding. At no load a portion of this winding is common to both primary and secondary circuits. When the transformer is loaded, a portion of the load current is obtained direct from the supply, and the remaining portion is obtained by transformer action. Hence, for a given volt-amperes in the load, the volt-ampere output from the secondary winding will be smaller than that for the corresponding case when an ordinary, or double-winding, transformer is employed.

Circuit diagrams showing no-load and loaded conditions in auto-transformers are given in Fig. 91. From the diagrams (*a*), (*b*), for a step-down transformer, we obtain, when the magnetizing current and losses are ignored—

$$V_1/V_2 = (N_1 + N_2)/N_2 = K$$
$$I_1N_1 = (I_2 - I_1)N_2 \text{ or } I_2/I_1 = (N_1 + N_2)/N_2 = K$$
$$\text{VA. ratings of windings} = V_2(I_2 - I_1) + (V_1 - V_2)I_1$$
$$= V_1I_1 + V_2I_2 - 2V_2I_1$$

Comparison of VA. Ratings of Windings of Auto-transformer and Ordinary Transformer. Assuming the same load currents and voltages in each case, and ignoring magnetizing currents and losses,

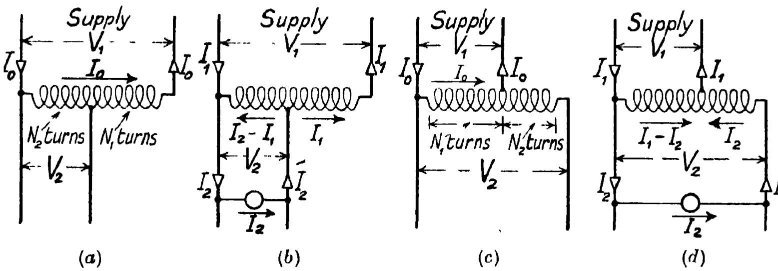


FIG. 91. CIRCUIT DIAGRAMS FOR AUTO-TRANSFORMERS
(*a*), (*b*), step-down, (*c*), (*d*), step-up

the volt-ampere ratings of the windings of an ordinary transformer are $V_1I_1 + V_2I_2 = 2V_1I_1$, and the corresponding ratings for an auto-transformer are $V_1I_1 + V_2I_2 - 2V_2I_1 = 2V_1I_1 - 2V_2I_1$.

Whence,

$$\frac{\text{VA. ratings of windings of auto-transformer}}{\text{VA. ratings of windings of ordinary transformer}} = 1 - \frac{V_2}{V_1}$$
$$= 1 - \frac{1}{K} \quad . \quad (52)$$

It can be shown that this expression also represents the approximate ratio of copper in the windings of the two transformers.

Hence, for *low ratios of transformation* the VA. rating of the windings of an auto-transformer (and also the amount of copper required for the windings) will be much lower than that of an ordinary transformer for the same load. For example, when the ratio of transformation is 2, the windings of an auto-transformer will require actually slightly less than one-half the amount of

copper required by the windings of an ordinary transformer for supplying the same load.

Applications and Uses of Auto-transformers. On account of the common connection between the primary and secondary circuits, the applications of auto-transformers in practice are limited to circuits where such interconnection is permissible. Examples of such cases are: Starting equipment (called auto-starters) for three-phase squirrel-cage induction motors; control equipment of single-phase and three-phase electric locomotives; voltage control of power and lighting circuits. On high-voltage transmission and distribution systems, however, it is usually desirable to have the low-voltage circuits entirely isolated electrically from the high-voltage circuits, and therefore auto-transformers cannot be employed.

Parallel Operation

In the distribution of power for industrial and general purposes from a substation or distribution centre, two or more transformers are usually provided in order that the varying demands may be supplied as economically as possible. For example, at light loads only one transformer would be employed, and as the load demand increased additional transformers would be connected in parallel as required.

Conditions for Parallel Operation. When transformers are to be operated in parallel, the primary windings are connected to the supply bus-bars and the secondary windings are connected to the load bus-bars. In making the connections, it is essential that terminals having similar markings or of similar polarity are connected to the same bus-bar. If this condition is not observed, the E.M.Fs. in the secondary windings of the transformers which are paralleled with incorrect polarity will act together in the local secondary circuits and produce the equivalent of a short circuit.

In order to avoid circulating currents, and to ensure that the transformers share the load in proportion to their kVA. ratings, the following conditions must be fulfilled—(1) the primary windings must be suitable for the supply system voltage and frequency; (2) the voltage ratio must have the same value for each transformer; (3) the impedance voltage triangles corresponding to the kVA. ratings of the transformers must all have the same shape and size, i.e. at the full-load currents the impedance voltages must have the same value for all the transformers, and the ratio (resistance/reactance) must also have the same value for all the transformers.

If condition (2) is not exactly satisfied, i.e. the transformers have *slight* differences in their ratios, parallel operation is possible, but

circulating currents will flow in the windings. Over-heating may, therefore, occur if the full kVA. output is taken from the transformers.

If condition (3) is not exactly fulfilled, i.e. the impedance triangles at the rated kVA.s are not identical in size and shape, parallel operation will be possible, but the power factors at which the transformers operate will differ from the power factor of the load. Hence, in this case the transformers will not share the load in proportion to their kVA. ratings.

Standard Terminal Markings. Transformers built to British Standard Specification have the terminals marked in accordance with

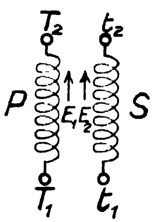


FIG. 92

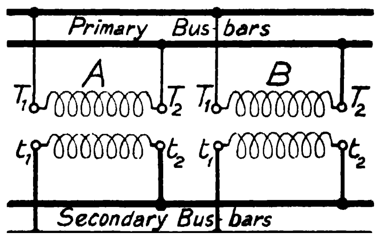


FIG. 93

B.S.I. TERMINAL MARKINGS AND CONNECTIONS OF TRANSFORMERS FOR PARALLEL OPERATIONS

the diagram in Fig. 92, and therefore in connecting such transformers in parallel, similarly marked terminals are connected to the same bus-bar as indicated in Fig. 93.

Test for Polarity. In the case of transformers with unmarked terminals, a preliminary test for polarity must be made before such transformers can be connected in parallel. For this test a voltmeter is required having a range *double* the normal secondary voltage.

Thus suppose we have two transformers A and B with terminals of unknown polarity. Permanent connections are made between the primary terminals and the supply, or primary, bus-bars. On the secondary side, one of the terminals of A is connected temporarily to a terminal of B, and the other terminals of A and B are connected through the voltmeter (i.e. the two secondary windings and the voltmeter form a closed series circuit). If, when the primary windings are excited, the voltmeter reads zero, the terminals which are temporarily connected are of similar or like polarity; but if the voltmeter reads twice the normal secondary voltage, these terminals are of unlike polarity.

Equivalent Circuits. The equivalent circuit for two transformers operating in parallel is shown in Fig. 94. By ignoring the no-load or exciting currents, this circuit reduces to the much simpler circuit shown in Fig. 95, which, for the majority of cases, gives results

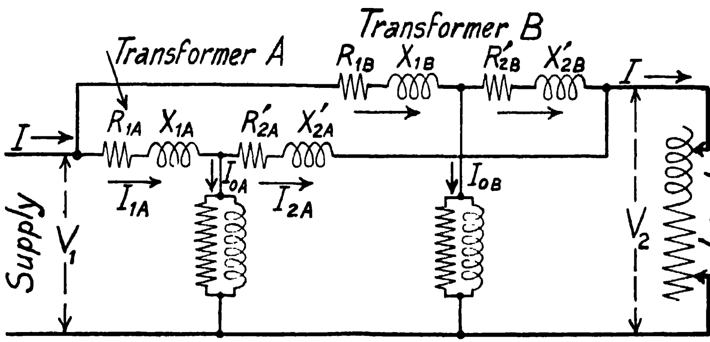


FIG. 94. EQUIVALENT CIRCUIT FOR TWO TRANSFORMERS OPERATING IN PARALLEL

which are sufficiently accurate for practical purposes when the ratios of transformation are equal. For the case when the ratios are not exactly equal, the more exact circuit of Fig. 94 should be used,

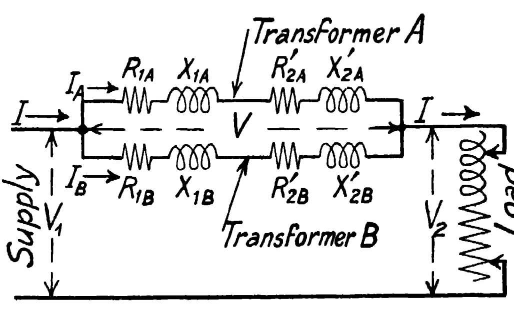


FIG. 95. APPROXIMATE EQUIVALENT CIRCUIT FOR TWO TRANSFORMERS OPERATING IN PARALLEL

in order that the appropriate values of impedance can be obtained for the purpose of calculating the circulating current.

In the simple equivalent circuit of Fig. 95, we have only three principal voltages, viz. the supply voltage, V_1; the equivalent impedance voltage, V, and the load voltage, V_2. Since these voltages are common to all the transformers, the vector diagram can easily be drawn.

Vector Diagrams for Parallel Operation. In all cases the primary windings are assumed to be suitable for the supply system as regards voltage and frequency. The diagrams (Figs. 96, 97, 98) are drawn for the parallel operation of two transformers. The vectors OV_1',

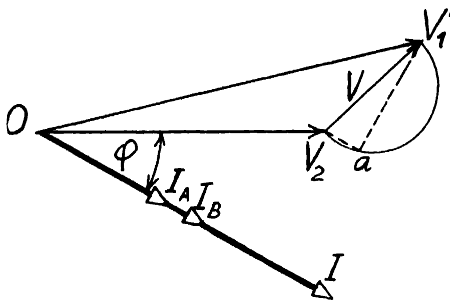


FIG. 96. VECTOR DIAGRAM REPRESENTING IDEAL CONDITIONS FOR PARALLEL OPERATION

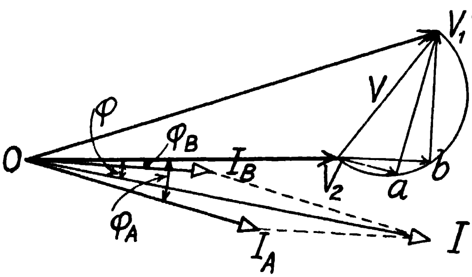


FIG. 97. VECTOR DIAGRAM FOR PARALLEL OPERATION OF TRANSFORMERS HAVING DISSIMILAR IMPEDANCE TRIANGLES

OV_2 represent the primary (reversed) and secondary terminal voltages respectively; OI, the load current, and OI_A, OI_B the secondary currents.

(1) *Ideal Case.* Fig. 96 shows the vector diagram. The impedance voltage triangles of the individual transformers are identical in size and shape, and are therefore represented by a single triangle $V_1'aV_2$ with the "resistance-drop" side, V_2a, parallel to the load current vector OI. The currents I_A, I_B, in the individual transformers are both in phase with the load current, I, and are inversely proportional to the respective impedances. Hence, $I = I_A + I_B$; $I_A/I_B = Z_B/Z_A$; $I_A = IZ_B/(Z_A + Z_B)$; $I_B = IZ_A/(Z_A + Z_B)$.

(2) *Equal Ratios, Dissimilar Impedance Triangles.* Fig. 97 shows the vector diagram. The impedance voltage triangles are now represented by two triangles V_2aV_1', V_2bV_1', having a common

hypotenuse V_2V_1'. The "resistance drop" sides V_2a, V_2b, are parallel to the vectors, OI_A, OI_B, of the respective secondary currents, and the sum of these vectors represents the load current, OI. The magnitudes of the secondary currents are, of course, inversely proportional to V_2a and V_2b.

(3) *Unequal Ratios, Similar Impedance Triangles.* (NOTE. Only relatively small inequalities in the ratios are permissible in practice.) Both primary windings are assumed to be suitable for the supply system, so that when each transformer is excited independently and the secondaries are not paralleled, the flux and magnetizing current are normal. The no-load secondary voltages will then be slightly unequal. These conditions are represented in the vector diagram (*a*), Fig. 98.

When the transformers are paralleled on the secondary side and there is no load on the secondary bus-bars, the difference between the two secondary induced E.M.Fs. causes a circulating current to flow in both secondary windings. This current will be almost wattless in relation to the E.M.F. producing it, and will cause voltage drops in both primary and secondary windings of such magnitude that the two secondary terminal voltages are equal and are in phase with each other when considering each internal circuit. These conditions are represented in the vector diagrams (*b*), (*c*), Fig. 98; diagram (*b*) referring to the transformer (*A*) having the higher no-load secondary voltage when independently excited; and diagram (*c*) referring to the other transformer (*B*). In these diagrams, the vectors OI_C represent the circulating current in the secondary windings; and the vectors OB_A, OB_B, the ampere-turns produced by this current. The exciting ampere-turns are represented by OC, and the corresponding primary ampere-turns and currents are represented by OA_A, OA_B, and OI_{1A}, OI_{1B} respectively. As the ampere-turns due to the circulating current are assumed to be greater than the exciting ampere-turns, the primary current in transformer B is shown in the third quadrant of the vector diagram, i.e. this transformer is feeding back into the supply system. In consequence, the voltage drop in the primary winding of this transformer is negative, and the induced E.M.F. (E_{2B}) is greater than the value shown in diagram (*a*). Actually, for the conditions shown in diagram (*c*), the internal E.M.F. in the primary is approximately equal to the supply voltage, OV_1.

The voltage drop in the primary winding of A results in an induced E.M.F., which is represented by the vector V_1G. The corresponding induced E.M.F. in the secondary is represented by OE_{2A}.

The common secondary terminal voltage, OV_2, is obtained by compounding OE_{2A} and OE_{2B} with the appropriate secondary impedance-voltages, which are of equal magnitude. Observe that the impedance voltage is compounded subtractively in one case (A) and additively in the other case (B).

The vector diagram (d), Fig. 98, refers to one condition when the secondary bus-bars are loaded. The load current is represented by

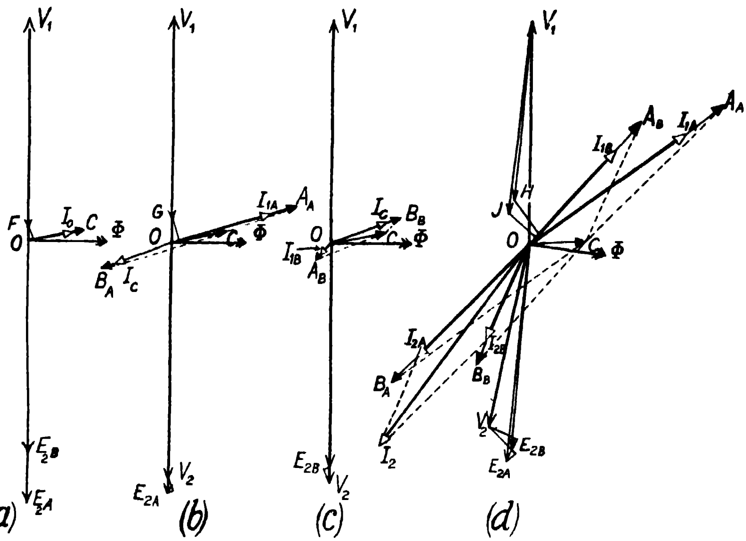


FIG. 98. VECTOR DIAGRAMS FOR PARALLEL OPERATION OF TRANSFORMERS HAVING UNEQUAL RATIOS OF TRANSFORMATION

OI_2; the secondary currents by OI_{2A}, OI_{2B}; and the primary currents by OI_{1A}, OI_{1B}. The internal E.M.Fs. in the primary windings are represented by OH, OJ, and the corresponding induced E.M.Fs. in the secondary windings by OE_{2A}, OE_{2B}. These E.M.Fs. when compounded with the appropriate impedance voltages give the common secondary terminal voltage OV_2.

Calculation of Loadings of Transformers for Given Load on Bus-bars. In practice the problem usually concerned with the parallel operation of transformers is the determination of the kVA. loading of the transformers when the kVA. and power factor of the load are known, together with the equivalent resistances and reactances of the transformers.

The case which we will consider is that of two transformers having equal ratios, but dissimilar impedance triangles. The

exciting currents will be ignored. Hence, the vector diagram of Fig. 97 will apply, from which it follows that

$$I_A/I_B = Z_B/Z_A = (R_B + jX_B)/(R_A + jX_A)$$
$$I = I_A + I_B;\ I_A Z_A = I_B Z_B \text{ or}$$

whence $I_A = I/(1 + Z_A/Z_B) = IZ_B/(Z_A + Z_B)$ . . . (53)

and $I_B = I/(1 + Z_B/Z_A) = IZ_A/(Z_A + Z_B)$ . . . (54)

Therefore I_A and I_B are determined.

If the load kVA. is given, this quantity is substituted for I, and the results will give directly the kVA. loadings (symbolically) of the transformers.

In many problems the resistances and reactances will not be given directly in ohms, but in the form of percentage voltage drops corresponding to full-load current. Since, however, the above equations show that only the ratio of the impedances is required, the calculations may be carried out directly with the percentage values when the transformers are of equal kVA. rating.

In the case of transformers of unequal ratings, however, the ratio of the ratings must be introduced into the above equations. Thus, let I_{Af}, I_{Bf} denote the full-load currents; V_2, the load voltage; % V_{ZAf}, % V_{ZBf}, the percentage impedance voltages corresponding to the full-load currents. Then $\%\ V_{ZAf} = 100 \times Z_A I_{Af}/V_2$, $\%\ V_{ZBf} = 100 \times Z_B I_{Bf}/V_2$.

Whence $Z_A/Z_B = (\%\ V_{ZAf}/\%\ V_{ZBf}) \times (\text{kVA}_B/\text{kVA}_A)$ . . (55)

$Z_B/Z_A = (\%\ V_{ZBf}/\%\ V_{ZAf}) \times (\text{kVA}_A/\text{kVA}_B)$ . . (56)

Example 1. Two transformers are connected in parallel to supply a load requiring 500 A. at 0·8 power factor (lagging), 400 V. The transformers have equal ratios of transformation and their equivalent impedances, referred to the secondary windings, are $2 + j3$ ohms and $2{\cdot}5 + j5$ ohms. Calculate the current and kVA. supplied by each transformer and the power factor at which it operates.

Let $Z_A = 2 + j3$, $Z_B = 2{\cdot}5 + j5$.

Then $Z_A + Z_B = 4{\cdot}5 + j8$.

Whence, from equations (53), (54),

$$I_A = \frac{IZ_A}{Z_A + Z_B} = \frac{500(0{\cdot}8 - j0{\cdot}6)\,(2{\cdot}5 + j5)\,(4{\cdot}5 - j8)}{4{\cdot}5^2 + 8^2}$$
$$= 252 - j170{\cdot}6$$

$\therefore$ $I_A = \sqrt{(252^2 + 170{\cdot}6^2)} = 304$ A.

$\cos \varphi_A = 252/304 = 0{\cdot}829$

$$I_B = I\frac{Z_A}{Z_A + Z_B} = \frac{500(0{\cdot}8 - j0{\cdot}6)\,(2 + j3)\,(4{\cdot}5 - j8)}{4{\cdot}5^2 + 8^2}$$
$$= 148 - j129{\cdot}4$$

$\therefore$ $I_B = \sqrt{(148^2 + 129{\cdot}4^2)} = 197$ A.

$\cos \varphi_B = 148/197 = 0{\cdot}75.$

Example 2. Two transformers, A and B, of equal ratio and rating, share a load of 200 kW. at 0·85 power factor (lagging). At full load the equivalent voltage drop in transformer A due to resistance is 1 per cent of the normal terminal voltage, and that due to reactance is 6 per cent. The corresponding values for transformer B are 2 per cent and 5 per cent. Determine the load on each transformer.

Now,
$$\frac{\dot{Z}_A}{\dot{Z}_B} = \frac{1 + j6}{2 + j5} = 1{\cdot}1 + j0{\cdot}241$$
$$\frac{\dot{Z}_B}{\dot{Z}_A} = \frac{2 + j5}{1 + j6} = 0{\cdot}865 - j0{\cdot}189$$
$$\text{kVA} = 200/0{\cdot}85 = 235$$
$$\text{k}\dot{\text{V}}\text{A} = 235(0{\cdot}85 - j0{\cdot}527) = 200 - j123{\cdot}8$$

Whence
$$\text{k}\dot{\text{V}}\text{A}_A = \frac{\text{k}\dot{\text{V}}\text{A}}{1 + \dot{Z}_A/\dot{Z}_B} = \frac{200 - j123{\cdot}8}{2{\cdot}1 + j0{\cdot}241} = 87{\cdot}6 - j69$$

$\therefore$
$$\text{kVA}_A = \sqrt{(87{\cdot}6^2 + 69^2)} = 111$$
$$\cos\varphi_A = 87{\cdot}6/111 = 0{\cdot}79$$
$$\text{k}\dot{\text{V}}\text{A}_B = \frac{\text{k}\dot{\text{V}}\text{A}}{1 + \dot{Z}_B/\dot{Z}_A} = \frac{200 - j123{\cdot}8}{1{\cdot}865 - j0{\cdot}189} = 112{\cdot}4 - j55$$

$\therefore$
$$\text{kVA}_B = \sqrt{(112{\cdot}4^2 + 55^2)} = 125$$
$$\cos\varphi_B = 112{\cdot}4/125 = 0{\cdot}9$$

Example 3. A 750-kVA. and a 500-kVA. transformer are connected in parallel to supply a load of 1000 kVA. at 0·8 power factor (lagging). At the rated kVA. the equivalent voltage drops in the transformers due to resistance and reactance are 3 per cent and 5 per cent respectively for the 750 kVA. transformer, and 2 per cent and 4 per cent respectively for the 500 kVA. transformer. Calculate the loading of each transformer.

Denoting the 750 kVA. transformer by A and the 500 kVA. transformer by B, we have, from equations (55), (56),

$$\frac{\dot{Z}_A}{\dot{Z}_B} = \left(\frac{500}{750}\right)\left(\frac{3 + j5}{2 + j4}\right) = 0{\cdot}867 - j0{\cdot}0667$$
$$\dot{Z}_B/\dot{Z}_A = 1{\cdot}147 + j0{\cdot}0883$$

Whence
$$\text{k}\dot{\text{V}}\text{A}_A = \frac{\text{k}\dot{\text{V}}\text{A}}{1 + \dot{Z}_A/\dot{Z}_B} = \frac{1000(0{\cdot}8 - j0{\cdot}6)}{1{\cdot}867 - j0{\cdot}0667} = 440 - j305$$

$\therefore$
$$\text{kVA}_A = \sqrt{(440^2 + 305^2)} = 535$$
$$\cos\varphi_A = 440/535 = 0{\cdot}822$$
$$\text{k}\dot{\text{V}}\text{A}_B = \frac{\text{k}\dot{\text{V}}\text{A}}{1 + \dot{Z}_A/\dot{Z}_B} = \frac{1000(0{\cdot}8 - j0{\cdot}6)}{2{\cdot}147 + j0{\cdot}0883} = 360 - j295$$

$\therefore$
$$\text{kVA}_B = \sqrt{(360^2 + 295^2)} = 465$$
$$\cos\varphi_B = 360/465 = 0{\cdot}774$$

CHAPTER IX

POLYPHASE CURRENTS AND SYSTEMS

Distinction between Single-phase and Polyphase Systems. Up to the present we have dealt exclusively with circuits supplied from systems having only two line wires between which an alternating E.M.F. was maintained. Such systems are called *single-phase* systems, and in this country they are employed chiefly for lighting and small-power industrial applications. But with large-power industrial applications, and when conversion to direct current is required, polyphase systems are employed because of the advantages which they possess for these purposes over single-phase systems.

A *polyphase* system is so called because a number of E.M.Fs. of the same magnitude and frequency, *but not of the same phase,* are produced by the generator and are utilized by the loads. In general, an n-phase system has n E.M.Fs., with mutual phase differences of $2\pi/n$ radians, and requires a minimum of n line wires.

Summary of Advantages of Polyphase Systems. The principal advantages of polyphase systems compared with single-phase systems are—

(1) A polyphase alternator is smaller and less costly than a single-phase alternator of the same output, voltage, and frequency, because in the former case the armature periphery may be utilized more effectively than is possible in the latter case. Moreover, in the polyphase alternator the armature reaction, with balanced loads, is constant, whereas in the single-phase alternator the armature reaction is pulsating, and this feature requires a more costly construction in the single-phase alternator than in the polyphase machine (see p. 243).

(2) Polyphase transmission requires less weight of copper in the line conductors than single-phase transmission (see p. 227).

(3) With polyphase currents rotating magnetic fields may be produced by means of stationary coils (see p. 181).

(4) The power in a polyphase balanced system is constant and non-pulsating (see p. 180).

(5) Polyphase motors and converting machinery have a higher efficiency and better performance than single-phase machines, these features being due to items (3) and (4).

The Simplest Form of Polyphase Alternator. In Chapter I, when considering the simple alternator, it was shown that if the conductors

forming the coil-sides of the rotating coil were distributed over the surface of the supporting cylinder, the E.M.Fs. generated in the several turns were not in phase with one another (see Figs. 6, 7). Hence, if a number of coils, displaced from one another, are arranged on the cylinder and each coil is provided with slip rings and brush gear, as shown in Figs. 99 (*a*) and 100 (*a*)—which show the arrangement for two and three coils respectively—we shall have the simplest form of polyphase alternator.

The mutual phase differences between the several E.M.Fs. are equal to the mutual angular displacements, in electrical degrees, between the respective coils. For example, with two coils fixed 90 electrical degrees apart [Fig. 99 (*a*)] the phase differences between the E.M.Fs. will be 90 degrees, and with three coils fixed 120 electrical degrees apart [Fig. 100 (*a*)] the phase differences will be 120 degrees.

Assuming the coils to be rotating with constant angular velocity in a magnetic field of uniform density, the E.M.Fs. will vary as a sine function of the time, and may be represented by the equations

$$e_1 = E_{1m} \sin \omega t, \qquad e_2 = E_{2m} \sin(\omega t - \tfrac{1}{2}\pi),$$

in the former case [Fig. 99 (*a*)]; and

$$e_1 = E_{1m} \sin \omega t, \; e_2 = E_{2m} \sin(\omega t - \tfrac{2}{3}\pi), \; e_3 = E_{3m} \sin(\omega t - \tfrac{4}{3}\pi)$$

in the latter case.

Graphical representations of the E.M.F. equations are given, in rectangular co-ordinates, in Figs. 99 (*b*), 100 (*b*); and vector diagrams showing the R.M.S. values of the several E.M.Fs. are given in Figs. 99 (*c*), 100 (*c*).

Conventional methods of representing the alternators in circuit diagrams are shown in Figs. 99 (*d*), 100 (*d*).

Two- and Three-phase Systems. If each coil of the alternators of Figs. 99, 100, is connected to a separate circuit as represented by the conventional diagrams in Figs. 101, 102, the currents, assuming these circuits to be similar, will have phase differences of 90 degrees in the former case and 120 degrees in the latter case. Each of these combinations constitutes a polyphase system, the former, Fig. 101, being called a *two-phase system*, and the latter, Fig. 102, a *three-phase system*.

Phases of a Polyphase System. In the present case the term "phase" refers to the separate circuits—forming part of, or being supplied by, a single alternator (or other source of electric power)—in which a phase difference exists between the generated, or impressed E.M.Fs. associated with the respective circuits. Thus

the circuits of a polyphase system are called the "phases" of the system.

Symmetrical Systems. A polyphase system is symmetrical when

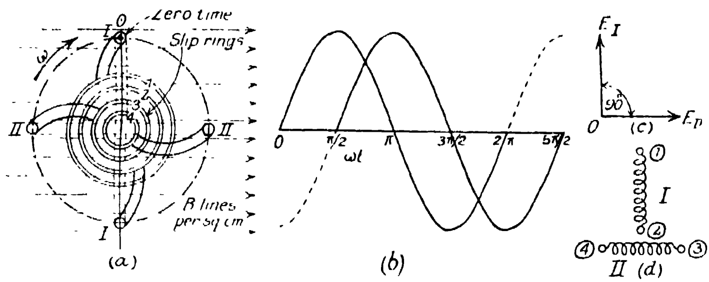


FIG. 99. SIMPLEST FORM OF TWO-PHASE ALTERNATOR

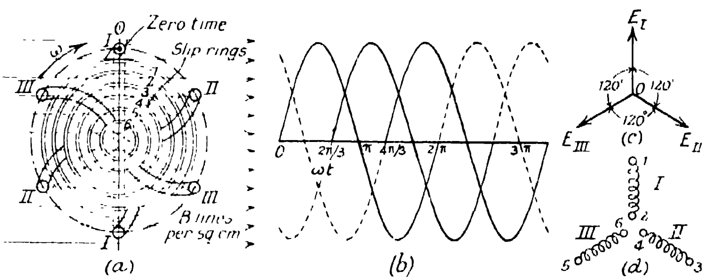


FIG. 100. SIMPLEST FORM OF THREE-PHASE ALTERNATOR

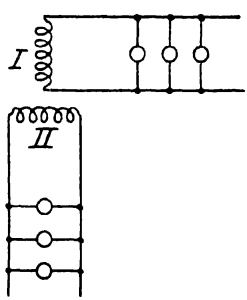


FIG. 101

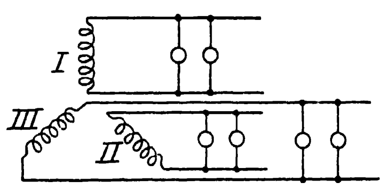


FIG. 102

CIRCUIT CONNECTIONS OF ALTERNATORS OF FIGS. 99 AND 100

the several E.M.Fs., of the same frequency, have equal maximum values and are displaced from one another by equal time angles. Thus in a symmetrical n-phase system the n E.M.Fs. are displaced

from one another by $1/n$th of a period. The E.M.Fs., if sinusoidal are represented by the equations

$$e_1 = E_m \sin \omega t, \; e_2 = E_m \sin\left(\omega t - \frac{2\pi}{n}\right), \; e_3 = E_m \sin\left(\omega t - 2\left[\frac{2\pi}{n}\right]\right),$$

$$\ldots \; e_n = E_m \sin\left[\omega t - (n-1)\frac{2\pi}{n}\right]$$

The E.M.F. vectors, therefore, form a regular closed polygon, and their vector sum is zero. Moreover, the algebraic sum of the instantaneous E.M.Fs. is zero at every instant. For example, with a three-phase system ($n = 3$) we have

$$e_1 = E_m \sin \omega t, \quad e_2 = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_3 = E_m \sin(\omega t - \tfrac{4}{3}\pi);$$

whence

$$\begin{aligned} e_1 + e_2 + e_3 &= E_m[\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin \omega t + 2\sin(\omega t - \pi) \,.\, \cos \tfrac{1}{3}\pi] \\ &= 0. \end{aligned}$$

The principal symmetrical systems in practical use are the three-phase and six-phase systems, of which the latter is used almost entirely in connection with converting machinery and apparatus, and is obtained from a three-phase system.

Balanced System. A polyphase system is balanced when the loads on the several circuits, or phases, are equal and have the same power factor. Under these conditions the instantaneous power in the system as a whole is constant, notwithstanding that the power in each phase is pulsating. This feature gives polyphase systems a great advantage over a single-phase system for the supply of power to motors and converting machinery.

Interconnection of the Phases of a Polyphase System. If the phases of a polyphase system supply separate circuits, as in Figs. 101, 102, then a pair of line wires is required for each circuit. But by suitably interconnecting the phases the number of line wires can be reduced without affecting the operation of the system Thus the three-phase six-wire system of Fig. 102 can be reduced. to one having only three line wires. Similarly, four- and six-phase systems can be operated with four and six-line wires respectively, and, in general, any symmetrical n-phase system can be operated with n line wires.

The interconnection must be carried out in such a manner that if closed circuits are formed the sum of the instantaneous E.M.Fs. in them must be zero in order that there shall be no circulating currents in these circuits.

The principal methods of interconnection are (1) the star connection, (2) the mesh, or ring, connection. These are shown diagrammatically in Figs. 103, 104.

Star Connection of Polyphase Systems. The star connection, Fig. 103, is formed by joining one end of each phase to a common point—which is called the "neutral point"—and connecting the other ends to the line wires. In making this connection for a generator, transformer, motor or apparatus in which the electric circuits are interlinked with magnetic circuits, the line wires must be connected to the terminals, or the ends of the phases, which have

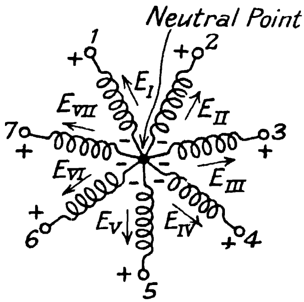


FIG. 103

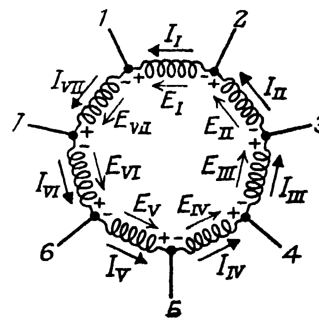


FIG. 104

STAR- AND MESH-CONNECTIONS OF POLYPHASE SYSTEM

like polarity at successive instants, as if this condition is not fulfilled the interconnected system will be unsymmetrical. [Examples of dissymmetry due to incorrect connections are, for the three-phase system, given on p. 189.]

Hence in these cases *similar ends* (i.e. either "starting" or "finishing" ends) of the phases—assuming the coils to be wound in the same direction and connected in the same manner—must be connected together to form the neutral point. For example, with the simple three-phase alternator, shown in Fig. 100, the neutral point may be formed either, as shown in Fig. 105, by connecting together the coil-ends which were originally connected to slip rings 2, 4, 6, in which case the line wires are connected to slip rings 1, 3, 5; or, as shown in the alternative method of Fig. 106, by connecting together the coil-ends which were originally connected to slip rings 1, 3, 5, in which case the line wires are connected to slip rings 2, 4, 6.

In the case of load circuits containing resistance or reactance, the particular ends of the circuits which must be connected together to form the neutral point are immaterial, provided that, with

inductively-reactive circuits the several magnetic circuits are not interlinked magnetically. If the magnetic circuits are interlinked, however, as in the case of the three-phase reactance, or choking

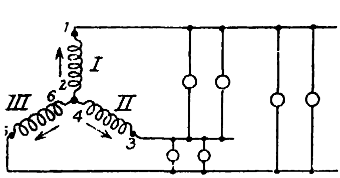


FIG. 105

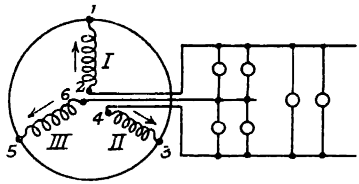


FIG. 106

METHODS OF INTERCONNECTING THE THREE-PHASE ALTERNATOR OF FIG. 102, TO OBTAIN THREE LINE WIRES

coil, shown in Fig. 107, then only similar ends of the coils may be connected together.

Line E.M.F. and Current Relationship for Star Connection. The magnitude of the E.M.F. between any pair of line wires, or terminals, of a star-connected polyphase alternator is given simply by the vector difference between the E.M.Fs. of the phases to which these lines are connected, since in making the star connection of the windings the ends of like instantaneous polarity were connected together. In the case of an unloaded n-phase symmetrical system with sinusoidal E.M.Fs. each of the line voltages is equal to

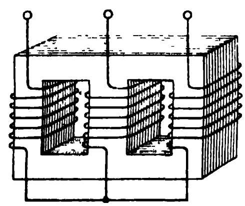


FIG. 107. CONNECTIONS OF THREE-PHASE, STAR-CONNECTED CHOKING COIL

$$2\sin(\pi/n) \times \text{phase voltage},$$

and leads the corresponding phase voltage by $[\frac{1}{2}(\pi - 2\pi/n)]$ radians, or $(90 - 180/n)$ degrees, i.e. the voltage between the lines connected to, say, phases 1 and 2 is $(90 - 180/n)$ degrees in advance of the voltage of phase 1.

Proof. Taking the positive direction of the E.M.F. generated in each phase to be outwards, or away from the neutral point, as indicated in Fig. 103, the R.M.S. values E_1, E_2, of the E.M.Fs. in two adjacent phases are represented in the vector diagram of Fig. 108, by the vectors OA, OB. The E.M.F. between the terminals of these phases is given by the difference of the vectors OA, OB, i.e. by the vector OC.

Now in a symmetrical system $E_1 = E_2 = E$, say. Hence, in Fig. 108, $OA = OB$, and $OC = AB$.

If AB is bisected at F and this point is joined to O, the line OF is perpendicular to AB and bisects the angle between OA and OB. Therefore, $AB = 2AF = 2OA \sin \frac{1}{2}(2\pi/n)$. Whence $OC = \underline{E}_1 - \underline{E}_2 = 2E\sin(\pi/n) = 2\sin(\pi/n) \times$ phase E.M.F. The angle between the vectors OC and OA is $[\frac{1}{2}\pi - \frac{1}{2}(2\pi/n)]$ radians, or $(90 - 180/n)°$.

The current in any line is equal to the current in the phase to which the line is connected.

Mesh, or Ring-connected Polyphase System. The mesh, or ring, connection (Fig. 104) is formed by joining the several phases in series to form a closed circuit and connecting the line wires to the junctions of the phases. In making this connection for a generator, transformer, motor, or apparatus in which the electric circuits are interlinked with magnetic circuits *dissimilar ends* of adjacent phases must be connected together. Hence the resultant E.M.F. acting in the closed circuit is equal to the vector sum of the several

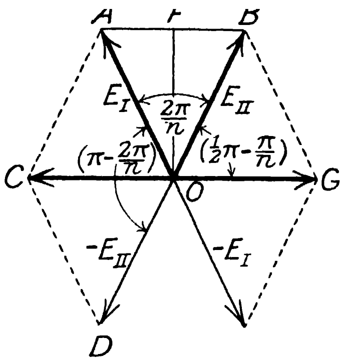


FIG. 108

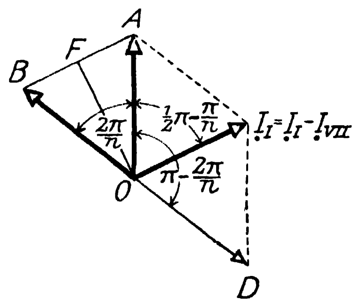


FIG. 109

VECTOR DIAGRAMS FOR STAR- AND MESH-CONNECTED POLYPHASE SYSTEMS

phase E.M.Fs. In the case of symmetrical systems with sinusoidal E.M.Fs., the resultant E.M.F. is zero, and therefore no circulating currents will flow in the closed circuit. With non-sinusoidal E.M.Fs. the resultant E.M.F. may not be zero, and in this case circulating currents will flow in the closed circuit.

Line E.M.F. and Current Relationship for Mesh Connection. The E.M.F. between two adjacent line wires is equal to the E.M.F. in the phase to which these line wires are connected.

The current in any line wire is equal to the vector difference of the currents in the phases connected to that line wire. In the case of a balanced n-phase system in which the currents are sinusoidal the line current is equal to

$$2 \sin(\pi/n) \times \text{phase current},$$

and leads the currents in the phases connected to that line wire by angles of $(\frac{1}{2}\pi + \pi/n)$ and $(\frac{1}{2}\pi - \pi/n)$ radians, as shown below.

Proof. Let the positive direction of the currents in the phases and line wires be that marked in the circuit diagram of Fig. 104. Then, assuming

balanced loads, the phase currents are represented, in the vector diagram of Fig. 109, by the vectors OA, OB, for the two phases under consideration. The current in the line wire connected to these phases is equal to $I_1 - I_2$, and is represented by the difference of the vectors OA and OB, i.e. by the vector OC.

Now for a symmetrical and balanced system $I_1 = I_2 = I$, say. Hence, in Fig. 109, $OA = OB$, and $OC = BA = 2\,OA \sin \frac{1}{2}(2\pi/n)$. Whence the line current $(= OC) = I_1 - I_2 = 2\,I \sin(\pi/n) = 2 \sin(\pi/n) \times$ phase current.

The angle between OC and OA—i.e. the phase difference between the current in line wire A, Fig. 109, and the current in phase I—is equal to $\frac{1}{2}(\pi - 2\pi/n) = (\frac{1}{2}\pi - \pi/n)$ radians, or $(90 - 180/)n^\circ$. Similarly, the angle between OC and OB is equal to $(\frac{1}{2}\pi + \pi/n)$ radians.

Power in a Balanced Polyphase System. The power in a balanced polyphase system is, at any instant, equal to the sum of the instantaneous power in the separate phases. Thus if in an n–phase system the several E.M.Fs. are given by the equations

$$e_1 = E_m \sin \omega t,\quad e_2 = E_m \sin(\omega t - 2\pi/n),\quad e_3 = E_m \sin(\omega t - 2(2\pi/n)),$$
$$\ldots\ e_n = E_m \sin(\omega t - 2\pi(n-1)/n),$$

and the currents are given by the equations

$$i_1 = I_m \sin(\omega t - \varphi),\quad i_2 = I_m \sin(\omega t - \varphi - 2\pi/n),$$
$$i_3 = I_m \sin(\omega t - \varphi - 2(2\pi/n)),\ \ldots$$

the instantaneous power is given by

$$\begin{aligned}
p &= e_1 i_1 + e_2 i_2 + e_3 i_3 + \ldots + e_n i_n \\
&= E_m I_m [\sin \omega t \,.\, \sin(\omega t - \varphi) + \sin(\omega t - 2\pi/n) \,.\, \sin(\omega t - \varphi - 2\pi/n) \\
&\qquad + \sin(\omega t - 4\pi/n) \,.\, \sin(\omega t - \varphi - 4\pi/n) + \ldots] \\
&= E_m I_m \tfrac{1}{2} [\cos \varphi - \cos(2\omega t - \varphi) + \cos \varphi - \cos(2\omega t - \varphi - 4\pi/n) \\
&\qquad + \cos \varphi - \cos(2\omega t - \varphi - 8\pi/n) + \ldots] \\
p &= E_m I_m \tfrac{1}{2} [n \cos \varphi - \cos(2\omega t - \varphi) - \{\cos(2\omega t - \varphi) \,.\, \cos 4\pi/n \\
&\qquad + \sin(2\omega t - \varphi) \,.\, \sin 4\pi/n\} - \{\cos(2\omega t - \varphi) \,.\, \cos 8\pi/n \\
&\qquad + \sin(2\omega t - \varphi) \,.\, \sin 8\pi/n\} - \ldots] \\
&= \tfrac{1}{2} E_m I_m [n \cos \varphi - \cos(2\omega t - \varphi) \{1 + \cos 4\pi/n + \cos 8\pi/n + .\} \\
&\qquad \sin(2\omega t - \varphi) \{\sin 4\pi/n + \sin 8\pi/n + \ldots\}] \\
&= \tfrac{1}{2} n E_m I_m \cos \varphi \\
&= n\,E\,I \cos \varphi \,. \qquad \qquad (57)
\end{aligned}$$

[Note.—$1 + \cos 4\pi/n + \cos 8\pi/n + \cos 12\pi/n + \ldots = 0$,

$\sin 4\pi/n + \sin 8\pi/n + \sin 12\pi/n + \ldots = 0$,

n being an integer.]

Hence the instantaneous power is expressed in terms of the

constant quantities E, I, n, $\cos\varphi$. Therefore the total power in any polyphase system with balanced loads is entirely free from pulsation.

Production of Rotating Magnetic Fields by Polyphase Currents. In addition to the above advantage a symmetrical polyphase system possesses the further advantage that a uniformly rotating magnetic field, of constant magnitude, may be produced by means of *stationary* coils.

The general conditions to be satisfied for the production of a bi-polar magnetic field from an n-phase system are (1) the number

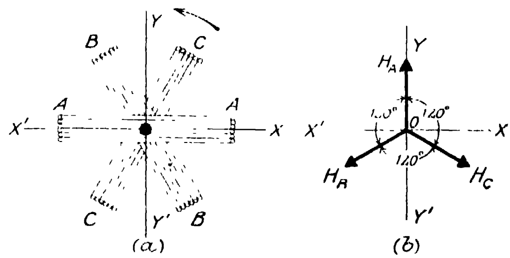


FIG. 110. SPACE AND VECTOR DIAGRAMS FOR ANALYTICAL PROOF OF THEOREM OF ROTATING MAGNETIC FIELD

of magnetizing coils must equal the number of phases, (2) these coils must be similar and must be arranged symmetrically with respect to a central axis, and (3) the angle between their magnetic axes, taken in order, must equal the mutual phase difference between the currents supplied to the coils (i.e. $360/n$ degrees).

The speed of rotation of the magnetic field is equal to the angular velocity of the supply currents, i.e. the speed in revolutions per second is equal to the frequency of the supply currents.

When a multipolar magnetic field is required there must be n similar coils per pair of poles, and the angle, measured in degrees, between the magnetic axes of the successive coils must equal $(360/\frac{1}{2}np)$, where p is the number of poles. In this case the speed of the field, in revolutions per second, is equal to $f/\frac{1}{2}p$, where f is the frequency of the supply currents.

In all cases the magnitude of the rotating field is given by

$$H = \tfrac{1}{2}nH_m,$$

where H_m is the maximum magneto-motive-force due to one coil.

Analytical Proof. Consider three coils arranged, as in Fig. 110, with their magnetic axes 120° apart. Let these coils be supplied from a symmetrical three-phase system, and let the instantaneous values of their M.M.Fs. be represented by the equations

$$h_A = H_m \sin\omega t, \qquad h_B = H_m \sin(\omega t - \tfrac{2}{3}\pi), \qquad h_C = H_m \sin(\omega t - \tfrac{4}{3}\pi).$$

These M.M.Fs. have directions in space along the magnetic axes of the respective coils and they are each alternating at the frequency of the supply currents.

The value of the resultant M.M.F., in space, due to the joint action of the coils may, at any particular instant, be obtained from a knowledge of their components, at this instant, along two arbitrary axes perpendicular to each other.

Thus, taking one axis (Y) along the magnetic axis of coil A, and measuring space angles in the counter-clockwise direction from the positive or right-hand horizontal axis, we have for the sum of the components of M.M.Fs. along the vertical axis at the instant t

$$\begin{aligned}
h_Y &= h_A \sin 90^\circ + h_B \sin 210^\circ + h_C \sin 330^\circ \\
&= H_m[\sin \omega t - \tfrac{1}{2}(\sin(\omega t - \tfrac{2}{3}\pi) - \tfrac{1}{2}\sin(\omega t - \tfrac{4}{3}\pi)] \\
&= H_m[\sin \omega t - \tfrac{1}{2}(\sin \omega t.\cos \tfrac{2}{3}\pi - \cos \omega t.\sin \tfrac{2}{3}\pi) \\
&\qquad - \tfrac{1}{2}(\sin \omega t.\cos \tfrac{4}{3}\pi - \cos \omega t.\sin \tfrac{4}{3}\pi)] \\
&= H_m\left[\sin \omega t - \frac{1}{2}\left(-\frac{1}{2}\sin \omega t - \frac{\sqrt{3}}{2}\cos \omega t\right) - \frac{1}{2}\left(-\frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t\right)\right] \\
&= \tfrac{3}{2}H_m \sin \omega t.
\end{aligned}$$

Similarly, the sum of the components of M.M.Fs. along the perpendicular axis (X) at the instant t are

$$\begin{aligned}
h_X &= h_A \cos 90^\circ + h_B \cos 210^\circ + h_C \cos 330^\circ \\
&= H_m\left[-\frac{\sqrt{3}}{2}\sin\left(\omega t - \frac{2}{3}\pi\right) + \frac{\sqrt{3}}{2}\sin\left(\omega t - \frac{4}{3}\pi\right)\right] \\
&= \frac{\sqrt{3}}{2}H_m\left[-\left(\sin \omega t.\cos \frac{2}{3}\pi - \cos \omega t.\sin \frac{2}{3}\pi\right) + \left(\sin \omega t.\cos \frac{4}{3}\pi\right.\right. \\
&\qquad\qquad\left.\left. - \cos \omega t.\sin \frac{4}{3}\pi\right)\right] \\
&= \frac{\sqrt{3}}{2}H_m\left(+\frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t - \frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t\right) \\
&= +\tfrac{3}{2}H_m \cos \omega t
\end{aligned}$$

Therefore the magnitude of the resultant M.M.F. at the instant t is given by

$$\begin{aligned}
h &= \sqrt{(h_X^2 + h_Y^2)} = \sqrt{\{(\tfrac{3}{2}H_m \sin \omega t)^2 + (\tfrac{3}{2}H_m \cos \omega t)^2\}} \\
&= \tfrac{3}{2}H_m.
\end{aligned}$$

i.e. the resultant M.M.F. is constant in magnitude and is equal to 3/2 $\times$ maximum M.M.F. of one coil.

Let θ represent the space angle which the resultant M.M.F. makes with the X axis at the time t.

Then $$\tan \theta = \frac{h_Y}{h_X} = \frac{\tfrac{3}{2}H_m \sin \omega t}{\tfrac{3}{2}H_m \cos \omega t} = \tan \omega t,$$

i.e. $$\theta = \omega t.$$

Therefore the magnetic field rotates in the counter-clockwise direction with an angular velocity of ω, and the number of revolutions per second is equal to the frequency of the supply currents.

Graphical proof. The construction for the graphical proof of the theorem is extremely simple for the case of two coils supplied with current from a two-phase system, the coils being arranged about a common centre with their magnetic axes perpendicular to each other. We shall, however, give the construction for the three-phase case.

First, a time diagram is required showing the instantaneous values of the M.M.Fs. of the coils at successive instants of the period. This diagram may be drawn either in rectangular co-ordinates, as shown in Fig. 111 (*a*), or in polar co-ordinates, as shown in Fig. 111 (*b*).

Second, a space vector diagram is required showing the directions of these M.M.Fs., as well as their magnitudes, in space.

The space vector diagram for the three-phase case under consideration is shown in Fig. 111 (*c*), and contains three axes of reference, OX, OY, OZ, having a mutual displacement of 120°.

To obtain the direction and magnitude of the resultant M.M.F. at successive instants, the instantaneous values of the M.M.Fs. of the coils are obtained

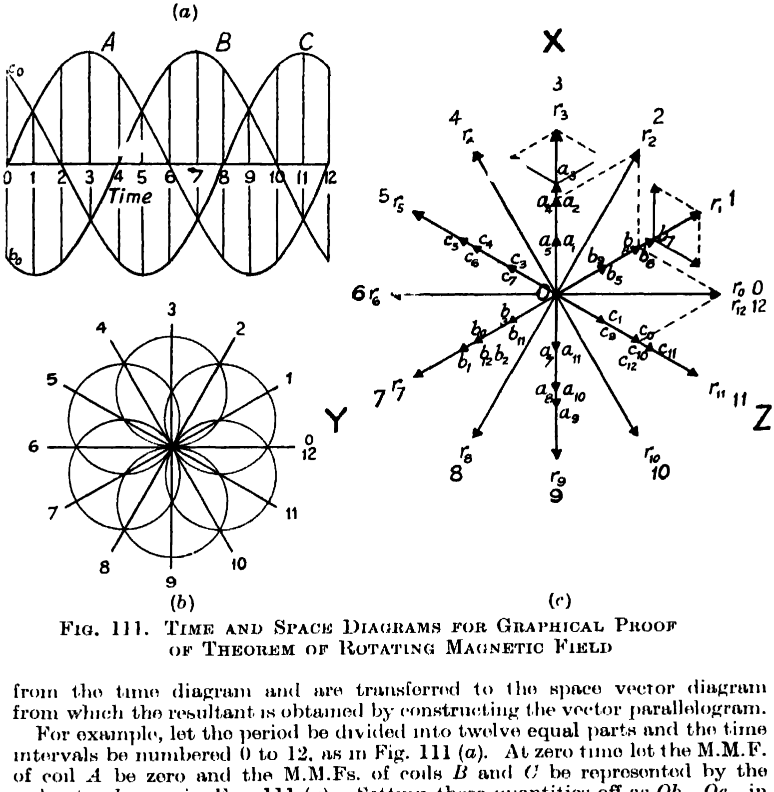


FIG. 111. TIME AND SPACE DIAGRAMS FOR GRAPHICAL PROOF OF THEOREM OF ROTATING MAGNETIC FIELD

from the time diagram and are transferred to the space vector diagram from which the resultant is obtained by constructing the vector parallelogram.

For example, let the period be divided into twelve equal parts and the time intervals be numbered 0 to 12, as in Fig. 111 (*a*). At zero time let the M.M.F. of coil A be zero and the M.M.Fs. of coils B and C be represented by the ordinates b_0, c_0, in Fig. 111 (*a*). Setting these quantities off as Ob_0, Oc_0, in the space vector diagram of Fig. 111 (*c*), we obtain, by constructing the vector parallelogram, the direction and magnitude of the resultant Or_0. Since $b_0 = -0{\cdot}866\,H_m$, and $c_0 = 0{\cdot}866\,H_m$, the resultant Or_0 is equal to

$$2 \times 0{\cdot}866\,H_m \cos 30° = \tfrac{3}{2}\,H_m$$

When one-twelfth of a period has elapsed the M.M.Fs. of the coils are represented in the time diagram by the ordinates a_1, b_1, c_1; and when these quantities are set off in their correct positions in the space vector diagram we obtain the resultant Or_1. Since $a_1 = \tfrac{1}{2}H_m$, $b_1 = -H_m$, $c_1 = \tfrac{1}{2}H_m$, the resultant Or_1 is equal to $H_m + 2 \times \tfrac{1}{2}H_m \cos 60° = \tfrac{3}{2}H_m$. Moreover, the angle between Or_0 and Or_1 is 30°, which is equal to the time angle (i.e. $\tfrac{1}{12} \times 360°$) between the points 0 and 1 in the time diagram. Similar results will be obtained for other points of the period.

Hence the magnitude of the resultant M.M.F. is constant and is equal to $\frac{3}{2} H_m$.

The locus of the resultant M.M.F. vector is a circle, and the radius vector makes one revolution during each period of the supply current.

Application of Polyphase Systems. Of the various polyphase systems available the three-phase system is the one most extensively employed at the present day, firstly, because this system requires the minimum number (three) of line conductors of any polyphase system, and, secondly, because it is possible, by means of *stationary* transformers, to obtain other polyphase systems—such as the two, six, nine, and twelve-phase systems—from a three-phase system.

COMMERCIAL POLYPHASE SYSTEMS

We will now consider the three-phase and six-phase systems more in detail.

THREE-PHASE SYSTEM

In a symmetrical system the phase E.M.Fs. are represented by the equations

$$e_{\text{I}} = E_m \sin \omega t, \quad e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\text{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

Star-connected System. The circuit diagram for a star-connected system is given in Fig. 112, in which the assumed positive directions for E.M.Fs. and currents are indicated by arrows. The instantaneous values of the terminal, or line, E.M.Fs. are therefore given by the equations

$$\begin{aligned} v_{1-2} &= e_{\text{I}} - e_{\text{II}} \\ &= E_m[\sin \omega t - \sin(\omega t - \tfrac{2}{3}\pi)] \\ &= 2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \tfrac{1}{3}\pi) \\ &= \sqrt{3} \,.\, E_m \cos(\omega t - \tfrac{1}{3}\pi) \\ &= \sqrt{3} \,.\, E_m \sin(\omega t + \tfrac{1}{6}\pi) \end{aligned}$$

$$\begin{aligned} v_{2-3} &= e_{\text{II}} - e_{\text{III}} \\ &= E_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= 2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \pi) \\ &= \sqrt{3} \,.\, E_m \cos(\omega t - \pi) \\ &= \sqrt{3} \,.\, E_m \sin(\omega t - \tfrac{1}{2}\pi) \end{aligned}$$

$$\begin{aligned} v_{3-1} &= e_{\text{III}} - e_{\text{I}} \\ &= E_m[\sin(\omega t - \tfrac{4}{3}\pi) - \sin \omega t] \\ &= 2E_m \sin \tfrac{1}{3}\pi \cos(\omega t - \tfrac{5}{3}\pi) \\ &= \sqrt{3} \,.\, E_m \cos(\omega t - \tfrac{5}{3}\pi) \\ &= \sqrt{3} \,.\, E_m \sin(\omega t - \tfrac{7}{6}\pi) \end{aligned}$$

Thus the line E.M.Fs. are equal to one another and have a mutual phase difference of 120 degrees, or $\frac{2}{3}\pi$ radians. Also, the line E.M.Fs. have a phase difference (leading) of 30 degrees, or $\frac{1}{6}\pi$ radians, with respect to the phase E.M.Fs. For example, the E.M.F. between lines 1 and 2 is 30 degrees in advance of the E.M.F. of phase I; that between lines 2 and 3 is 30 degrees in advance of the E.M.F. of phase II; and that between lines 3 and 1 is 30 degrees in advance of the E.M.F. of phase III.

The R.M.S. values of the line E.M.Fs. are

$$V_{1-2} = \sqrt{3} \,.\, E\,; \quad V_{2-3} = \sqrt{3} \,.\, E\,; \quad V_{3-1} = \sqrt{3} \,.\, E.$$

The vector diagram for a star-connected symmetrical system with balanced loads is shown in Fig. 113, in which the vectors OE_{I}, OE_{II}, OE_{III}, represent the phase E.M.Fs., and the vectors OV_{1-2}, OV_{2-3}, OV_{3-1}, represent the line E.M.Fs.; OV_{1-2} being the difference between the vectors OE_{I} and OE_{II}; OV_{2-3}, the

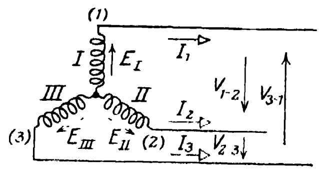


FIG. 112

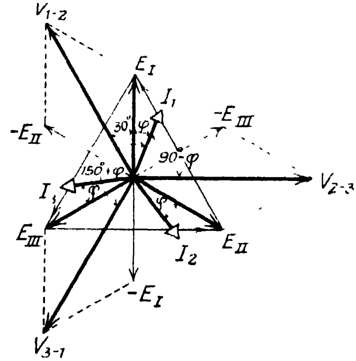


FIG. 113

CIRCUIT AND VECTOR DIAGRAMS OF THREE-PHASE, STAR-CONNECTED (THREE WIRE) SYSTEM

difference between the vectors OE_{II} and OE_{III} OV_{3-1}, the difference between the vectors OE_{III} and OE_{I}. These vector differences may also be represented by the sides of the triangle formed by joining the extremities of the vectors representing the phase E.M.Fs. For example, the side $E_{\text{II}}E_{\text{I}}$, taken in the direction $E_{\text{II}} - E_{\text{I}}$, is equal and parallel to OV_{1-2}. Similarly, the sides $E_{\text{III}}\,E_{\text{II}}$ and $E_{\text{I}}\,E_{\text{III}}$ are equal and parallel to the vectors OV_{2-3} and OV_{3-1} respectively. Hence in a star-connected system the line E.M.Fs. may be represented by the triangle (or polygon, when the number of phases exceeds three) formed by joining the extremities of the vectors representing the phase E.M.Fs.

The line and phase currents are shown, in Fig. 113, by the vectors OI_1, OI_2, OI_3, each of which has a phase difference of φ with respect to the phase E.M.Fs., φ being the power factor of each of the balanced loads. Hence the phase difference between the current in any line and the voltage across adjacent pairs of line wires may be $(30 + \varphi)^\circ$, $(150 + \varphi)^\circ$, or $(90 - \varphi)^\circ$, according to the particular phase and line wires considered. For example, the phase difference between the current in line 1 and the voltage across lines 1 and 2 is

$(30 + \varphi)°$ lagging; that between the current in line 1 and the voltage across lines 3 and 1 is $(150 + \varphi)°$ lagging; but that between the current in line 1 and the voltage across lines 2 and 3 is $(90 - \varphi)°$ leading, the appropriate line E.M.F. vector being considered as the vector of reference in each case. These phase differences are shown clearly in the vector diagram, which has been drawn for φ lagging. If φ is leading the phase differences become $(30 - \varphi)°$, $(150 - \varphi)$, $(90 + \varphi)°$.

Star-connected System with Neutral Wire. This system, which is shown diagrammatically in Fig. 114 and is called the *three-phase four-wire system*, is used in cases where unbalanced loads have to be supplied from a three-phase system. The principal application of the system in practice is for supplying single-phase lighting networks from a three-phase system which also supplies a power load; and its advantage over the ordinary three-phase star-connected system without neutral wire (which is usually called the *three-phase three-wire system*) is that the power load may be supplied at a higher voltage than the lighting load, since the latter is supplied at the "phase" voltage, while the former may be supplied at the "line" voltage of the system. For example, if the lighting load is supplied at a pressure of 230 volts, the power load may be supplied at a pressure of $230 \times \sqrt{3} = 400$ volts.

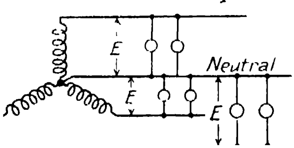


FIG. 114. CIRCUIT DIAGRAM OF THREE-PHASE, FOUR-WIRE SYSTEM

The current in the neutral wire is equal to the reversed vector sum of the currents in the line wires. Thus when the single-phase loads are balanced there is no current in the neutral wire, since the vector sum of three equal quantities having a mutual phase difference 120° is zero. When, however, the single-phase loads are unbalanced the magnitude of the current in the neutral wire and its phase relations with respect to the line currents depend upon the manner in which the unbalanced loads are distributed on the system. Vector diagrams for a number of cases are shown in Fig. 115, and a worked example is given in Chapter XIX. This example shows also the manner in which the voltage drop in the neutral wire affects the symmetry and balance of the voltages across the loads.

Delta-connected System. The circuit diagram for a delta-connected* three-phase system is shown in Fig. 116 (*a*), in which the

* The mesh connection of three-phase circuits is usually called the "delta" connection, and is represented by the symbol Δ.

phases are drawn in the same relative positions as in the diagram (Fig. 112) for the star-connected system. A comparison of the diagrams will show what changes in connections are necessary to convert one system to the other.

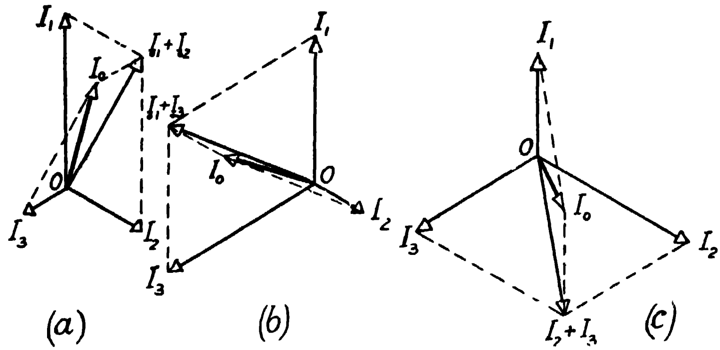


FIG. 115. VECTOR DIAGRAMS SHOWING METHOD OF DETERMINING CURRENT (I_0) IN NEUTRAL WIRE OF THREE-PHASE, FOUR-WIRE SYSTEM

OI_0 should be reversed in direction

The conventional circuit diagram for the delta-connected system is shown in Fig. 116 (*b*), this form of the circuit diagram being used in practice in preference to that of Fig. 116 (*a*). In both diagrams the assumed positive directions for E.M.Fs. and currents are indicated by arrows.

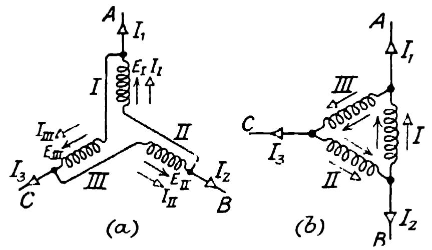


FIG. 116

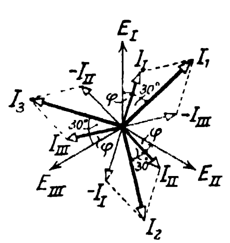


FIG. 117

CIRCUIT AND VECTOR DIAGRAMS OF THREE-PHASE, DELTA-CONNECTED SYSTEM

In the delta-connected three-phase system the line E.M.Fs. are equal to the phase E.M.Fs., and their instantaneous values are therefore represented by the equations

$$v_{1-2} = e_{\text{I}} = E_m \sin \omega t,$$
$$v_{2-3} = e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi),$$
$$v_{3-1} = e_{\text{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi).$$

Hence the R.M.S. values of the line E.M.Fs. are

$$V_{1-2} = E; \quad V_{2-3} = E; \quad V_{3-1} = E.$$

If the system is balanced the instantaneous values of the phase currents may be represented by the equations

$$i_{\mathrm{I}} = I_m \sin(\omega t - \varphi), \quad i_{\mathrm{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi), \quad i_{\mathrm{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$$

and the instantaneous values of the line currents will then be represented by the equations

$$\begin{aligned} i_1 = i_{\mathrm{I}} - i_{\mathrm{III}} &= I_m \sin(\omega t - \varphi) - I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi) \\ &= \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{6}\pi - \varphi) \\ i_2 = i_{\mathrm{II}} - i_{\mathrm{I}} &= I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi) - I_m \sin(\omega t - \varphi) \\ &= \sqrt{3}.I_m \sin(\omega t - \tfrac{5}{6}\pi - \varphi) \\ i_3 = i_{\mathrm{III}} - i_{\mathrm{II}} &= I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi) - I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi) \\ &= \sqrt{3}.I_m \sin(\omega t - \tfrac{9}{6}\pi - \varphi) \end{aligned}$$

Thus the line currents are equal to one another and have a mutual phase difference of 120°, or $\frac{2}{3}\pi$ radians. Also the current in any line has a phase difference, lagging, of 30°, or $\frac{1}{6}\pi$ radians, with respect to the current in the lagging phase connected to that line. For example, the current in line 1 (which is connected to the junction of phases I and III) has a phase difference of 30°, lagging, with respect to the current in phase I. Similarly, the current in line 2 has a phase difference of 30°, lagging, with respect to the current in phase II, and the current in line 3 has a phase difference of 30°, lagging, with respect to the current in phase III.

The R.M.S. values of the line currents are

$$I_1 = I\sqrt{3}; \quad I_2 = I\sqrt{3}; \quad I_3 = I\sqrt{3}.$$

The *vector diagram* for a delta-connected system with balanced loads is shown in Fig. 117, in which the vectors OE_{I}, OE_{II}, OE_{III} represent the phase E.M.Fs.—and also the E.M.Fs. between the line wires—OI_{I}, OI_{II}, OI_{III} represent the phase currents, each lagging φ with respect to the appropriate phase E.M.F., and OI_1, OI_2, OI_3 represent the line currents. Observe that the phase difference between the current in any line wire, e.g. 1, and the voltage across the adjacent line wires, e.g. 1 and 2, is $(30 + \varphi)°$, and is lagging with respect to the line voltage.

Therefore, the phase relations between line currents and line voltages are the same for star- and delta-connected systems; and in every symmetrical and balanced system the line-voltage vectors are displaced from the line-current vectors by an angle of $(30 \pm \varphi)°$, *φ being the*

phase difference between the E.M.F. and current in each phase, the plus sign to be employed when φ is lagging and the minus sign when φ is leading.

Conversion of a Balanced Star-connected Load into an Equivalent Mesh-connected Load and Vice Versa. In consequence of the above relationship between the line and phase currents and voltages, any symmetrical and balanced star-connected system may be replaced by an equivalent mesh-connected system, and *vice versa.* For example, a three-phase star-connected system in which the line voltage is V and the line current is I may be replaced by a delta-connected system in which the phase voltage is V and the phase current is $I/\sqrt{3}$.

Similarly a balanced star-connected load for which the impedance of each branch is equal to $Z\underline{/\varphi^\circ}$ may be replaced by an equivalent

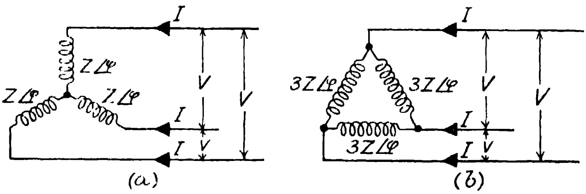


FIG. 118. EQUIVALENT STAR AND DELTA CIRCUITS

delta-connected load for which the impedance per phase is equal to $3Z\underline{/\varphi^\circ}$. These equivalent conditions are shown in the circuit diagrams of Fig. 118.

Proof. Let V = line voltage, I = line current, and Z_Y = impedance per phase for the star-connected system. Then the phase voltage is equal to $V/\sqrt{3}$, and the phase current is equal to I. Whence $Z_Y = V/(\sqrt{3}.I)$.

Now in the equivalent delta-connected system the line voltage and current must have the same values as in the star-connected system, and therefore we must have

$$\text{phase voltage} = V, \qquad \text{phase current} = I/\sqrt{3},$$

$$\text{impedance per phase} = Z_\Delta = V/(I/\sqrt{3}) = \sqrt{3}.V/I = 3Z_Y,$$

since $V/I = \sqrt{3}.Z_Y$.

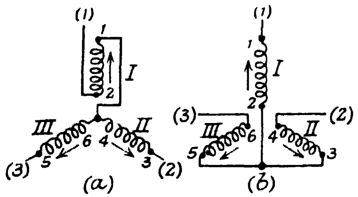
Moreover, as the phase difference between the line voltage and line current must be the same in both systems, we must have, therefore, the same phase difference between the phase voltages and currents. Hence $Z_\Delta\underline{/\varphi^\circ} = 3Z_Y\underline{/\varphi^\circ}$.

The cases of unbalanced loads are considered in Chapter XIX.

Incorrect Star Connections. Two cases of incorrect star connections are shown in Fig. 119, and the vector diagrams are shown in Fig. 120. In one case, Fig. 119 (*a*), one phase has been incorrectly connected, and in the other case, Fig. 119 (*b*), two phases have been incorrectly connected. In both cases the vector diagrams show that the line voltages are unbalanced and are not 120° apart.

Thus, with a symmetrical system the equality of the line voltages is an absolute check on the correctness of the connections.

Incorrect Delta Connections. Two cases of incorrect delta connections are shown in Fig. 121, and the vector diagrams are shown in Fig. 122. In both cases the resultant E.M.F. acting in the closed circuit is equal to *twice* the phase E.M.F. For example, if phase



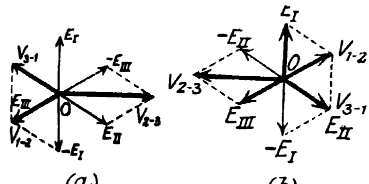
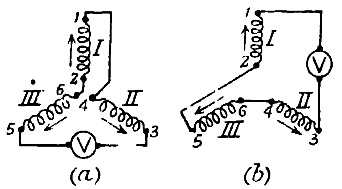


FIG. 119 FIG. 120

CIRCUIT AND VECTOR DIAGRAMS OF INCORRECT STAR CONNECTIONS



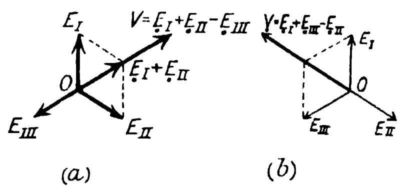


FIG. 122

CIRCUIT AND VECTOR DIAGRAMS OF INCORRECT DELTA CONNECTIONS

III is connected incorrectly, i.e. reversed, the resultant E.M.F. is given by

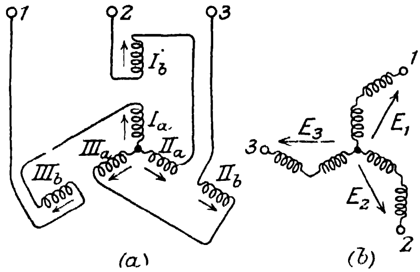
$$\begin{aligned} e = e_{\mathrm{I}} + e_{\mathrm{II}} - e_{\mathrm{III}} &= E_m[\sin\omega t + \sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin(\omega t - \tfrac{1}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin(\omega t - \tfrac{1}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi + \pi)] \\ &= 2E_m \sin(\omega t - \tfrac{1}{3}\pi). \end{aligned}$$

If phases II and III are reversed the resultant E.M.F. is given by

$$\begin{aligned} e = e_{\mathrm{I}} - e_{\mathrm{II}} - e_{\mathrm{III}} &= E_m[\sin\omega t - \sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= 2E_m \sin\omega t. \end{aligned}$$

Thus the correctness of the connections may be checked by leaving one of the junctions of the phases open and connecting a voltmeter between these phase ends. If the reading on the voltmeter is zero the phase ends to which the instrument is connected may be permanently connected together, but if the reading is equal to twice the phase E.M.F. then the connections are incorrect.

With non-sinusoidal wave-forms, however, a reading will be obtained on the voltmeter, even if the connections are correct, but this reading will usually be less than twice the phase E.M.F. In such cases the non-sinusoidal E.M.F. results in circulating currents in the closed circuit, and the value of this current will depend upon the magnitudes and frequencies of the harmonic components of the



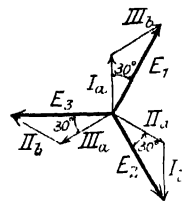


FIG. 123 FIG. 124

CIRCUIT AND VECTOR DIAGRAMS OF ZIGZAG CONNECTION OF THREE-PHASE SYSTEM

E.M.F. wave, as well as the impedances of the closed circuit at these frequencies. A worked example is given on page 309. On account of this feature the Δ-connection is not usually employed in connection with alternators.

Zigzag Star and Open-delta Connections of Three-phase Systems. These connections are modifications of the star and delta connections and are occasionally employed in connection with generators and transformers.*

The zigzag-star connection (called also the "inter-connected-star" connection) is shown diagrammatically in Fig. 123, and requires each phase of the generator, or transformer, to be wound in two equal sections, which are connected in the manner shown in Fig. 123 (*a*)

The vector diagram for this connection is shown in Fig. 124, from which it follows that if the E.M.F. of each half-section of a phase-winding is equal to $\frac{1}{2}E$, the phase voltage of the system is equal to $\sqrt{3}(\frac{1}{2}E) = 0{\cdot}866E$, and the line voltage is equal to $\sqrt{3}(\frac{1}{2}E.\sqrt{3}) = 1{\cdot}5E$. Therefore, with this connection, the line E.M.F. is only 86·6 per cent of that which would be obtained from the same machine with the normal star connection.

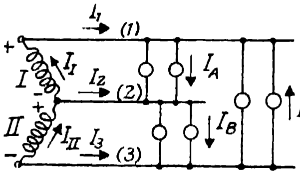
The vector diagram also shows that the vectors representing the phase voltages of the system have a phase difference of 30°, lagging, with respect to the vectors representing the induced E.M.Fs.; and that the vectors representing the line voltages are in phase with the latter. Hence if φ is the phase difference between the phase E.M.Fs. and currents, the phase difference between the line E.M.Fs. and currents will be $(30 + \varphi°)$, as in an ordinary three-phase system.

The zigzag-star connection therefore results in a reduction of the output of the alternator, or transformer, in which it is employed, and its use is restricted to cases where the ordinary star connection is inadmissible. For example, if the E.M.F. wave-form of an alternator is known to contain a component of triple frequency, this component may be eliminated from the phase E.M.F.

* For other connections which are suitable for instrument transformers, see the Author's *Power Wiring Diagrams*, Third edition (Pitman), p. 156.

by employing the zigzag-star connection, as the triple-frequency components of the E.M.Fs. in the sections of the phases which are connected in series neutralize each other, due to these sections being 120° apart and reversed relatively to each other.

The Open-delta Connection (called also the "V"-connection) is occasionally employed in connection with transformers, but it is never employed with alternators. It is obtained from the delta connection by removing one phase, as shown in the circuit diagram of Fig. 125. When equal E.M.Fs., having a phase difference of 120°, are induced in the remaining two phases,



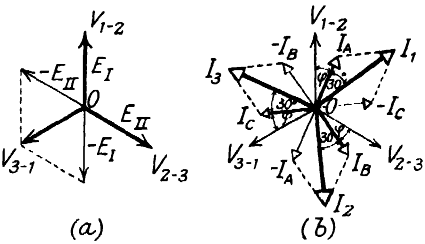


FIG. 125 FIG. 126

CIRCUIT AND VECTOR DIAGRAMS OF OPEN-DELTA OR V-CONNECTION OF THREE-PHASE SYSTEM

the E.M.Fs. between the line wires are equal and have a phase difference of 120°, as in a symmetrical three-phase system.

The vector diagram for this connection is shown in Fig. 126. The E.M.Fs. induced in phases 1 and II are represented by the vectors $OV_{1\text{-}2}$, and OV_2 [Fig. 126 (*a*)], which also represent the E.M.Fs. between lines 1 and 2, and 2 and 3, respectively. The E.M.F. between lines 3 and 1 is equal to the reversed vector sum of the E.M.Fs. induced in phases I and II, and is represented by the vector $OV_{3\text{-}1}$. Thus, the line E.M.Fs. are equal and have a mutual phase difference of 120°.

If a balanced load is connected to the line wires the line currents will be equal to one another and will have a mutual phase difference of 120°, as in a symmetrical three-phase system. The currents in the phase windings will be equal to the currents in the line wires. Thus, if I_A, I_B, I_C, are the load currents (see Fig. 125), the current, I_1, in line 1 is given by $I_1 = I_A - I_C$, that, I_2, in line 2 is given by $I_2 = I_B - I_A$; and that, I_3, in line 3 is given by $I_3 = I_C - I_B$. These currents are represented, in the vector diagram of Fig. 126(*b*), by the vectors OI_1, OI_2, OI_3; and the currents in each phase of the load are represented by the vectors OI_A, OI_B, OI_C.

The current in phase I of the transformer is equal to that in line 1, and is represented by the vector OI_1; but the current in phase II is equal to the vector sum of the currents in phase I and line 2, i.e. $I_{II} = I_1 + I_2 = -I_3$.

Hence the currents in the phases of the transformer have a phase difference of 60°, instead of 120° as in the normal delta connection. Moreover, the current in phase I has a phase difference of $(30 + \varphi)°$, lagging, with respect to the voltage between lines 1 and 2; but the current in phase II has a phase difference of $(30 - \varphi)°$, leading, with respect to the voltage between lines 2 and 3.

The total power supplied by the two phases is, therefore, equal to $VI\cos(30 + \varphi)° + VI\cos(30 - \varphi)° = \sqrt{3}.VI\cos\varphi$. Thus the power supplied by each phase is equal to $\frac{1}{2}(\sqrt{3}.VI\cos\varphi)$; whereas, with the normal delta connection (i.e. with three phases) and the same phase currents, the power supplied by each phase is equal to $VI\cos\varphi$. Hence the output per phase with the open-delta connection is only $\frac{1}{2}\sqrt{3} = 0{\cdot}866$ of that obtainable with the delta connection and same phase current. In consequence of this

reduced output the open-delta connection is employed only in cases where the saving of the cost of the third transformer is important, as in transformer-starting apparatus for polyphase motors. (See *Power Wiring Diagrams*, p. 104.) This connection, however, is useful in practice for obtaining a temporary (three-phase) supply from a group of three delta-connected transformers in the event of one transformer becoming defective.

Power in Three-phase Circuits. The power in a balanced polyphase system has already (p. 180) been shown to be equal to $nEI \cos \varphi$, where E, I, are the phase E.M.Fs. and currents, and φ is the phase difference between them. Hence in a balanced three-phase system the power is given by $P = 3EI \cos \varphi$.

In the case of a star-connected system the line voltage, V, is equal to $\sqrt{3}E$, and the power is given by $P = \sqrt{3} \,.\, VI \cos \varphi$. Similarly, with a delta-connected system the line current (I') is equal to $\sqrt{3} \,.\, I'$, and the power is given by $P = \sqrt{3} \,.\, EI' \cos \varphi$. Therefore, in general, the power in a balanced three-phase system is given by

$$\sqrt{3} \times \text{line voltage} \times \text{line current} \times \cos \varphi.$$

In the case of an *unbalanced system* the total power must be obtained by taking the sum of the powers in the separate phases. Thus if the instantaneous values of the phase E.M.Fs. are given by the equations

$$e_{\text{I}} = E_{\text{I}m} \sin \omega t, \quad e_{\text{II}} = E_{\text{II}m} \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\text{III}} = E_{\text{III}m} \sin(\omega t - \tfrac{4}{3}\pi)\,;$$

and the instantaneous values of the phase currents are given by

$$i_{\text{I}} = I_{\text{I}m} \sin(\omega t - \varphi_1), \quad i_{\text{II}} = I_{\text{II}m} \sin(\omega t - \tfrac{2}{3}\pi - \varphi_2), \quad i_{\text{III}} = I_{\text{III}m} \sin(\omega t - \tfrac{4}{3}\pi - \varphi_3)\,;$$

the instantaneous power will be given by

$$\begin{aligned}
p &= e_{\text{I}} i_{\text{I}} + e_{\text{II}} i_{\text{II}} + e_{\text{III}} i_{\text{III}} \\
&= E_{\text{I}m} I_{\text{I}m} \sin \omega t \,.\, \sin(\omega t - \varphi_1) + E_{\text{II}m} I_{\text{II}m} \\
&\qquad \sin(\omega t - \tfrac{2}{3}\pi) \,.\, \sin(\omega t - \tfrac{2}{3}\pi - \varphi_2) + E_{\text{III}m} I_{\text{III}m} \\
&\qquad \sin(\omega t - \tfrac{4}{3}\pi) \,.\, \sin(\omega t - \tfrac{4}{3}\pi - \varphi_3) \\
&= \tfrac{1}{2} E_{\text{I}m} I_{\text{I}m} [\cos \varphi_1 - \cos(2\omega t - \varphi_1)] + \tfrac{1}{2} E_{\text{II}m} I_{\text{II}m} \\
&\qquad [\cos \varphi_2 - \cos(2\omega t - \tfrac{4}{3}\pi - \varphi_2)] + \tfrac{1}{2} E_{\text{III}m} I_{\text{III}m} \\
&\qquad [\cos \varphi_3 - \cos(2\omega t - \tfrac{8}{3}\pi - \varphi_3)] \\
&= E_{\text{I}} I_{\text{I}} [\cos \varphi_1 - \cos(2\omega t - \varphi_1)] + E_{\text{II}} I_{\text{II}} \\
&\qquad [\cos \varphi_2 \{1 - \cos(2\omega t - \tfrac{4}{3}\pi)\} - \sin \varphi_2 \,.\, \sin(2\omega t - \tfrac{4}{3}\pi)] \\
&\quad + E_{\text{III}} I_{\text{III}} [\cos \varphi_3 \{1 - \cos(2\omega t - \tfrac{8}{3}\pi)\} - \sin \varphi_3 \,.\, \sin(2\omega t - \tfrac{8}{3}\pi)]
\end{aligned}$$

Hence the mean power is given by the mean value of the above expression taken over a period. Thus

$$P = E_{\text{I}} I_{\text{I}} \cos \varphi_1 + E_{\text{II}} I_{\text{II}} \cos \varphi_2 + E_{\text{III}} I_{\text{III}} \cos \varphi_3 \quad . \quad . \quad (58)$$

since all terms of double frequency become zero when averaged over a period

Therefore the mean power in an unbalanced three-phase, or any polyphase, system is equal to the sum of the mean power in each phase.

[NOTE.—The case for which the E.M.F. and current are non-sinusoidal is discussed in Chapter XIV.]

Measurement of Power in Three-phase Circuits. With *balanced circuits* the total power may be measured by a single wattmeter in a number of ways.* For example, if the neutral point of the system is available the wattmeter may be connected as shown in

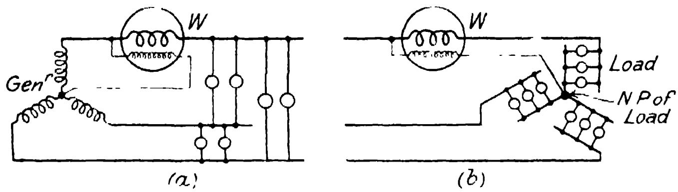


FIG. 127. METHODS OF MEASURING POWER IN THREE-PHASE, STAR-CONNECTED SYSTEM AND BALANCED LOADS

Fig. 127, in which the current coil of the wattmeter is inserted in one of the line wires and the potential coil is connected between that line wire and the neutral point. The total power is equal to three times the power indicated by the wattmeter. When, however, this method is employed in practice it is customary to mark the scale of the instrument in terms of the total power instead of the actual power indicated by the instrument.

If the neutral point of the system is not available, the total power may still be measured by means of a single wattmeter, but in this case it is necessary to impress on the potential coil of the wattmeter a voltage which is equivalent to the phase voltage of the system. With low-voltage circuits and the dynamometer type of wattmeter this equivalent phase voltage may be obtained by the use of a star-connected potential circuit, as shown in Fig. 128, in which one branch consists of the potential coil of the wattmeter and the other branches consist of resistances equal in value to that of the potential coil. For extreme accuracy the inductance of each of these branches must equal that of the potential coil. Thus the

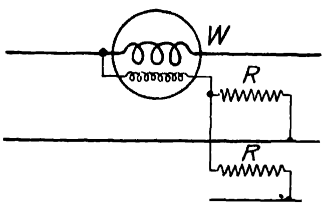


FIG. 128. METHOD OF MEASURING POWER IN BALANCED THREE-PHASE SYSTEM, WITH ISOLATED NEUTRAL POINT

* See *Power Wiring Diagrams*, Third Edition, p. 158.

wattmeter indicates the true power in one phase of the system, as in the previous case.

With *unbalanced* circuits the total power may be determined by measuring the power in each phase, by separate wattmeters, and adding the results. Such a method requires three wattmeters and is not used in practice because the same result may be obtained by means of two wattmeters, or a single wattmeter of the polyphase type (p. 381), connected in the manner shown in Fig. 129. The total power is then given by the *algebraic* sum of the readings of the wattmeters, when two instruments are employed. When a polyphase wattmeter is employed, the total power is given by a single reading of the instrument, and this feature renders this type of instrument very serviceable for commercial circuits. It should be noted that in each case the results are true, whether the system is balanced or unbalanced.

The theory of the *two-wattmeter method* of power measurement is best developed by considering the case when two separate wattmeters are employed, as we are then able to show how the power factor of a balanced system may be obtained from the readings of the instruments. Moreover, since a polyphase wattmeter consists essentially of two single-phase instruments with a common moving system, the proof developed for the case of two separate wattmeters may readily be interpreted for the case of the polyphase wattmeter.

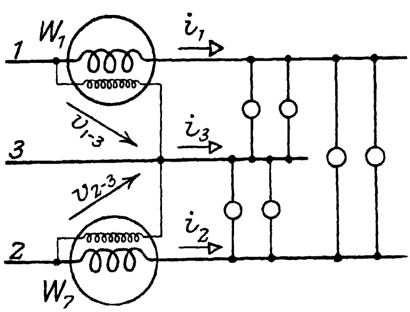


FIG. 129. CIRCUIT DIAGRAM FOR TWO-WATTMETER METHOD OF MEASURING POWER IN THREE-PHASE SYSTEM

Theory of the Two-wattmeter Method of Measuring Power in Three-phase Circuits. Let the positive direction of E.M.Fs. and currents in the circuit be that marked by the arrows in Fig. 129. Then if the instantaneous values of the currents in lines 1 and 2 (in which the current coils of the wattmeters W_1, W_2 are inserted) are denoted by i_1, i_2 respectively, the instantaneous power measured by wattmeter W_1 is equal to $i_1 v_{1-3}$, and that measured by wattmeter W_2 is equal to $i_2 v_{2-3}$. Observe that the positive directions for the E.M.Fs. impressed on the potential coils of the wattmeters are: v_{1-3} for wattmeter W_1, and v_{2-3} for wattmeter W_2.

The sum of these quantities is

$$i_2 v_{2-3} + i_1 v_{1-3}.$$

But if the phase E.M.Fs. of the system are e_I, e_{II}, e_{III}, we have $v_{2-3} = e_{II} - e_{III}$, $v_{1-3} = e_I - e_{III}$, and therefore

$$i_2 v_{2-3} + i_1 v_{1-3} = i_2(e_{II} - e_{III}) + i_1(e_I - e_{III}) = i_1 e_I + i_2 e_{II} - e_{III}(i_1 + i_2)$$

Now in any three-phase three-wire system, whether the loads are balanced or unbalanced, the instantaneous sum of the currents must be zero, i.e. $i_1 + i_2 + i_3 = 0$, or $i_1 + i_2 = -i_3$.

Therefore the sum of the instantaneous quantities measured by the wattmeters is given by

$$i_1 e_{\mathrm{I}} + i_2 e_{\mathrm{II}} + i_3 e_{\mathrm{III}}$$

and is equal to the total power in the system.

With a *balanced* system

$$i_1 = I_m \sin(\omega t - \varphi), \qquad i_2 = I_m \sin(\omega t - \varphi - \tfrac{2}{3}\pi);$$

$$\begin{aligned}
v_{1-3} &= E_m \sin \omega t - E_m \sin(\omega t - \tfrac{4}{3}\pi) \\
&= E_m[2 \cos \tfrac{1}{2}(2\omega t - \tfrac{4}{3}\pi).\sin(\tfrac{1}{2} \times \tfrac{4}{3}\pi)] \\
&= \sqrt{3}.E_m \sin(\omega t - \tfrac{1}{6}\pi) \\
v_{2-3} &= E_m \sin(\omega t - \tfrac{2}{3}\pi) - E_m \sin(\omega t - \tfrac{4}{3}\pi) \\
&= E_m[2 \cos \tfrac{1}{2}(2\omega t - \tfrac{6}{3}\pi).\sin(\tfrac{1}{2} \times \tfrac{2}{3}\pi)] \\
&= \sqrt{3}.E_m \sin(\omega t - \tfrac{1}{2}\pi)
\end{aligned}$$

Hence the instantaneous power measured by wattmeter W_1 is given by

$$\begin{aligned}
p_1 = i_1 v_{1-3} &= \sqrt{3}.E_m I_m \sin(\omega t - \tfrac{1}{6}\pi).\sin(\omega t - \varphi) \\
&= \tfrac{1}{2}\sqrt{3}.E_m I_m[\cos(\tfrac{1}{6}\pi - \varphi) - \cos(2\omega t - \tfrac{1}{6}\pi - \varphi)],
\end{aligned}$$

and the reading on this instrument is given by

$$\begin{aligned}
P_1 &= \tfrac{1}{2}\sqrt{3}.E_m I_m \cos(\tfrac{1}{6}\pi - \varphi) \\
&= \sqrt{3}.EI \cos(30 - \varphi)^\circ \\
&= VI \cos(30 - \varphi)^\circ, \text{ where } V \text{ is the line voltage of the system.}
\end{aligned}$$

Similarly the instantaneous power measured by wattmeter W_2 is given by

$$\begin{aligned}
p_2 = i_2 v_{2-3} &= \sqrt{3}.E_m I_m \sin(\omega t - \tfrac{1}{2}\pi).\sin(\omega t - \tfrac{2}{3}\pi - \varphi) \\
&= \tfrac{1}{2}\sqrt{3}.E_m I_m[\cos(\tfrac{1}{6}\pi + \varphi) - \cos(2\omega t - \tfrac{7}{6}\pi - \varphi)]
\end{aligned}$$

and the reading on this instrument is given by

$$\begin{aligned}
P_2 &= \tfrac{1}{2}\sqrt{3}.E_m I_m \cos(\tfrac{1}{6}\pi + \varphi) \\
&= \sqrt{3}.EI \cos(30 + \varphi)^\circ \\
&= VI \cos(30 + \varphi)^\circ
\end{aligned}$$

[*Note*. If φ is leading instead of lagging, $P_1 = VI \cos(30 + \varphi)^\circ$, and $P_2 = VI \cos(30 - \varphi^\circ)$.]

Now the ratio of the readings of the wattmeters gives

$$\frac{P_2}{P_1} = \frac{VI \cos(30 + \varphi)^\circ}{VI \cos(30 - \varphi)^\circ} = \frac{\cos 30^\circ.\cos\varphi - \sin 30^\circ.\sin\varphi}{\cos 30^\circ.\cos\varphi + \sin 30^\circ.\sin\varphi}$$

$$= \frac{\tfrac{1}{2}\sqrt{3}.\cos\varphi - \tfrac{1}{2}\sin\varphi}{\tfrac{1}{2}\sqrt{3}.\cos\varphi + \tfrac{1}{2}\sin\varphi}$$

Cross multiplying, we have

$$P_2(\sqrt{3}.\cos\varphi + \sin\varphi) = P_1(\sqrt{3}.\cos\varphi - \sin\varphi).$$

Whence

$$(P_1 + P_2)\sin\varphi = \sqrt{3}(P_1 - P_2)\cos\varphi$$

Hence

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \sqrt{3}\left(\frac{P_1 - P_2}{P_1 + P_2}\right) = \sqrt{3}\left(\frac{1 - P_2/P_1}{1 + P_2/P_1}\right)$$

and

$$\cos\varphi = \frac{1}{\sqrt{(1 + \tan^2\varphi)}} = \frac{1 + P_2/P_1}{2\sqrt{(1 - P_2/P_1 + (P_2/P_1)^2)}} \quad . \quad . \quad . \quad (59)$$

Therefore the power factor of a balanced system may be obtained from the readings of the wattmeters without a knowledge of the volt-amperes.

A curve, calculated from equation (59), connecting power factor and the ratio of the wattmeter readings is given in Fig. 130.

Variation of Readings of Wattmeters with Variation of Power Facter. Another interesting feature of the two-wattmeter method of power measurement, when applied to balanced systems and when two wattmeters are employed, is the manner in which the readings vary when the power factor of the system is varied, the volt-amperes being constant. Thus, since one wattmeter, W_1,

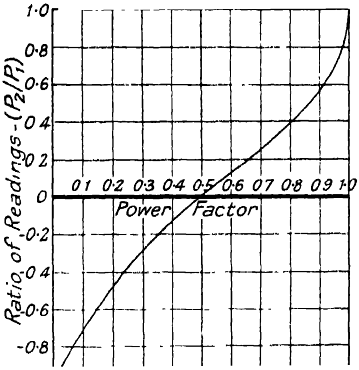


FIG. 130

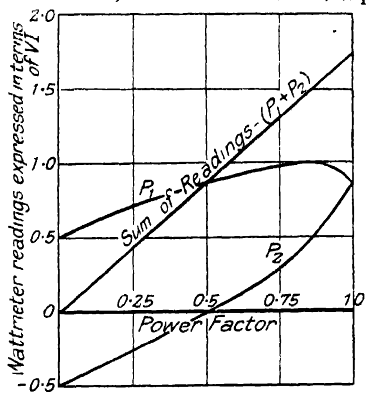


FIG. 131

VARIATION OF WATTMETER READINGS (IN TWO-WATTMETER METHOD) WITH POWER FACTOR (BALANCED LOADS AND SINUSOIDAL CURRENT AND PRESSURE)

measures the product $VI\cos(30-\varphi)°$, and the other wattmeter, W_2, measures the product $VI\cos(30+\varphi)°$, equal readings will only be obtained on the instruments, with balanced loads, when $\cos(30-\varphi)° = \cos(30+\varphi)°$, i.e. when $\varphi = 0$.*

Under these conditions,

$$P_1 = VI\cos 30° = 0{\cdot}866\,VI\ ;\ P_2 = VI\cos 30° = 0{\cdot}866\,VI$$
$$P = P_1 + P_2 = 1{\cdot}732\ VI.$$

For all other values of φ, except $\varphi = 90°$, the readings of the instruments will be unequal.

For example,

when $\varphi = 30°$, i.e. the power factor $= \cos 30° = 0{\cdot}866$,

$$\left.\begin{aligned} P_1 &= VI\cos(30° - 30°) = VI \\ P_2 &= VI\cos(30° + 30°) = 0{\cdot}5\ VI \end{aligned}\right\} P = P_1 + P_2 = 1{\cdot}5VI(= 0{\cdot}866 \times 1{\cdot}732VI)$$

When $\varphi = 60°$, i.e. the power factor $= \cos 60° = 0{\cdot}5$,

$$\left.\begin{aligned} P_1 &= VI\cos(30° - 60°) = 0{\cdot}866\ VI \\ P_2 &= VI\cos(30° + 60°) = 0 \end{aligned}\right\} P = P_1 + P_2 = 0{\cdot}866VI(= 0{\cdot}5 \times 1{\cdot}723\ VI)$$

When $\varphi = 90°$, i.e. the power factor $= \cos 90° = 0$,

$$\left.\begin{aligned} P_1 &= VI\cos(30° - 90°) = 0{\cdot}5\ VI \\ P_2 &= VI\cos(30° + 90°) = -0{\cdot}5\ VI \end{aligned}\right\} P = P_1 + P_2 = 0.$$

* If φ is leading instead of lagging, wattmeter W_1 measures the product $VI\cos(30+\varphi)°$, and wattmeter W_2 measures the product $VI\cos(30-\varphi)°$. In this case, assuming balanced loads, wattmeter W_2, Fig. 129, will show the larger reading (except in the special case when $\varphi = 0$).

A curve showing the manner in which the readings of the wattmeters vary as the power factor is varied is shown in Fig. 131.

Hence at unity power factor equal readings, both positive, are obtained on the wattmeters; at a power factor of 86·6 per cent the reading on one instrument is double that on the other; at a power factor of 50 per cent one instrument reads zero; and at zero power factor equal readings are obtained, but one is now positive and the other is negative. Moreover, at power factors above 50 per cent both wattmeters are reading positively, and the total power is therefore obtained by adding the readings. But at power factors below 50 per cent one wattmeter is reading in the negative direction, and the connections of its pressure coil, or, alternatively, the connections of its current coil, must be reversed in order to obtain the reading. Hence, under these conditions, the total power is obtained by subtracting the readings.

Therefore, when using separate wattmeters for measuring power by the two-wattmeter method, it is very important to know whether the smaller reading (P_2) is positive or negative, as otherwise the total power cannot be computed. In cases where the polarity of the current and potential terminals is marked on the instruments, or where the manufacturer's diagram showing the correct connections of the instrument is available, or where the power factor of the load is known to be above, or below, 50 per cent, there will be no difficulty in interpreting the readings of the wattmeters. But in other cases a test will have to be made to ascertain the connections which give a positive reading on each wattmeter.*

Measurement of Power in a Three-phase Four-wire System. The two-wattmeter method may also be adapted to measure the power in a three-phase four-wire system, but in this case it is necessary to supply the current coils of the wattmeters from current transformers† inserted in the principal line wires in order to obtain the correct magnitudes and phase differences of the currents in the current coils of the wattmeters, since in the three-phase four-wire system the instantaneous sum of the currents in the principal line wires, or "outers," is not necessarily equal to zero, as is the case for a three-wire system. In general, in the four-wire system, the sum of the instantaneous currents in the "outers" is equal to the instantaneous current in the neutral wire, and the reversed vector sum of the R.M.S. currents in the "outers" is equal to the R.M.S. current in the neutral wire.

* For example, each instrument is connected to a single phase circuit. Alternatively, if it is permissible to open-circuit the line wires (1 and 2, Fig. 129) in which the current coils of the wattmeters are connected, single-phase power may be supplied to each instrument without disconnecting it from the circuit; e.g. single-phase power may be supplied to wattmeter W_1 by open-circuiting line wire 2, and single-phase power may be supplied to wattmeter W_2 by open-circuiting line 1.

† A current transformer is a special type of transformer designed to give a constant ratio between the currents in the primary and secondary windings when operated with a closed secondary circuit. The primary winding is connected in *series* with the main circuit, and the secondary winding is connected to the current coil of a wattmeter, ammeter, or other current-measuring instrument, which is therefore supplied with current proportional to that in the primary circuit. Instrument (current and potential) transformers are considered in detail in Chapter XVII.

The connections of the wattmeters and current transformers for measuring the power in a four-wire system by the two-wattmeter method is shown in Fig. 132.

Assuming, for simplicity, the current transformers to have a ratio of transformation of unity, and neglecting the small phase displacement between the currents in primary and secondary windings, the currents in the secondary windings, which supply the current coils of the wattmeters, will be equal in magnitude to the currents

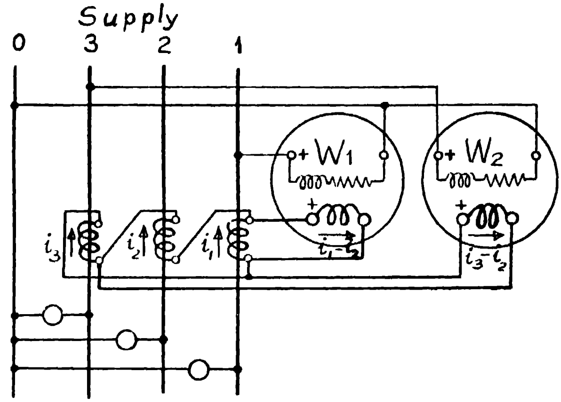


FIG. 132. CIRCUIT DIAGRAM FOR TWO-WATTMETER METHOD OF MEASURING POWER IN THREE-PHASE, FOUR-WIRE SYSTEM

in the line wires to which the primary windings are connected, and the mutual phase difference between the secondary currents will be equal to the mutual phase difference between the primary currents. It is necessary to observe, however, that the secondary currents are reversed in direction relatively to the primary currents.

If the instantaneous values of the currents in the secondary windings of the current transformers are denoted by i_1, i_2, i_3, the current in the current coil of wattmeter W_1 is equal to $i_1 - i_2$, and that in the current coil of wattmeter W_2 is equal to $i_3 - i_2$. Hence if the phase voltages of the system are denoted by e_{I}, e_{II}, e_{III}, and the assumed positive directions are those shown by the arrows in Fig. 132, the instantaneous power measured by wattmeter W_1 is given by $p_1 = e_{\text{I}}(i_1 - i_2)$, and that measured by wattmeter W_2 is given by $p_2 = e_{\text{III}}\ (i_3 - i_2)$. The sum of these quantities is therefore given by

$$\begin{aligned} p = p_1 + p_2 &= e_{\text{I}}(i_1 - i_2) + e_{\text{III}}(i_3 - i_2) \\ &= e_{\text{I}}i_1 + e_{\text{III}}i_3 - i_2(e_{\text{I}} + e_{\text{III}}) \\ &= e_{\text{I}}i_1 + e_{\text{II}}i_2 + e_{\text{III}}i_3, \end{aligned}$$

since, in a symmetrical system, $e_{\text{I}} + e_{\text{III}} = -e_{\text{II}}$.

But the expression $(e_{\text{I}}i_1 + e_{\text{II}}i_2 + e_{\text{III}}i_3)$ represents the instantaneous power in the system. Therefore two wattmeters, or a polyphase wattmeter, connected as shown in Fig. 132, will measure the total power in a three-phase four-wire system whether the loads are balanced or unbalanced.

An extreme case of unbalanced loading occurs when a single-phase load is connected across two of the line wires. Thus, assume the load to be connected across lines 1 and 2, as in Fig. 133 (*a*), and

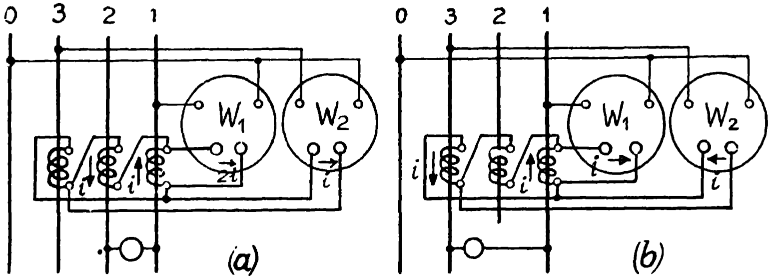


FIG. 133. CIRCUIT DIAGRAMS SHOWING CURRENTS IN WATTMETER COILS WITH SINGLE-PHASE LOADS

let i denote the instantaneous value of the line current. Then the current in the current coil of wattmeter W_1 is equal to $2i$, and that in the current coil of wattmeter W_2 is equal to i. Hence the instantaneous power measured by wattmeter W_1 is given by $p_1 = 2ie_{\text{I}}$, and that measured by wattmeter W_2 is given by $p_2 = ie_{\text{III}}$. Whence the sum of these quantities gives

$$\begin{aligned} p = p_1 + p_2 = 2ie_{\text{I}} + ie_{\text{III}} &= i(2e_{\text{I}} + e_{\text{III}}) \\ &= i(e_{\text{I}} + (e_{\text{I}} + e_{\text{III}})\,) \\ &= i(e_{\text{I}} - e_{\text{II}}) \\ &= iv_{1\text{-}2}, \end{aligned}$$

where $v_{1\text{-}2}$ is the voltage between the line wires, 1 and 2, across which the load is connected.

But the expression $iv_{1\text{-}2}$ represents the instantaneous power supplied to the load. Therefore the sum of the readings of the wattmeters, or the reading on the polyphase wattmeter, when such an instrument is used, gives the power supplied to the single-phase load.

Similarly, if a single-phase load is connected across lines 1 and 3, as in Fig. 133 (*b*), and i denotes the instantaneous value of the line current, the current in the current coil of wattmeter W_1 is equal to i, and that in the current coil of wattmeter W_2 is equal

to $-i$. Hence the instantaneous power measured by wattmeter W_1 is given by $p_1 = ie_{\text{I}}$, and that measured by wattmeter W_2 is given by $p_2 = ie_{\text{III}}$. Whence the sum of these quantities gives

$$p = p_1 + p_2 = ie_{\text{I}} - ie_{\text{III}} = i(e_{\text{I}} - e_{\text{III}})$$
$$= iv_{1\text{-}3},$$

where $v_{1\,3}$ is the voltage between the lines, 1 and 3, across which the load is connected. Therefore, in this case also, the sum of the readings of the wattmeters gives the power supplied to the single-phase load.

Again, if the load is connected between one of the principal line wires and the neutral wire (e.g. between lines 1 and 0), the current i circulates only in the current coil of wattmeter W_1. The power measured by this wattmeter is, therefore, given by $p = e_1 i$, and that measured by the other wattmeter (W_2) is zero.

When the load is connected between lines 2 and 0 the current i circulates (in the negative direction) in the coils of both wattmeters, and accordingly $p_1 = -ie_1$, $p_2 = -ie_3$; $p_1 + p_2 = -i(e_1 + e_3) = ie_2$.

Hence in general two wattmeters, or a polyphase wattmeter connected according to Fig. 132, may be employed for measuring the power in any three-phase system, whether three-wire or four-wire, under any condition of loading. In the case of the three-phase three-wire system, however, the connections, when current transformers are necessary,* may be simplified to those shown in Fig. 134, for which only two current transformers are required.

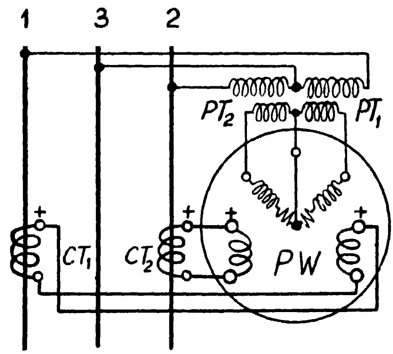


FIG. 134. CIRCUIT DIAGRAM OF POLYPHASE WATTMETER WITH INSTRUMENT TRANSFORMERS

Reactive Power in Three-phase Circuits. In Chapter V (p. 71), the reactive power in a single-phase circuit was defined as: the

* Current transformers are necessary with all high-voltage systems in order to avoid the direct connection of the instrument to the high-voltage circuit. In such cases the potential coils of the instruments are supplied from potential transformers. Current transformers are also necessary in cases where the line currents are larger than those for which the current coils of instruments can be conveniently wound. Further details of the uses of current transformers are given in Chapter XVII.

product of the impressed E.M.F. and the component of the current which is perpendicular to it, i.e. the product $EI \sin \varphi$. This definition may be extended to include polyphase circuits under conditions of balanced loads. Thus the reactive power in a polyphase circuit of n phases is given by the product $nEI \sin \varphi$, where E, I, denote the E.M.Fs. and currents of each phase. Hence for a balanced three-phase system the reactive power is given by

$$P_x = 3EI \sin \varphi = \sqrt{3}. VI \sin \varphi.$$

As in the case of single-phase circuits, the reactive power in a polyphase circuit represents the rate at which energy must be supplied to the circuit to maintain the magnetic and electrostatic fields, i.e. the rate at which energy is stored in the circuit; but in the polyphase circuit with balanced loads the stored energy is constant and does not surge between the generator and the circuit.

Measurement of Reactive Power. In a balanced three-phase circuit the reactive power may be measured directly on a single wattmeter by so connecting the current and potential coils that at unity power factor the currents in them have a phase difference of 90°. For example, if the current coil of the wattmeter is connected in one line wire, No. 1, and the potential coil is connected across the other line wires, Nos. 2 and 3, as shown in Fig. 135, the reading on the wattmeter will represent the product

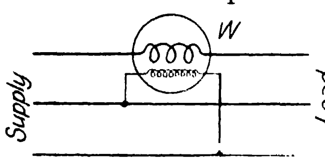


FIG. 135. CONNECTIONS OF A WATTMETER TO MEASURE THE REACTIVE POWER IN A THREE-PHASE SYSTEM

$$VI \cos (90 - \varphi)^\circ = VI \sin \varphi,$$

and therefore the reactive power in the system will be given by $\sqrt{3}$ times the reading on the wattmeter.

When, however, the two-wattmeter method of power measurement is employed, and separate wattmeters are used, the reactive power may be obtained from the readings on the wattmeters. Thus, if P_1, P_2, are the readings on the wattmeters, then

$$P_1 = VI \cos (30 - \varphi)^\circ, \qquad P_2 = VI \cos (30 + \varphi)^\circ$$

Whence,

$$\begin{aligned} P_1 - P_2 &= VI[\cos(30 - \varphi)^\circ - \cos(30 + \varphi)^\circ] \\ &= VI[\cos 30^\circ . \cos \varphi + \sin 30^\circ . \sin \varphi - \cos 30^\circ . \cos \varphi \\ &\qquad + \sin 30^\circ . \sin \varphi] \\ &= VI \sin \varphi \end{aligned}$$

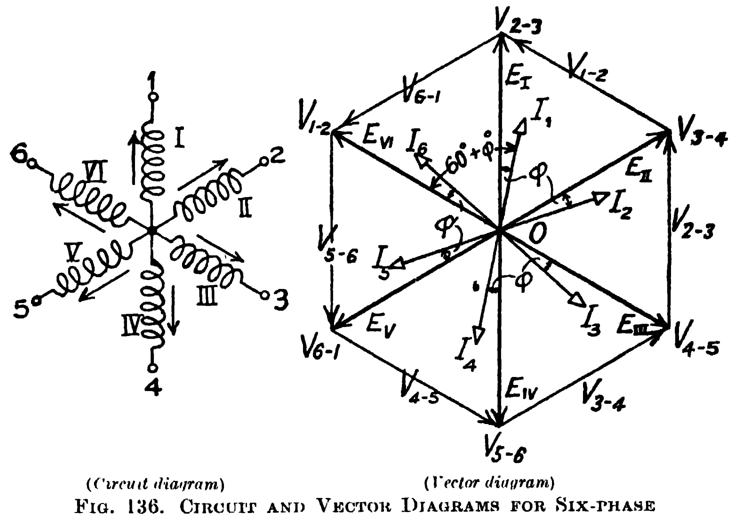
Therefore the reactive power in the system is given by $\sqrt{3}$ times the difference of the readings on the wattmeters.

SIX-PHASE SYSTEM

In a symmetrical system the phase E.M.Fs. are represented by the equations

$$e_{\mathrm{I}} = E_m \sin \omega t,\ e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{1}{3}\pi),\ e_{\mathrm{III}} = E_m \sin(\omega t - \tfrac{2}{3}\pi),$$
$$e_{\mathrm{IV}} = E_m \sin(\omega t - \pi),\ e_{\mathrm{V}} = E_m \sin(\omega t - \tfrac{4}{3}\pi),\ e_{\mathrm{VI}} = E_m \sin(\omega t - \pi)$$

Star-connected System. With a star-connected system the line E.M.Fs. are equal to the phase E.M.Fs. (since the E.M.F. between any two adjacent line wires is equal to the vector difference of the E.M.Fs. in the phases to which these lines are connected), and the line E.M.F. vectors are displaced 60° (leading) with respect to the



(*Circuit diagram*) (*Vector diagram*)

FIG. 136. CIRCUIT AND VECTOR DIAGRAMS FOR SIX-PHASE STAR-CONNECTED SYSTEM

[NOTE. The vectors OE_{I}, OE_{II}, . . . represent the phase E.M.Fs., and the vectors OV_{1-2}, OV_{2-3}, . . . (which are coincident with OE_{VI}, OE_{I}, . . .) represent the line E.M.Fs. Observe that the line E.M.Fs. may be drawn in the form of a regular hexagon.]

phase E.M.F. vectors, as shown by the vector diagram of Fig. 136, and also by the equations

$$
\begin{aligned}
v_{1\text{-}2} = e_{\mathrm{I}} - e_{\mathrm{II}} &= E_m \sin \omega t - E_m \sin(\omega t - \tfrac{1}{3}\pi) \\
&= 2E_m[\cos(\omega t - \tfrac{1}{6}\pi) \,.\, \sin \tfrac{1}{6}\pi] = E_m \sin(\omega t + \tfrac{1}{3}\pi) \\
v_{2\text{-}3} = e_{\mathrm{II}} - e_{\mathrm{III}} &= E_m \sin(\omega t - \tfrac{1}{3}\pi) - E_m \sin(\omega t - \tfrac{2}{3}\pi) \\
&= 2E_m[\cos(\omega t - \tfrac{1}{2}\pi) . \sin \tfrac{1}{6}\pi] = E_m \sin \omega t, \\
v_{3\text{-}4} = e_{\mathrm{III}} - e_{\mathrm{IV}} &= E_m \sin(\omega t - \tfrac{2}{3}\pi) - E_m \sin(\omega t - \pi) \\
&= 2E_m[\cos(\omega t - \tfrac{5}{6}\pi) . \sin \tfrac{1}{6}\pi] = E_m \sin(\omega t - \tfrac{1}{3}\pi),
\end{aligned}
$$

etc., etc.

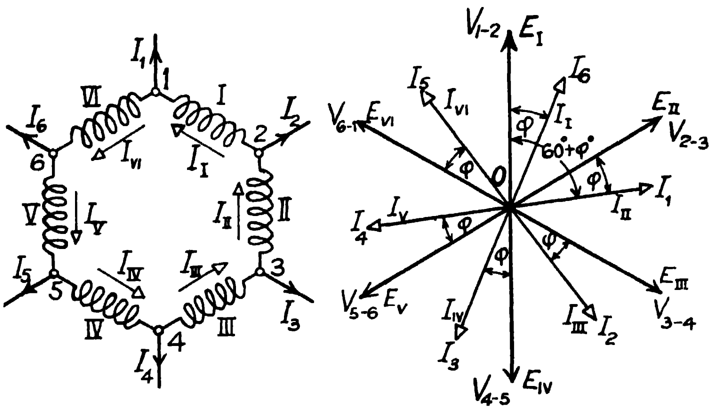
The R.M.S. values of the line E.M.Fs. are equal to those of the phase E.M.Fs. Thus

$$V_{1\cdot2} = E,\ V_{2\cdot3} = E,\ V_{3\text{-}4} = E,\ V_{4\cdot5} = E,\ V_{5\cdot6} = E,$$

where E is the R.M.S. value of each of the phase E.M.Fs.

The line currents are equal to the phase currents, and the phase difference between the line-E.M.F. vectors and the line-current vectors is equal to $(60 + \varphi)°$, lagging, where φ is the phase difference between phase E.M.Fs. and currents.

Mesh, or Ring-connected System. With balanced loads the line currents are equal to the phase currents, and the line-current



(*Circuit diagram*) (*Vector diagram*)

FIG. 137. CIRCUIT AND VECTOR DIAGRAMS FOR SIX-PHASE MESH-CONNECTED SYSTEM

vectors are displaced 60 degrees, lagging, with respect to the phase-current vectors, since the current in any line is equal to the vector difference of the currents in the phases connected to that line, as shown in the vector diagram of Fig. 137. Thus the phase difference between the line E.M.Fs. and the line currents is equal to $(60 + \varphi)$ degrees, lagging, where φ is the phase difference between the phase E.M.Fs. and currents.

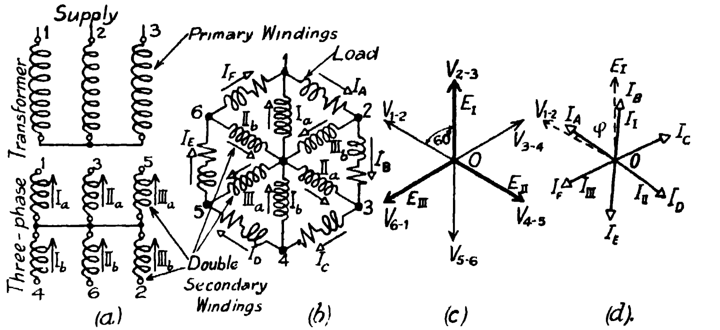
Applications of Six-phase System. The six-phase system is not used for general power distribution, as it possesses no advantages over the three-phase system for power supply to motors, and has the great disadvantage of requiring double the number of line wires of a three-phase system.

For supplying rotary converters, however, the increased number of phases of the six-phase system is an advantage, as a larger output

can be obtained from a given size of armature than when the supply is three-phase, due to the lower resultant I^2R losses in the former case, other conditions being equal.

The six-phase system is also advantageous for the power supply to mercury-arc power rectifiers, as the larger number of phases and anodes compared with a three-phase, three-anode rectifier result in a smoother output voltage and a larger output from a given size of bulb or tank.

In the majority of cases where the six-phase system is used in practice the six-phase current is obtained from a three-phase system by means of static transformers as shown below. But in



(*Circuit diagrams*) (*Vector diagrams*)

FIG. 138. CIRCUIT AND VECTOR DIAGRAMS FOR THE STAR METHOD OF SUPPLYING A SIX-PHASE LOAD FROM A THREE-PHASE TRANSFORMER

(*a*) Transformer Connections; (*b*) Circuit Diagram showing Directions of Currents in Secondary Windings and Load, (*c*) Vector Diagram of E.M.Fs.; (*d*) Vector Diagram of Currents

special cases (e.g. when a large amount of power in the direct-current form is to be generated, using steam turbines as prime movers) the generators are wound to supply six-phase current directly to the rotary converters, which are located close to the generators, thus minimizing the disadvantages of a six-wire distribution.

Methods of Obtaining a Symmetrical Six-phase System from a Three-phase System. A symmetrical six-phase system may be obtained from a symmetrical three-phase system by means of a three-phase transformer, or, alternatively, three single-phase transformers, provided with double secondary windings. Both the star and mesh methods of interconnecting the secondary windings may be employed, the interconnections for the star connection being shown in Fig. 138, and those for the mesh connection being shown in Fig. 139. With both methods the line voltage is equal

to the phase voltage (i.e. the voltage at the terminals of each of the half-sections of the secondary windings), and the line current is equal to the phase current. Moreover, with balanced loads, the line-current vectors are displaced $(60 + \varphi)^\circ$, lagging, with respect to the line-voltage vectors.

With the *star connection* the E.M.F. between any pair of line wires is equal to the vector sum of the E.M.Fs., or the reversed E.M.Fs., as the case may be, in the two half-sections of the secondary windings to which these lines are connected. For example, if the E.M.Fs. in the double secondary windings are represented by the equations

$$e_{\text{I}} = E_m \sin \omega t, \quad e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\text{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the line E.M.Fs. are given by the equations

$$v_{1\text{-}2} = e_{\text{I}} + e_{\text{III}} = E_m \sin(\omega t + \tfrac{1}{3}\pi), \; v_{2\text{-}3} = -e_{\text{II}} - e_{\text{III}} = E_m \sin \omega t,$$
$$v_{3\text{-}4} = e_{\text{I}} + e_{\text{II}} = E_m \sin(\omega t - \tfrac{1}{3}\pi),$$
$$v_{4\text{-}5} = -e_{\text{I}} - e_{\text{III}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \; v_{5\text{-}6} = e_{\text{II}} + e_{\text{III}} = E_m \sin(\omega t - \pi),$$
$$v_{6\text{-}1} = -e_{\text{I}} - e_{\text{II}} = E_m \sin(\omega t - \tfrac{4}{3}\pi).$$

The vector diagram is shown in Fig. 138 (*c*), in which the vectors OE_{I}, OE_{II}, OE_{III} represent the E.M.Fs. in the half-sections of the secondary windings, and the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, . . . represent the E.M.Fs. between the line wires.

If the six-phase load is balanced, the currents in the secondary windings will be balanced, and may be represented by the vectors OI_{I}, OI_{II}, OI_{III} [Fig. 138 (*d*)] and the equations

$$i_{\text{I}} = I_m \sin(\omega t - \varphi);$$
$$i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi); \; i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$$

The currents in the line wires 1, 2, 3 . . . are equal to i_{I}, $-i_{\text{III}}$, i_{II}, $-i_{\text{I}}$, i_{III}, $-i_{\text{II}}$, as shown in the circuit diagram (*b*).

The currents in the branches of the load are given by

$$i_{\text{A}} = i_{\text{I}} + i_{\text{III}} = I_m[\sin(\omega t - \varphi) + \sin(\omega t - \tfrac{4}{3}\pi - \varphi)]$$
$$= I_m \sin(\omega t + \tfrac{1}{3}\pi - \varphi)$$
$$i_{\text{B}} = -i_{\text{III}} - i_{\text{II}} = -I_m[\sin(\omega t - \tfrac{4}{3}\pi - \varphi) + \sin(\omega t - \tfrac{2}{3}\pi - \varphi)]$$
$$= I_m \sin(\omega t - \varphi)$$

and so on. They are represented in the vector diagram, Fig. 138 (*d*), by the vectors OI_{A}, OI_{B}, . . .

With the *mesh connection* the line E.M.Fs. are equal to the E.M.Fs. in the half-sections of the secondary windings, and by

interconnecting these sections in the manner shown in Fig. 139 a phase difference of 60° is obtained between the E.M.Fs. of adjacent series-connected sections, as shown in the vector diagram of Fig 139 (*c*).

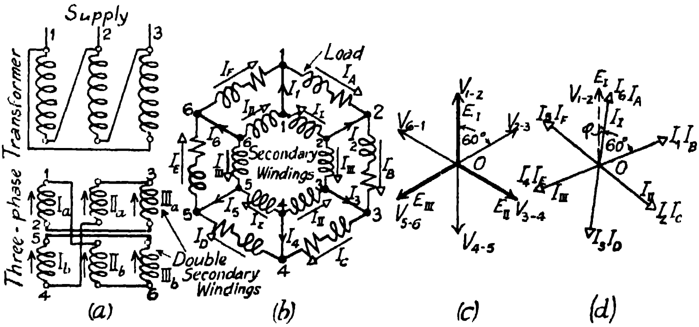
For example,

$$v_{1-2} = e_{\text{I}} = E_m \sin \omega t$$

$$v_{2-3} = -e_{\text{III}} = -E_m \sin(\omega t - \tfrac{4}{3}\pi) = E_m \sin(\omega t - \tfrac{1}{3}\pi)$$

$$v_{3-4} = e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi)$$

and so on. These E.M.Fs. are represented in the vector diagram [Fig. 139 (*c*)] by the vectors OV_{1-2}, OV_{2-3}, OV_{3-4}, . . ., the



(*Circuit diagrams*) (*Vector diagrams*)

FIG. 139. CIRCUIT AND VECTOR DIAGRAMS FOR THE MESH METHOD OF SUPPLYING A SIX-PHASE LOAD FROM A THREE-PHASE TRANSFORMER

(*a*) Transformer Connections; (*b*) Circuit Diagram showing Directions of Currents in Secondary Windings and Load; (*c*) Vector Diagram of E.M.Fs.; (*d*) Vector Diagram of Currents

E.M.Fs. in the secondary windings being represented by OE_{I}, OE_{II}, OE_{III}.

With a balanced system, i.e. with equal currents in each of the sections of the secondary winding, the current in any line wire is equal to the vector sum of the currents, or the reversed currents, as the case may be, in the sections to which that line is connected. For example, if the currents in the secondary windings are

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the line currents are given by the equations

$$i_1 = i_{\text{I}} + i_{\text{II}} = I_m \sin(\omega t - \tfrac{1}{3}\pi), \; i_2 = -i_{\text{I}} - i_{\text{III}} = I_m \sin(\omega t - \tfrac{2}{3}\pi)$$

$$i_3 = i_{\text{II}} + i_{\text{III}} = I_m \sin(\omega t - \pi),$$

$$i_4 = -i_{\text{I}} - i_{\text{II}} = I_m \sin(\omega t - \tfrac{4}{3}\pi), \; i_5 = i_{\text{I}} + i_{\text{III}} = I_m \sin(\omega t - \tfrac{5}{3}\pi),$$

$$i_6 = -i_{\text{II}} - i_{\text{III}} = I_m \sin \omega t;$$

and the load currents are

$$i_A = i_1 - i_2 = I_m[\sin(\omega t - \tfrac{1}{3}\pi) - \sin(\omega t - \tfrac{2}{3}\pi)] = I_m \sin \omega t$$
$$i_B = i_2 - i_3 = I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \pi)] = I_m \sin(\omega t - \tfrac{1}{3}\pi)$$

and so on.

Fig. 139 (d) is a vector diagram of currents. The vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings, and are shown lagging $\varphi°$ with respect to the corresponding E.M.F. vectors in the diagram (a); the vectors OI_1, OI_2, . . . represent the line currents; and the vectors OI_A, OI_B, . . . represent the load currents.

Alternative Connections. When the transformer windings supply a balanced mesh-connected load, as for example the armature of a

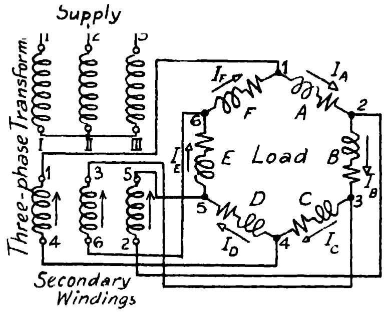


Fig. 140

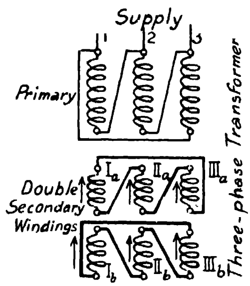


Fig. 141

Transformer Connections for the Diametrical and Double-delta Methods of Supplying a Six-phase Load from a Three-phase Transformer

[Note. With the double delta method (Fig. 141) the line wires are connected to the top terminals of the upper and lower sections of the secondary windings in the order (left to right) 1, 3, 5 for the upper windings and 6, 2, 4 for the lower windings.]

rotary converter, the connections of Figs. 138, 139, may be replaced by those shown in Figs. 140, 141 respectively, which possess the advantage, over those of the former figures, of a higher line voltage and a smaller cross-section of line wire for a given power in the two cases.

The connection of Fig. 140 is called the "*diametrical*" *connection* and does not require double secondary windings on the transformers. With the load open circuited, the secondary windings are not inter-connected and their E.M.Fs. are equivalent to three equal single-phase E.M.Fs. having a mutual phase difference of 120°. But when the windings are connected to the load the currents in the phases of the latter have a mutual phase difference of 60°, and a 60° phase difference exists between the E.M.Fs. across successive pairs of line wires, so that a six-phase system is produced. These

conditions are represented in the circuit and vector diagrams of Fig. 142.

If the E.M.Fs. in the secondary windings are

$$e_{\text{I}} = 2E_m \sin \omega t,\ e_{\text{II}} = 2E_m \sin(\omega t - \tfrac{2}{3}\pi),\ e_{\text{III}} = 2E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

the voltage across phase A of the (balanced) load is

$$v_{1\cdot 2} = \tfrac{1}{2}(e_{\text{I}} + e_{\text{III}}) = E_m[\sin \omega t + \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= E_m \sin(\omega t + \tfrac{1}{3}\pi)$$

that across phase B of the load is

$$v_{2\cdot 3} = -\tfrac{1}{2}(e_{\text{II}} + e_{\text{III}}) = -E_m[\sin(\omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= E_m \sin \omega t$$

and so on

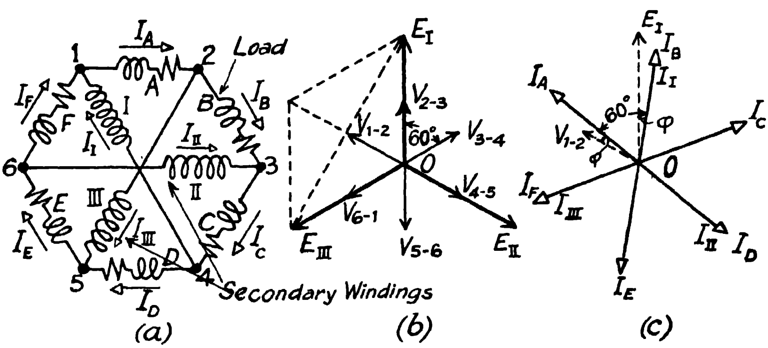


FIG. 142. CIRCUIT AND VECTOR DIAGRAMS FOR "DIAMETRICAL" METHOD OF SUPPLYING A SIX-PHASE SYSTEM

In Fig. 142 (*b*) the E.M.Fs. in the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III}, and the voltages across the load are represented by the vectors $OV_{1\cdot 2}$, $OV_{2\cdot 3}$ . . .

If the currents in the secondary windings are represented by the equations

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the current in phase A of the load is given by

$$i_A = i_{\text{I}} + i_{\text{III}} = I_m[\sin \omega t + \sin(\omega t - \tfrac{4}{3}\pi)] = I_m \sin(\omega t + \tfrac{1}{3}\pi)$$

Similarly, the current in phase B is given by

$$i_B = -i_{\text{II}} - i_{\text{III}} = -I_m[\sin \omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)] = I_m \sin \omega t,$$

the current in phase C by

$$i_C = i_{\text{I}} + i_{\text{II}} = I_m [\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi) = I_m \sin(\omega t - \tfrac{1}{3}\pi),$$

and so on.

In Fig. 142 (*c*) the vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings (which are shown lagging $\varphi°$ with respect to the corresponding E.M.F. vectors in the diagram (*a*)), and the vectors OI_{A}, OI_{B}, . . . represent the load currents.

The connection of Fig. 141 is called the "*double delta*" *connection* : it requires double secondary windings on the transformers as in the six-phase connections of Figs. 138, 139. This connection, so far as the secondary windings of the transformer are concerned, is equivalent to two three-phase delta connections displaced 180° in relation to each other, as represented in the conventional diagram of Fig 143.

If the currents in secondary windings are represented by the equations

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the currents in the line wires are given by

$$\begin{aligned}
i_1 &= i_{\text{I}} - i_{\text{III}} &&= I_m[\sin \omega t - \sin(\omega t - \tfrac{4}{3}\pi)] = \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{6}\pi),\\
i_2 &= i_{\text{II}} - i_{\text{III}} &&= I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] = \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{2}\pi),\\
i_3 &= i_{\text{II}} - i_{\text{I}} &&= I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin \omega t] = \sqrt{3}.I_m \sin(\omega t - \tfrac{5}{6}\pi),\\
i_4 &= i_{\text{III}} - i_{\text{I}} &&= I_m[\sin(\omega t - \tfrac{4}{3}\pi) - \sin \omega t] = \sqrt{3}.I_m \sin(\omega t - \tfrac{7}{6}\pi),
\end{aligned}$$

and so on.

The currents in the phases of the mesh-connected load are given by

$$\begin{aligned}
i_{\text{A}} &= i_1 - i_2 = \sqrt{3}.I_m[\sin(\omega t - \tfrac{1}{6}\pi) - \sin(\omega t - \tfrac{1}{2}\pi)]\\
&= \sqrt{3}.I_m \sin(\omega t + \tfrac{1}{6}\pi),\\
i_{\text{B}} &= i_2 - i_3 = \sqrt{3}.I_m[\sin(\omega t - \tfrac{1}{2}\pi) - \sin(\omega t - \tfrac{5}{6}\pi)]\\
&= \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{6}\pi),\\
i_{\text{C}} &= i_3 - i_4 = \sqrt{3}.I_m[\sin(\omega t - \tfrac{5}{6}\pi) - \sin(\omega t - \tfrac{7}{6}\pi)]\\
&= \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{2}\pi),
\end{aligned}$$

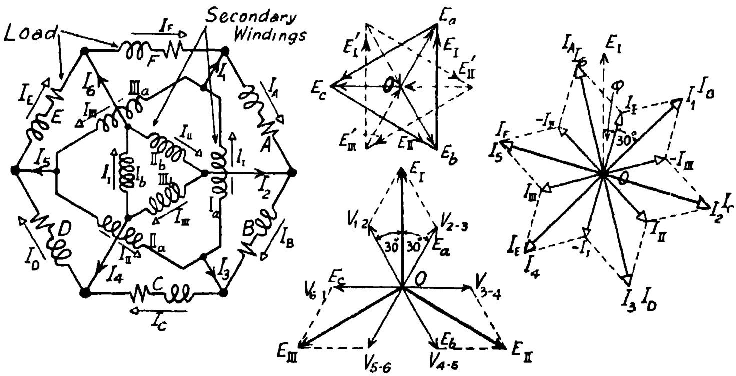
and so on.

The vector diagrams are shown in Fig. 143. In the vector diagram of currents, the vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings—these vectors being shown lagging $\varphi°$ with respect to the corresponding E.M.F. vectors OE_{I}, OE_{II}, OE_{III}—the vectors OI_1, OI_2, . . . represent the line currents, and the vectors OI_{A}, OI_{B}, . . . represent the load currents.

The determination of the voltage across each phase of the load is not so simple as the determination of the load currents, and is best effected by replacing the two delta-connected circuits by

equivalent star-connected circuits. Thus considering the delta circuit I_a, II_a, III_a, the E.M.Fs. of the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III} and by the vector triangle shown in full lines. The phase E.M.Fs. of the equivalent star-connected circuit are represented by the vectors OE_a, OE_b, OE_c, which have a phase difference of 30°, lagging, with respect to the vectors OE_{I}, OE_{II}, OE_{III}. Observe that in the upper diagram OE_a, OE_b, OE_c are drawn from the centre, O, of the triangle formed by the vectors E_{I}, E_{II}, E_{III}.

For the delta circuit I_b, II_b, III_b, the E.M.Fs. of the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III} and by the vector triangle (shown dotted) formed by the vectors E_{I}',



Circuit Diagram *Vector Diagrams of E.M.Fs.* *Vector Diagram of Currents*

FIG. 143. CIRCUIT AND VECTOR DIAGRAMS FOR "DOUBLE-DELTA" METHOD OF SUPPLYING A SIX-PHASE SYSTEM

E_{II}', E_{III}'.* The phase E.M.Fs. of the equivalent star-connected circuit are represented by the vectors OE_a, OE_b, OE_c in the lower diagram and by the dotted vectors drawn from the corners to the centre, O, of the triangle E_{I}', E_{II}', E_{III}'.

Hence if the neutral points of the equivalent star circuits are connected together, the conditions are equivalent to Fig. 138. Thus the voltages across the phases of the load are equal in magnitude to the phase voltages of the star system, and are represented by the vectors OV_{1-2}, OV_{2-3}, . . . in Fig. 143. Observe that $\dot{V}_{1-2} + \dot{V}_{2-3} = \dot{E}_{I}$, $\dot{V}_{3-4} + \dot{V}_{4-5} = \dot{E}_{II}$, and so on.

* This triangle is reversed relatively to the triangle E_{I}, E_{II}, E_{III}, as the interconnections of the two sets of secondary windings are reversed relatively to each other.

To obtain analytical expressions for the load voltages, let the E.M.Fs. in the secondary windings be given by

$$e_{\text{I}} = E_m \sin \omega t, \; e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \; e_{\text{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

Then if the E.M.Fs. of the equivalent star circuit are denoted by e_a, e_b, e_c, we have

$$e_a - e_b = e_{\text{I}}, \; e_b - e_c = e_{\text{II}}, \; e_c - e_a = e_{\text{III}}$$

Hence, since $e_{\text{I}} + e_{\text{II}} + e_{\text{III}} = 0$, the solution of these equations gives

$$e_a = \tfrac{1}{3}(e_{\text{I}} - e_{\text{III}}), \; e_b = \tfrac{1}{3}(e_{\text{II}} - e_{\text{I}}), \; e_c = \tfrac{1}{3}(e_{\text{III}} - e_{\text{II}})$$

The voltage across phase A of the load is then given by

$$v_{1-2} = e_a + e_c = \tfrac{1}{3}(e_{\text{I}} - e_{\text{II}}) = \tfrac{1}{3}E_m[\sin \omega t - \sin(\omega t - \tfrac{2}{3}\pi)]$$

$$= \frac{1}{\sqrt{3}} E_m \sin(\omega t + \tfrac{1}{6}\pi)$$

that across phase B is given by

$$v_{2-3} = -e_b - e_c = \tfrac{1}{3}(e_{\text{I}} - e_{\text{III}}) = \tfrac{1}{3}E_m[\sin \omega t - \sin(\omega t - \tfrac{4}{3}\pi)]$$

$$= \frac{1}{\sqrt{3}} E_m \sin(\omega t - \tfrac{1}{6}\pi)$$

and so on.

The voltage, E_t, at the terminals of any half-section of the transformer winding for the double-delta connection is equal to the vector sum of the voltages across the two phases of the load to which that section is connected. Since the latter have a phase difference of 60°, therefore, $E_t = \sqrt{3}.E_l$, where E_l is the voltage across each phase of the load, the voltage drop in the line wires being neglected.

With the diametrical connection, however, the voltage across each phase of the secondary winding of the transformer is equal to the vector sum of the voltages across three phases of the load, and since these have a mutual phase difference of 60°, the secondary voltage is equal to $2E_l$, where E_l is the voltage across each phase of the load and the voltage drop in the line wires is neglected.

But whereas the VA.-output from each delta-connected group of windings in the double-delta connection is one-half of the total VA. supplied to the load, the VA.-output from each winding in the diametrical connection is one-third of the total VA. supplied to the load. Hence the line currents will be the same for both connections.

Measurement of Power in Six-phase Systems. Although very few cases occur in practice where the measurement of power in six-phase circuits is required, since this measurement is usually effected on the three-phase system from which the six-phase power is obtained, we shall consider briefly some methods by which the power in six-phase systems may be measured.

If the system is *balanced* the total power may be measured by a single wattmeter, as in a balanced three-phase system, by inserting the current coil in one line wire and supplying the potential coil with a voltage equivalent to the phase voltage of the system. For example, if the neutral point of the system is available the potential coil is connected between the neutral point and the line wire in which the current coil of the wattmeter is inserted

If the neutral point is not available, a number of alternative connections are possible. For example, if the current coil is inserted in line No. 1, the potential coil may be connected across line wires 2 and 3, since the voltage between these lines is equal to, and is in phase with, the phase voltage of phase I (see vector diagram Fig. 136).

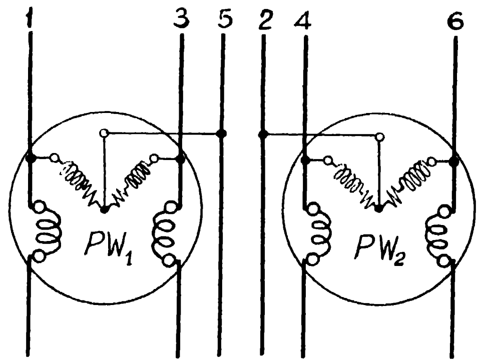


FIG. 144. DOUBLE TWO-WATTMETER METHOD OF MEASURING POWER IN A SIX-PHASE SYSTEM (POLYPHASE WATTMETERS)

Alternatively, a star-connected potential circuit for the wattmeter, similar to that shown in Fig. 128, may be employed. In the present case the potential branch containing the potential coil must be connected to the line wire in which the current coil of the wattmeter is inserted, and the other branches must be connected to line wires which have a phase difference of 120° from that containing the current coil. Thus, if the current coil is connected in line 1, the end of the potential coil must be connected to this line and the other ends of the star-connected potential resistances must be connected to lines 3 and 5.

In all the above cases the wattmeter measures the power in one phase of the system, and the total power is therefore given by six times the reading of the wattmeter.

If the system is unbalanced the double two-wattmeter method, using two polyphase wattmeters connected as shown in Fig. 144, may be employed. The total power in the system is then given by the sum of the readings of the wattmeters. Thus, denoting instantaneous values of the phase voltages by

$$e_{\mathrm{I}},\ e_{\mathrm{II}},\ e_{\mathrm{III}},\ e_{\mathrm{IV}},\ e_{\mathrm{V}},\ e_{\mathrm{VI}},$$

line voltages by

$$v_{1\text{-}2},\ v_{2\text{-}3},\ v_{3\text{-}4},\ v_{4\text{-}5},\ v_{5\text{-}6},\ v_{6\text{-}1},$$

and line currents by

$$i_1,\ i_2,\ i_3,\ i_4,\ i_5,\ i_6,$$

the instantaneous power measured by wattmeter PW_1 is given by

$$p_1 = i_1 v_{1\text{-}5} + i_3 v_{3\text{-}5},$$

and that measured by wattmeter PW_2 is given by

$$p_2 = i_4 v_{4\text{-}2} + i_6 v_{6\text{-}2}.$$

Now $v_{1\text{-}5} = e_{\text{I}} - e_{\text{V}},\ v_{3\text{-}5} = e_{\text{III}} - e_{\text{V}},\ v_{4\text{-}2} = e_{\text{IV}} - e_{\text{II}},\ v_{6\text{-}2} = e_{\text{VI}} - e_{\text{II}}.$

Hence,

$$\begin{aligned} p_1 + p_2 &= i_1 v_{1\text{-}5} + i_3 v_{3\text{-}5} + i_4 v_{4\text{-}2} + i_6 v_{6\text{-}2} \\ &= i_1(e_{\text{I}} - e_{\text{V}}) + i_3(e_{\text{III}} - e_{\text{V}}) + i_4(e_{\text{IV}} - e_{\text{II}}) + i_6(e_{\text{VI}} - e_{\text{II}}) \\ &= i_1 e_{\text{I}} + i_3 e_{\text{III}} + i_4 e_{\text{IV}} + i_6 e_{\text{VI}} - e_{\text{V}}(i_1 + i_3) - e_{\text{II}}(i_4 + i_6) \end{aligned}$$

But, since $i_1,\ i_3,\ i_5$ have a mutual phase difference of 120°

$$i_1 + i_3 = -i_5$$

and, since $i_2,\ i_4,\ i_6$ also have a mutual phase difference of 120°

$$i_4 + i_6 = -i_2$$

Therefore,

$$p_1 + p_2 = i_1 e_{\text{I}} + i_2 e_{\text{II}} + i_3 e_{\text{III}} + i_4 e_{\text{IV}} + i_5 e_{\text{V}} + i_6 e_{\text{VI}},$$

which is equal to the total power in the six-phase system.

Two-phase System

Independent Two-phase System. The independent, or four-wire, two-phase system has already been considered in connection with the simple polyphase alternator (p. 175). This system, however, may be obtained from a four-phase, star-connected system by disconnecting the neutral point and connecting alternate phases in series, as shown in Fig. 145. In this manner two equal E.M.Fs., having a phase difference of 90°, are obtained.

Interconnected Two-phase System. The two-phase system may be interconnected, so that only three line wires are necessary, by joining one end of each phase to a common line wire as shown in Fig. 146, this interconnected system being called the two-phase three-wire system. The common line wire is called the neutral wire and the other line wires are called the "outers."

The vector diagram for this system is shown in Fig. 147, from which we observe that in a balanced system the E.M.F. across the

"outers" is equal to the vector difference of the phase E.M.Fs., and is numerically equal to $\sqrt{2}$ (= 1·414) times the phase E.M.F. Moreover, the E.M.F. across the "outers" has a phase difference of 45°, leading, with respect to the E.M.F. of phase I.

The current in the neutral wire is equal to the reversed vector sum of the currents in the "outers," and with balanced loads the

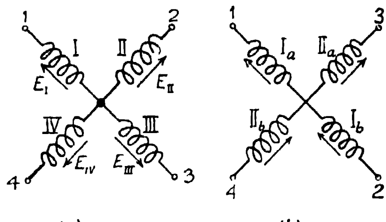


FIG. 145. METHOD OF CONVERTING A FOUR-PHASE, STAR-CONNECTED ALTERNATOR INTO A TWO-PHASE ALTERNATOR

former is equal to $\sqrt{2}$ times the current in each "outer." Moreover, the current in the neutral has a phase difference of 135° with respect to the current in the "outers," being leading with respect to the

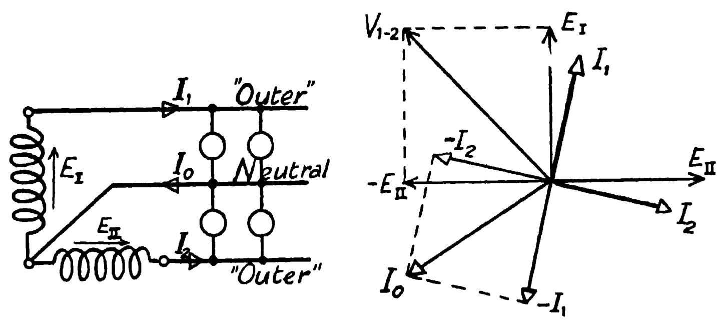


FIG. 146 FIG. 147
CIRCUIT AND VECTOR DIAGRAMS FOR TWO-PHASE, THREE-WIRE SYSTEM

current in one "outer," and lagging with respect to the current in the other "outer," as shown in Fig. 147.

Due to these large phase differences between the currents in the neutral and "outers" of a two-phase three-wire system the voltage drop in the neutral wire will have large phase differences with respect to the voltage drops in the "outers," and therefore when these voltage drops are compounded with the phase E.M.Fs. to obtain

the voltages at the load, the latter will, in general, be unequal and will not have a phase difference of 90°. Thus, the two-phase three-wire system becomes unsymmetrical when loaded, and for this reason the system does not possess much practical value for power transmission.

Comparison of Two-phase and Three-phase Systems. The two-phase system is not so advantageous as the three-phase system for power supply, as four line wires are necessary to maintain a symmetrical two-phase system, whereas only three wires are necessary in the case of a three-phase system. The two-phase four-wire system, however, is used to some extent in this country and abroad for mixed lighting and power loads, the lighting load being single phase and the power load being two phase.

The majority of these installations were originally single-phase systems supplying lighting loads, and were converted into two-phase systems for the purpose of enabling a power load to be supplied in addition to the lighting load. For modern installations, however, the three-phase four-wire (star-connected) system is employed for mixed loads, as with this system the lighting portion of the load is supplied at the phase voltage of the system, and the power portion of the load is supplied at the "line" voltage of the system, which is $\sqrt{3}$ times the phase voltage. Moreover, a two-phase supply may be obtained from such a system by means of transformers connected in the manner described in Chapter XIII.

CHAPTER X

TRANSMISSION CIRCUITS AND THEIR CONSTANTS

In this chapter we shall determine the constants (resistance, inductance, capacitance) for simple transmission circuits and discuss some factors (such as comparison of systems of transmission, equivalent circuits, voltage drop, etc.) concerned with such circuits.

Resistance. For the usual sizes of conductors employed in power-transmission circuits, the resistance under working conditions may be calculated from the usual formula—$R = l\rho/a$, where R is the resistance in ohms; l, a, the length and cross-section respectively, and ρ, the resistivity, all of which must be expressed in the same system of units. But with large solid conductors (above 1 in. in diameter) the resistance under working conditions will be higher than that calculated from this formula owing to the skin effect, which has already been explained (p. 48).

The *increase in resistance due to the skin effect* can be calculated by the method explained in Chapter XV, but for ordinary power frequencies the general equation may be replaced by a simple approximate expression. Thus the resistance, R_a, under working (i.e. alternating-current) conditions is given by

$$R_a = R\left\{1 + \frac{1}{12}\left(\frac{\omega a}{\rho 10^9}\right)^2 - \frac{1}{180}\left(\frac{\omega a}{\rho 10^9}\right)^4\right\} \quad . \quad . \quad (60)$$

where R is the resistance calculated from the formula ($R = l\rho/a$), $\omega = 2\pi \times$ frequency, and the factors a, ρ, denote the cross-section and resistivity respectively in centimetre units. For round copper conductors of diameter d cm., and resistivity $= 1{\cdot}7 \times 10^{-6}\ \Omega$ per cm. cube, equation (60) reduces to

$$R_a = R\{1 + 0{\cdot}7(fd^2/1000)^2 - 0{\cdot}4(fd^2/1000)\} . \quad . \quad . \quad (60a)$$

Inductance of Single-phase Transmission Line. The inductance of two parallel conductors forming a simple single-phase transmission line is given by

$$L = (0{\cdot}92 \log_{10}D/r + 0{\cdot}1) \times 10^{-8} \text{ henries,}$$

per cm. of the line, or by

$$L = 12 \times 5280 \times 2{\cdot}54(0{\cdot}92 \log_{10}D/r + 0{\cdot}1) \times 10^{-8} \text{ H.}$$
$$= 1{\cdot}61(0{\cdot}92 \log_{10}D/r + 0{\cdot}1) \text{ mH.} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

per mile of line, where r is the radius of each conductor and D is

the spacing of the conductors, which is assumed to be large in comparison with r.

Proof. To calculate the inductance of two parallel conductors, we must first determine the flux linked with unit length of the conductors when the current in the circuit is 1 A. We assume that (1) the current is distributed uniformly over the cross-section of each conductor; (2) the conductors are non-magnetic and surrounded by air.

Thus, considering the magnetic effect due to a current of i amperes in one conductor, the magnetic force (or density of magnetic field in lines per cm.²) at a point P, Fig. 148, external to the conductor, distant x cm. from its axis is

$$H_x = \frac{0{\cdot}4\pi i}{2\pi x} = \frac{0{\cdot}2i}{x}$$

or, when $i = 1$,

$$H_x = 0{\cdot}2/x.$$

The magnetic force at a point inside the conductor, distant y cm. from its axis, is

$$H_y = \frac{0{\cdot}4\pi i(\pi y^2/\pi r^2)}{2\pi y} = 0{\cdot}2i\left(\frac{y}{r^2}\right)$$

or, when $i = 1$,

$$H_y = 0{\cdot}2(y/r^2)$$

where r = radius of the conductor.

The variation of the magnetic force in the plane perpendicular to that containing the axes of the conductors is shown in Fig. 148, curve I.

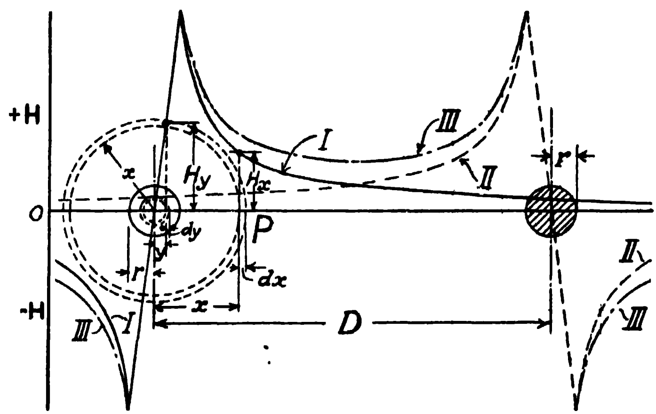


FIG. 148. PARALLEL CYLINDRICAL CONDUCTORS AND THEIR ASSOCIATED MAGNETIC FIELDS

The effect of the current in the other conductor is represented by the curve II, Fig. 148. Hence when both conductors are carrying current the resultant magnetic force in the space between them is represented by the sum of curves I and II, and is shown by curve III. The area between this curve and the abscissa axis is therefore proportional to the flux linked with each cm. length of the circuit when both conductors are carrying current.

The external flux in the space between the conductors is, per cm. length of circuit and per ampere,

$$\Phi_e = 2\int_r^{D-r}(H_x \times 1)dx = 2\int_r^{D-r}\frac{0{\cdot}2}{x}\,dx = 2\times 0{\cdot}2\Big[\log_\varepsilon x\Big]_r^{D-r}$$

$$= 0{\cdot}4\log_\varepsilon[(D-r)/r]$$

$$= 0{\cdot}4 \times 2{\cdot}3\log_{10} D/r$$

when D is large in comparison with r.

The internal flux per cm. length of each conductor, due to unit current in it, is, for an element dy distant y from the axis,

$$d\Phi_i = H_y(dy \times 1) = 0{\cdot}2(y/r^2)dy$$

This flux is linked with the portion πy^2 of the cross section of the conductor Whence the linkage is

$$d\Phi_i \times \pi y^2/\pi r^2 = 0{\cdot}2(y^3/r^4)dy$$

Hence the total linkage for each conductor due to the internal flux is

$$\int_0^r 0{\cdot}2(y^3/r^4)dy = \frac{0{\cdot}2}{r^4}\int_0^r y^3\,dy = 0{\cdot}2/4 = 0{\cdot}05$$

Therefore the inductance per cm. of circuit is

$$L = \{(0{\cdot}4 \times 2{\cdot}3\log_{10}D/r) + 2 \times 0{\cdot}05\} \times 10^{-8}$$

$$= \{0{\cdot}92\log_{10}D/r + 0{\cdot}1\} \times 10^{-8} \text{ henry.}$$

Inductance of Three-phase Lines. The conductors of a three-phase line may be arranged either symmetrically, e.g. at the corners of an equilateral triangle, or parallel to one another in the same plane. With the symmetrical arrangement of conductors, the inductance per line, or per phase, has the same value for each line, and is one-half of that for a corresponding two-wire line. Thus

$$L = 1{\cdot}61(0{\cdot}46\log_{10}D/r + 0{\cdot}05) \text{ mH.} \quad . \quad . \quad . \quad (62)$$

per conductor per mile of line, where D is the spacing of the conductors and r is the radius of each conductor.

Proof. Let the actual three-conductor system be replaced by an equivalent six-conductor system, in which the three return conductors are arranged along the neutral axis of the original three-conductor system, as indicated in Fig. 149. Thus the current in conductor A' is equal and opposite to that in conductor A, and so on. The sum of the currents in A', B', C' is zero at every instant, and if these conductors are assumed to have zero resistance and to be coincident with the neutral axis of A, B, C, their presence will not affect the constants of the original circuit.

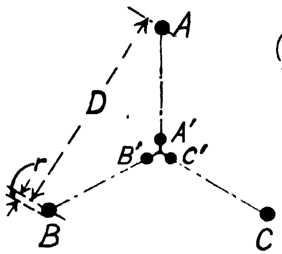
In calculating the inductance of a phase-loop (e.g. AA') of this system, the effect of mutual inductance from the other phase-loops, e.g. BB' and CC', must be taken into account. Thus, let L' denote the inductance of each phase-loop when isolated and M the mutual inductance between a pair of phase-loops. Then considering a symmetrical system and denoting the currents in A, B, C by $i_1 = I_m \sin \omega t$, $i_2 = I_m \sin(\omega t - \frac{2}{3}\pi)$, $i_3 = I_m \sin(\omega t - \frac{4}{3}\pi)$, the total linkages in the loop AA' due to these currents is equal to

$$L'i_1 - Mi_2 - Mi_3 = L'I_m \sin\omega t - MI_m \sin(\omega t - \tfrac{2}{3}\pi) - MI_m \sin(\omega t - \tfrac{4}{3}\pi)$$

$$= (L' + M)I_m \sin\omega t = (L' + M)i_1.$$

Hence the inductance of each phase-loop when forming part of the three-phase system (i.e. the inductance *per* phase) is $L = L' + M$.

Now L can be expressed in terms of the inductance of a loop formed by any pair of the original line conductors, e.g. AB. Thus, if the phase-loops AA' and BB' are connected in series and are supplied with single-phase current—the currents in A' and B' flowing in opposite directions—then the inductance of this double loop, when isolated, is equal to $L' + 2M + L' = 2(L' + M) = 2L$. But since the magnetic effect of the currents in A' and



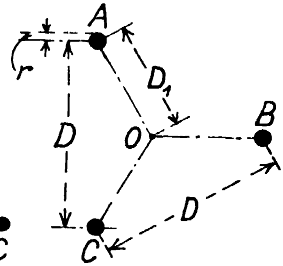


FIG. 149 FIG. 150

ARRANGEMENT OF CONDUCTORS FOR CALCULATIONS OF INDUCTANCE (FIG. 149) AND CAPACITANCE (FIG. 150) OF THREE-PHASE TRANSMISSION LINES

B' now cancel, the inductance of the double loop $AA'B'B$ will be the same as that of the loop AB when isolated and carrying single-phase current.

Hence the inductance per phase, or per line, of a three-conductor three-phase system, with symmetrically spaced conductors, is one-half of that of an isolated pair of the line conductors.

With the semi-symmetrical, or co-planar, arrangement of conductors and a symmetrically-loaded system, the inductance of the middle line has the same value as that of one of the lines of a two-wire line—viz. $L = 1{\cdot}61(0{\cdot}46 \log_{10}D/r + 0{\cdot}05)$ mH. per mile—but that of each of the outer lines has a higher value, viz.

$$L = 1{\cdot}61(0{\cdot}46 \log_{10}2D/r + 0{\cdot}05)$$
$$= 1{\cdot}61(0{\cdot}46 \log_{10}D/r + 0{\cdot}19) \text{ mH. per mile.}$$

In practice, with the semi-symmetrical arrangement of conductors and transposed lines, it is sufficient to calculate the inductance on the assumption that the lines are symmetrically spaced with an equivalent spacing equal to the geometric mean of the actual spacings. Thus the inductance per line, or per phase of the system, per mile of conductor, is given by

$$L = 1{\cdot}61(0{\cdot}46 \log_{10}[(\sqrt[3]{D_{12}D_{23}D_{31}})/r] + 0{\cdot}05) \text{ mH.} \quad . \quad (63)$$

where D_{12}, D_{23}, D_{31} denote the actual spacings of the conductors taken in the order 1–2, 2–3, 3–1.

Inductance of Concentric Cables. Although a concentric cable

produces no external magnetic field, the internal magnetic field, due both to the separation of the conductors as well and their dimensions, causes inductive effects which may require to be taken into account in long cable lines. The linkages due to the magnetic field internal to the inner conductor, and also the linkages in dielectric separating the conductors, can be calculated by the same principles as were applied to parallel cylinders if the current is assumed to be uniformly distributed over the cross-section of the conductors. The calculation of the partial linkages due to the internal field in the outer conductor is a little more difficult, although the same fundamental principles are involved.

The inductance per mile of cable at ordinary power frequencies on the assumption of uniform distribution of the current is given by

$$L = 1{\cdot}61[0{\cdot}46 \log_{10}(r_1/r) + 0{\cdot}05 + \tfrac{1}{15}t/r_1] \text{ mH.} \qquad (64)$$

where r is the radius of the inner conductor; r_1, t, the internal radius and thickness respectively of the outer conductor, and t is small in comparison with r.

Inductance of Two-core and Three-core Cables. For unarmoured cables, the inductance of a two-core cable is obtained by the application of equation (61), and the inductance of a three-core cable is obtained from equation (62). The effect of armouring is to increase the inductance, but as the calculation of the inductance is now somewhat involved, and as the values so obtained may not be very accurate, it is better to rely upon manufacturer's data.

Capacitance of Single-phase Transmission Lines. The capacitance of parallel cylindrical isolated conductors has already been deduced on p. 57. The results can be applied directly to single-phase transmission lines as normally erected, as the height of the conductors above earth is such that the proximity effect of the earth may be ignored. Hence the capacitance per mile of the two-conductor line is given approximately by

$$C = 0{\cdot}0195/\log_{10}(D/r) \ \mu\text{F.} \qquad (65)$$

where r is the radius of each conductor, and D is the spacing of the conductors (i.e. the distance between their centres), which is considered to be large in comparison with r.

Capacitance of Three-phase Lines. With symmetrical spacing (i.e. triangular arrangement) of the conductors, the capacitance of each conductor with respect to the neutral axis, O (Fig. 150), i.e. the *capacitance per phase* (called also the *star capacitance*), is given approximately by

$$C = 0{\cdot}039/\log_{10}(D/r) \ \mu\text{F.} \qquad (66)$$

per mile of line, where D is the distance between the centres of the conductors and is considered to be large in comparison with the radius r.

Proof. Consider a symmetrical system. Let $e = E_m \sin \omega t$ denote the phase voltage of conductor A at the instant t, $q_A = Q_m \sin \omega t$ the charge on this conductor, and $q_B = Q_m \sin(\omega t - \frac{2}{3}\pi)$, $q_C = Q_m \sin(\omega t - \frac{4}{3}\pi)$ the charges on the other conductors B and C at this instant. Then the work done on unit positive charge when it is moved from the neutral axis to conductor A is

$$2Q_m[\sin \omega t \log_e(D_1/r) + \sin(\omega t - \tfrac{2}{3}\pi) \log_e(D_1/D) + \sin(\omega t - \tfrac{4}{3}\pi) \log_e(D_1/D)]$$
$$= 2Q_m \sin \omega t \log_e(D/r),$$

where D_1 is the distance between the neutral axis and the axis of each conductor.

Whence the capacitance per cm. length of conductor

$$= Q_m \sin \omega t / 2Q_m \sin \omega t \log_e(D/r) = 1/[2 \log_e(D/r)] \text{ electrostatic units.}$$

Capacitance of Cables. (1) *Concentric Cable.* The capacitance of a concentric cable has already been deduced on p. 56, and is given by

$$C = 0{\cdot}039\kappa/\log_{10}(r_1/r)\ \mu\text{F. per mile} \quad . \quad . \quad . \quad (67)$$

where κ is the dielectric constant of the dielectric, r the radius of the inner core, and r_1 the internal radius of the outer core (Fig. 151).

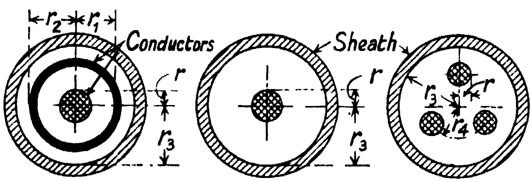


FIG. 151. CROSS-SECTIONS OF CABLES SHOWING DIMENSIONS REQUIRED FOR CALCULATION OF CAPACITANCE

If the cable is lead sheathed, and the sheath is earthed, the capacitance between the outer conductor and the sheath is

$$C = 0{\cdot}039\kappa/\log_{10}(r_3/r_2)\ \mu\text{F. per mile} \quad . \quad . \quad . \quad (68)$$

where r_2 is the external radius of the outer core and r_3 is the inner radius of the sheath (Fig. 151).

(2) *Single-core Cable.* Assuming a homogeneous dielectric and a lead sheath, the capacitance per mile is given by

$$C = 0{\cdot}039\kappa/\log_{10}(r_3/r)\ \mu\text{F.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

(3) *Multi-core Cables, Conductors of Circular Cross-section.* The capacitance of multi-core cables should preferably be obtained

from manufacturers' data (which is based upon measurements of the capacitance of completed cables), as the calculation of the capacitance from fundamental principles is complicated by the close proximity of the sheath to the conductors and by the close spacing of the conductors themselves.

The following empirical formula is used by cable-makers for calculating the approximate star capacitance of three-core paper-insulated cables with conductors of circular cross-section—

$$C = 0{\cdot}133/\log_{10}(r_3/r)\ \mu\text{F. per mile} \qquad (70)$$

where r is the radius of each conductor and r_3 is the inner radius of the sheath.

Equivalent Star and Delta Capacitances of a Three-core Cable Transmission System. The equivalent circuit of the capacitances of

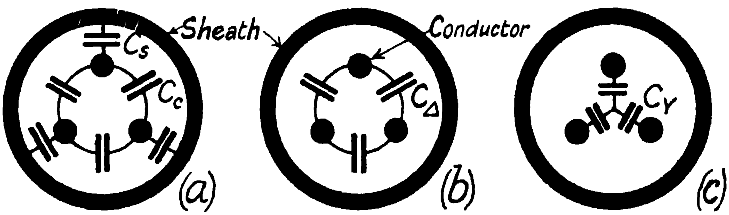


FIG. 152. EQUIVALENT CAPACITIVE CIRCUIT OF THREE-PHASE CABLE (a), AND THE EQUIVALENT DELTA (b) AND STAR (c) CAPACITANCES

a three-core cable is shown in Fig. 152 (a), which consists of a delta-star arrangement of condensers, the delta-connected condensers, C_c, representing the capacitances between the cores themselves, and the star-connected condensers, C_s, representing the capacitances between each core and the sheath. The circuit of Fig. 152 (a) may be replaced either by a group of delta-connected condensers, C_Δ (Fig. 152 (b)), or by a group of star-connected condensers, C_Y (Fig. 152 (c)). Thus,

$$C_\Delta = C_c + \tfrac{1}{3}C_s; \; C_Y = 3C_c + C_s.$$

The capacitances of the equivalent condensers C_c, C_s, are obtained from two capacitance tests on the cable. Thus (1) measure the capacitance between two cores, with the third core and sheath free; (2) connect two cores to the earthed sheath, and measure the capacitance between these cores and the other core. The capacitance obtained from the first test is $C_1 = C_c + \frac{1}{2}C_c + \frac{1}{2}C_s = \frac{1}{2}(3C_c + C_s)$, and that obtained from the second test is $C_2 = 2C_c + C_s$. Whence $C_c = 2C_1 - C_2$; $C_s = 3C_2 - 4C_1$.

Charging Current of Three-core Cable Transmission System. The charging current (I_c) is easily obtained when the equivalent star capacitance, C_Y, is known. Thus

$$I_c = \omega C_Y V/\sqrt{3} = \omega(3C_c + C_s)V/\sqrt{3} = \omega(2C_1)V/\sqrt{3},$$

where V is the line voltage of the system.

Observe that the equivalent capacitance (C_Y) for this calculation may be obtained from a single measurement of the capacitance of the cable, viz. test (1) above.

Equivalent Transmission Circuits. With short overhead transmission lines capacitance effects can be ignored. Hence the equivalent circuit for such a line is a simple series circuit containing resistance and inductance. The general vector and load diagram for

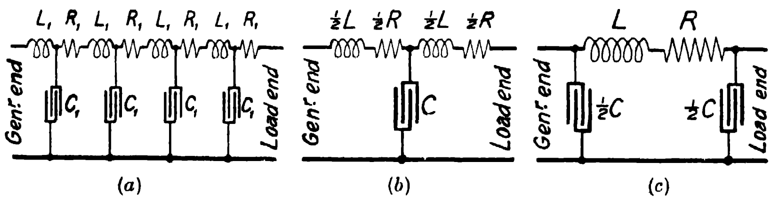


FIG. 153. EQUIVALENT CIRCUITS OF LONG TRANSMISSION LINE
(a) Equivalent Circuit; (b) Approximate Equivalent Circuit (T), (c) Approximate Equivalent Circuit (Π)

this type of circuit is given in Chapter VII, and a worked example is given on pp. 129–131.

With long overhead lines and all cable systems, however, capacitance effects must be considered. In the actual line or cable, the resistance, inductance, and capacitance are uniformly distributed over the whole length of the line, and the closest approximation to the actual conditions is given by an equivalent circuit of the complex series-parallel type shown in Fig. 153 (*a*), which, so far as the currents and voltages at the extreme ends of the line are concerned, can be simplified to the equivalent circuits shown in diagrams (*b*) and (*c*), Fig. 153. The latter are called the equivalent T and Π circuits, and the values of the equivalent impedances are deduced on p. 230.

For approximate calculations, the values of the equivalent impedances are calculated from the lumped values of the resistance, inductance, and capacitance, and the equivalent circuits in which these quantities are used are called the *nominal* Π and *nominal* T circuits. Thus in the nominal Π circuit, one-half of the total capacitance is concentrated at each end of the line, and in the

nominal T circuit the whole of the capacitance is concentrated at the mid point of the line.

Voltage Drop. The voltage drop can be easily calculated for the simpler equivalent circuits when the vector diagrams are drawn.

Thus for a short overhead line, in which the effects of capacitance can be ignored, the vector diagram is shown in Fig. 154, in which OI represents the current; OV_2, the voltage at the load end of the line; V_2aV_1, the impedance voltage triangle for the line; and OV_1,

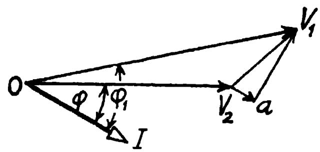


FIG. 154. VECTOR DIAGRAM FOR SHORT TRANSMISSION LINE

the voltage at the generating end of the line. The voltage drop—i.e. the arithmetic difference between the voltages at the generating end and the load, or receiver, end—is given by

$$V_1 - V_2 = \sqrt{[(V_2 + IR\cos\varphi + IX\sin\varphi)^2 + (IX\cos\varphi - IR\sin\varphi)^2]} - V_2 \quad . \quad . \quad (71)$$

If φ is leading instead of lagging,

$$V_1 - V_2 = \sqrt{[(V_2 + IR\cos\varphi - IX\sin\varphi)^2 + (IX\cos\varphi + IR\sin\varphi)^2]} - V_2 \quad . \quad (72)$$

where V_1, V_2 are the voltages at the generator and load ends of the line respectively, I the current, and φ the phase difference between the load voltage and the current.

For *cable transmission*, in cases where the inductance can be ignored, the voltage drop is given by equation (72) when the load power factor is lagging, and by equation (71) when the load power factor is leading.

For a *long overhead line*, or any line in which the effects of capacitance must be considered, the vector diagrams for the nominal Π and T equivalent circuits are shown in Figs. 155, 156, from which the voltage drop can be readily calculated.

In drawing the vector diagram (Fig. 155) for a nominal Π line, the charging current, OI_{c2}, at the load end of the line is combined vectorially with the load current, OI, to obtain the line current, OI_L. The impedance voltage triangle, V_2aV_1, is then drawn with reference to this current (OI_L), and the voltage (OV_1) at the generating end of the line is determined. The current, OI_G, at this end

of the line is the vector sum of the line current and the charging current, OI_{C1}, at the generating end of the line.

In drawing the vector diagram (Fig. 156) for a nominal T line, the impedance voltage triangle, V_2aV_M, for the load-end half of the line is compounded with the load voltage, OV_2, to obtain the voltage, OV_M, at the mid point of the line. The charging current, OI_C, is then calculated and is compounded with the load current to obtain the line current, OI_G, in the generating-end half of the line. The impedance voltage triangle, V_MbV_1, for this portion of the line

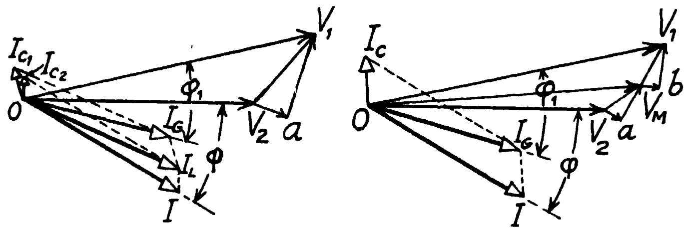


FIG. 155 FIG. 156

VECTOR DIAGRAMS FOR Π AND T EQUIVALENT CIRCUITS OF LONG TRANSMISSION LINES

is then compounded with the mid-point voltage to obtain the voltage, OV_1, at the generating end.

Example. A three-phase line, 50 miles long, has a resistance of 22 Ω. per line, an inductance of 95 mH. per line, and a star capacitance of 0·75 μF. Calculate the voltage at the generating end to supply a load of 10,000 kVA. at 132,000 V., 0·9 power factor (lagging), 50 cycles.

Approximate Solution for a Nominal T *Equivalent Circuit*—

Line current at load
$= 10{,}000/(\sqrt{3} \times 132) = 43{\cdot}7$ A.

Actual voltage drop due to resistance of load-end half of line
$= 43{\cdot}7 \times \frac{1}{2} \times 22 = 480$ V.

Actual voltage drop due to inductance of load-end half of line
$= 43{\cdot}7 \times \frac{1}{2} \times 0{\cdot}095 \times 314 = 652$ V.

Voltage at mid-point of line

$$= \sqrt{3}\sqrt{\{[(132{,}000/\sqrt{3}) + 480 \times 0{\cdot}9 + 652 \times 0{\cdot}435]^2 + (652 \times 0{\cdot}9 - 480 \times 0{\cdot}435)^2\}}$$

$$= \sqrt{3}\sqrt{[(76{,}100 + 432 + 284)^2 + (587 - 209)^2]} = 133{,}245 \text{ V.}$$

(This calculation shows that the angle between the vectors OV_2 and OV_M, Fig. 156, is so small that it need not be taken further into account.)

Hence, charging current
$= 314 \times 0{\cdot}75 \times 133{,}245/(\sqrt{3} \times 10^6) = 18{\cdot}1$ A.

Line current at generator end of line
$= \sqrt{[(0{\cdot}436 \times 43{\cdot}7 - 18{\cdot}1)^2 + (0{\cdot}9 \times 43{\cdot}7)^2]} = 39{\cdot}3$ A.

Phase difference between OV_M and OI_G is approximately zero.

∴ Voltage at generator
$133{,}245 + \sqrt{3} \times 480 \times 39{\cdot}3/43{\cdot}7 = 133{,}995$ V.

Weight of Line Conductors for Single-phase and Polyphase Transmission Systems. In comparing the weight of copper required for the line conductors of single-phase and polyphase systems, it is essential to base the comparison upon similar conditions in each case. Thus the total power, the power factor, and the distance over which the power is transmitted must obviously be the same for the several cases under consideration. Other conditions which must be co-related are: the efficiency of the transmission, the percentage drop in voltage, the current density in the line wires,

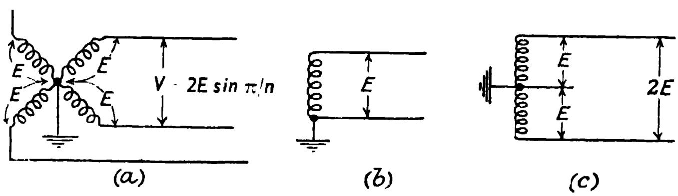


FIG. 157. CIRCUIT DIAGRAMS OF POLYPHASE AND SINGLE-PHASE TRANSMISSION SYSTEMS WITH EARTHED NEUTRAL POINT

the voltage between the line wires, and the voltage between each line wire and earth. Some of these conditions are independent of one another, while others are inter-dependent. For example, in systems in which the neutral point is earthed the voltage between line wires depends upon the number of phases and the phase voltage. Hence, if the voltage between any line wire and earth (i.e. the phase voltage of the system) is fixed, the voltage between the line wires of a given system will depend upon the number of phases in that system.

Again, if the reactance of the line is neglected the percentage drop in voltage in the transmission system is equal to the I^2R loss in the line wires expressed as a percentage of the total power transmitted, and since the efficiency of transmission is given by [1 - (power expended in line wires/total power transmitted)], the percentage drop in voltage and the efficiency of transmission are inter-dependent.

Case I. Star-connected Systems with Earthed Neutral Point (Fig. 157).

Let n = number of phases, P = total power supplied to the transmission lines, E = phase voltage, $\cos\varphi$ = power factor. Then, assuming sinusoidal E.M.Fs., the voltage between adjacent line wires, taken in order, is equal to $2E\sin\pi/n$. Also with balanced loads the power transmitted by each phase is equal to P/n, and the current in each wire is equal to $P/(nE\cos\varphi)$. Hence the I^2R

loss in each line is given by $R_p(P/nE \cos\varphi)^2$, where R_p is the resistance of each line. If the efficiency is to be the same for each case the loss in the line wires must bear a fixed ratio to the power transmitted, and therefore

$$\frac{R_p(P/nE \cos\varphi)^2}{P/n} = 1 - \eta = \zeta, \text{ a constant,}$$

where η is the efficiency of transmission.

Now $R_p = \rho l/a$, where ρ is the specific resistance of the material of the conductors and l, a, are the length and cross section, respectively, of each. Hence

$$\zeta = \frac{\rho P l}{E^2 \cos^2\varphi} \cdot \frac{1}{na}$$

i.e. the product na is a constant. But na is proportional to the total weight of the line conductors.

Thus the amount of copper required for the line wires for the transmission of a given power under similar conditions is the same for all star-connected polyphase systems with earthed neutral point. Since, however, the three-phase system gives the lowest overall cost when insulators, poles, towers, labour, etc., are considered, this system is always employed in practice in preference to other polyphase systems.

COMPARISON WITH SINGLE-PHASE SYSTEM. To obtain a comparison with the single-phase system under similar conditions, we shall consider the case in which one terminal of the generator is earthed (Fig. 157 (*b*)). The line current $= P/E \cos\varphi$, and the I^2R loss in line wires $= 2R_s(P/E \cos\varphi)^2$, where R_s is the resistance of each line.

Hence for the same efficiency of transmission as in the polyphase systems we must have

$$\frac{2R_s(P/E \cos\varphi)^2}{P} = \frac{R_p(P/nE \cos\varphi)^2}{P/n}$$

Whence $$2R_s = R_p/n,$$

i.e. $$a_s = 2na_p,$$

where a_s, a_p, are the cross-sections of the line conductors for the single and polyphase systems respectively.

Hence

$$\frac{\text{Weight of line conductors for single-phase system}}{\text{Weight of line conductors for polyphase system}}$$

$$= \frac{2a_s}{na_p} = \frac{2 \times 2na_p}{na_p} = 4,$$

i.e. the weight of the line conductors for a single-phase system with one terminal earthed is four times that for a star-connected polyphase system with earthed neutral point.

The current density in the conductors of the single-phase system, however, is only one-half of that in the conductors of the polyphase system.

For the special, and unusual, case in which the mid-point of the generator winding is earthed (Fig. 157 (c)), the weight of the line conductors is the same as that for a polyphase system.

Case II. Non-earthed Systems (Fig. 158). The comparison will

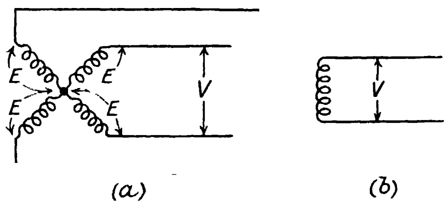


FIG. 158. CIRCUIT DIAGRAMS OF POLYPHASE AND SINGLE-PHASE TRANSMISSION SYSTEMS WITH INSULATED NEUTRAL POINT

be based upon equal line voltages in all cases, the other governing conditions being the same as before.

Considering star-connected systems for convenience, let V = line voltage. Then the phase voltage, E, of an n-phase system is equal to $V/2 \sin \pi/n$, and the line current is equal to

$$P/nE \cos \varphi = (2P \sin \pi/n)/(nV \cos \varphi).$$

If R_p' is the resistance of one line, the I^2R loss in each line is given by $R_p'[(2P \sin \pi/n)/(nV \cos \varphi)]^2$. Hence for the same efficiency of transmission as before we must have

$$\frac{R_p'[(2P \sin \pi/n)/(nV \cos \varphi)]^2}{P/n} = \zeta, \text{ a constant,}$$

i.e. $$4R_p' \frac{\sin^2 \pi/n}{n} \cdot \frac{P}{V^2 \cos^2 \varphi} = \zeta,$$

or $$\frac{4\sin^2 \pi/n}{na_p'} \cdot \frac{P\rho l}{V^2 \cos^2 \varphi} = \zeta,$$

whence $na_p' = 4\sin^2 \pi/n \times$ a constant,

where a_p' = cross-section of each line conductor.

Now na'_p is proportional to the total weight of the line conductors Hence in this case the weight of the line conductors varies with the number of phases, becoming smaller as the number of phases is increased.

The decrease in weight of the line conductors due to the increase in the number of phases results in an increase in the current density.

COMPARISON WITH SINGLE-PHASE SYSTEM. In the single-phase system, the line current is given by $P/V \cos \varphi$, and if the resistance of each line is R_s'' the total line I^2R loss is $2R_s''\ (P/V \cos \varphi)^2$.

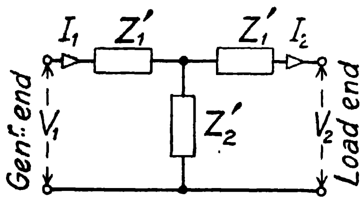


FIG. 159

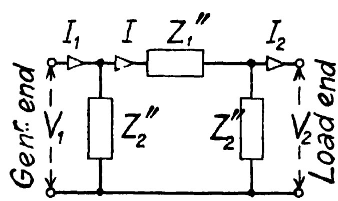


FIG. 160

CIRCUIT DIAGRAMS FOR NOMINAL T AND NOMINAL Π TRANSMISSION LINES

Hence for the same efficiency of transmission as in the polyphase systems, we must have

$$\frac{2R''_s(P/V \cos \varphi)^2}{n} = 4R_p'\,\frac{\sin^2 \pi/n}{n} \cdot \frac{P}{V^2\cos^2\varphi}$$

Whence $R_s'' = (2R_p' \sin^2 \pi/n)/n$

and $\quad 2a_s'' = na_p'/\sin^2 \pi/n$

i.e. $$\frac{\text{Weight of line conductors for single-phase system}}{\text{Weight of line conductors for polyphase system}} = \frac{1}{\sin^2 \pi/n}$$

Therefore for non-earthed systems and equal voltages of transmission the weight of the line conductors for the single-phase system is 1·33 times that of the line conductors for the three-phase system.

The Constants of a Long Transmission Line Having Distributed Resistance, Inductance, and Capacitance. Consider a single-phase, or two-wire, line, and let R, L, C denote the resistance, inductance, and capacitance per mile of line. Then for an element, dx, of the line, the incremental values of the series and shunt impedances will be $(R + j\omega L)dx$ and $-j\omega Cdx$ respectively.

Hence, if distances are taken in the direction from the generator to the load, and if the current in the element is $\underset{.}{I}$, the voltage drop or decrease of voltage across the element will be given by $d\underset{.}{V} = \underset{.}{I}(R + j\omega L)dx$. Similarly, if $\underset{.}{V}$ is the voltage at the generator end of the element, the decrease of line current will be given by $d\underset{.}{I} = -\underset{.}{V}(j\omega C)dx$. In both cases the assumption is made that $\underset{.}{I}$ and $\underset{.}{V}$ are constant over the infinitesimal length dx.

Whence $\quad d\underset{.}{V}/dx = -\underset{.}{I}(R + j\omega L)$

$\quad d\underset{.}{I}/dx = -\underset{.}{V}(j\omega C)$

By differentiation and substitution, we obtain

$$d^2V/dx^2 = Vj\omega C(R + j\omega L)$$
$$d^2I/dx^2 = Ij\omega C(R + j\omega L),$$

the solutions of which are

$$V = A_1 \cosh \beta x + A_2 \sinh \beta x \quad . \quad . \quad . \quad (73)$$
$$I = -(A_2/Z_0) \cosh \beta x - A_1/Z_0 \sinh \beta x \quad . \quad . \quad . \quad (74)$$

where A_1, A_2 are constants of integration, the values of which depend upon the terminal conditions of the line, $\beta(= \sqrt{[j\omega C(R + j\omega L)]})$ is called the *propagation constant* of the line, and $Z_0(= \sqrt{[(R + j\omega L)/j\omega C]})$ is called the *characteristic impedance* (or *surge impedance*) of the line. [NOTE. When R can be neglected $Z_0 = \sqrt{(L/C)}$.]

At the generator end of the line we have $x = 0$, $V = V_1$, $I = I_1$, whence $A_1 = V_1$ and $A_2 = -I_1Z_0$.

Substituting these values in equations (73), (74), we have

$$V_x = V_1 \cosh \beta x - I_1Z_0 \sinh \beta x \quad . \quad . \quad . \quad (75)$$
$$I_x = I_1 \cosh \beta x - V_1/Z_0 \sinh \beta x \quad . \quad . \quad . \quad (76)$$

where V_x and I_x are the voltage and current respectively at a distance x miles from the generating end of the line.

Hence the voltage and current at the load end of a line, of length l miles, are

$$V_2 = V_1 \cosh \beta l - I_1Z_0 \sinh \beta l \quad . \quad . \quad . \quad (77)$$
$$I_2 = I_1 \cosh \beta l - (V_1/Z_0) \sinh \beta l \quad . \quad . \quad . \quad (78)$$

If the load voltage (V_2) and current (I_2) are known and distances, x', are measured from the load end of the line, then

$$V_x' = V_2 \cosh \beta x' + I_2Z_0 \sinh \beta x' \quad . \quad . \quad . \quad (79)$$
$$I_x' = I_2 \cosh \beta x' + V_2/Z_0 \sinh \beta_x' \quad . \quad . \quad . \quad (80)$$

Constants of Equivalent T and Π Lines. From the circuit diagram of Fig. 159, we have

$$V_1 = I_1Z_1' + (I_1 - I_2)Z_2'; \; V_2 = (I_1 - I_2)Z_2' - I_2Z_1',$$

from which we obtain

$$I_2 = I_1(1 + Z_1'/Z_2') - V_1/Z_2'.$$

For this expression to be identical with (78), we must have

$$(1 + Z_1'/Z_2') = \cosh \beta l \text{ and } (1/Z_2') = (1/Z_0) \sinh \beta l.$$

Whence $Z_1' = Z_0 \sinh \tfrac{1}{2}\beta l$ and $Z_2' = Z_0/\sinh \beta l$ . . . . (81)

From the circuit diagram of the equivalent Π circuit (Fig. 160), we have

$$V_1 = (I_1 - I)Z_2''; \; V_2 = (I - I_2)Z_2''; \; IZ_1'' = (I_1 - I)Z_2'' - (I - I_2)Z_2''$$

Whence $I_2 = I_1(1 + Z_1''/Z_2'') - V_1(Z_1'' + 2Z_2'')/Z_2''^2$

For this expression to be identical with (78), we must have

$$Z_1'' = Z_0 \sinh \beta l; \; Z_2'' = Z_0(1 + \cosh \beta l)/\sinh \beta l = Z_0 \coth \tfrac{1}{2} \beta l \quad . \quad (82)$$

CHAPTER XI

ALTERNATORS—SINGLE-PHASE AND POLYPHASE

Practical Forms of Alternators. The elementary method of generating alternating E.M.Fs. by rotating a conductor, or a system of conductors, in a stationary magnetic field is, in practice, only employed for alternators of very small output, because of the difficulties involved in the design and construction of the windings, and the collection of current by sliding contacts, in large machines. These difficulties are avoided when the alternative method—with stationary conductors and a rotating magnetic field—is employed. All alternators for outputs above about 10 kW. are now, therefore, built with stationary armature conductors and rotating field magnets.

The stationary armature conductors are located in slots in the inner periphery of a ring of alloyed-steel laminations built into a frame of either cast iron or welded steel plate, the whole arrangement being called the *stator*.

The revolving field magnets, known as the *rotor*, may be built either in the form of salient poles, each with its exciting coil, fitted to a central hub on a shaft, or in the form of a cylinder of forged steel, with slots in the outer periphery for the excitation winding. The latter form of construction is employed in machines of large output, which are direct-coupled to steam turbines. With both forms, the exciting current is supplied to the winding by means of slip-rings and brushes, but as only relatively small currents at low voltage are required, these sliding contacts present no difficulties.

Number of Poles, Speed, and Frequency. In an alternator with two poles, one cycle of E.M.F. is produced at each revolution, and therefore the frequency is equal to the number of revolutions per second. If the number of poles is doubled, the frequency will be equal to twice the number of revolutions per second. Hence, in general, if p = number of poles, f = frequency, n = revolutions per second, then $f = \frac{1}{2}pn$, or $n = f/\frac{1}{2}p$.

For the standard frequency of 50 cycles per second, the operating speeds, in revolutions per minute, corresponding to different numbers of poles are as follows—

Number of poles	2	4	6	8	10	12	14	16	18	20	22	24
Speed, r.p.m.	3000	1500	1000	750	600	500	428	375	333	300	272·7	250

Conductor E.M.F. The E.M.F. generated in a single conductor when cut by a magnetic field is, at any instant, given by $e = Blv \times 10^{-8}$ volts, where B is the flux density in which the conductor is situated at the instant under consideration; l, the active length of the conductor, and v, the speed of the magnetic field relative to the conductor.

Now, in an alternator working under normal conditions, l and v are constant, so that $e \propto B$. Hence the instantaneous value of the E.M.F. is directly proportional to the actual value of the flux density sweeping past the conductor at that instant. Since one cycle of E.M.F. corresponds to the movement of two magnetic poles (i.e. two-pole pitches) past the conductor, the wave-form of the E.M.F. will be similar to that of the space distribution of the flux.

For example, a flat-topped flux distribution curve due to a uniform air gap will result in a flat-topped conductor-E.M.F., and a sine-wave flux distribution over the pole pitch will result in a sine-wave E.M.F.

With salient poles, therefore, the pole shoes should be shaped or chamfered so as to give an air gap which is a minimum at the centre of the pole face. With a cylindrical rotor, an approximation to a sine-wave flux distribution is obtained with a uniform air gap by distributing the turns of the exciting winding over the pole pitch which is slotted for this purpose.

Coil E.M.F. Since alternator windings are always of the drum type, the two coil-sides of a *full-pitch* coil will always be situated in flux densities of the same magnitude, but of opposite sign, because of the symmetrical construction of the field magnets. The instantaneous E.M.Fs. in the coil-sides will, therefore, be equal in magnitude; and since these E.M.Fs. act in the same direction round the coil, the coil-E.M.F. will be equal to twice the conductor-E.M.F., and the wave-form of the coil-E.M.F. will be the same as that of the conductor-E.M.F.

R.M.S. Value of Coil-E.M.F. for Full-pitch Coil. Let the coil consist of one turn of full pitch, and let Φ denote the flux per pole; p, the number of poles; n, the revolutions per second; f, the frequency. Then during one revolution, each conductor will be cut by a flux $p\Phi$. Hence the mean value of the E.M.F. in the coil will be given by

$$E_a = 2p\Phi n \times 10^{-8}$$

$$= 4\Phi f \times 10^{-8}$$

since $pn = 2f$.

If E denotes the R.M.S. value of this E.M.F., and K_f denotes the form factor (i.e. $K_f = E/E_a$ = R.M.S. value/mean value), then

$$E = 4K_f\Phi f \times 10^{-8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

For a sine wave $K_f = (1/\sqrt{2})/(2/\pi) = \pi/2\sqrt{2} = 1{\cdot}11$, so that when the flux distribution is sinusoidal,

$$E = 4{\cdot}44\Phi f \times 10^{-8}$$

Effect of Coil Pitch. With a fractional-pitch coil, for which the pitch = θ electrical degrees, the E.M.Fs. in the two coil-sides, or conductors, will have the same wave-form, but will not have equal

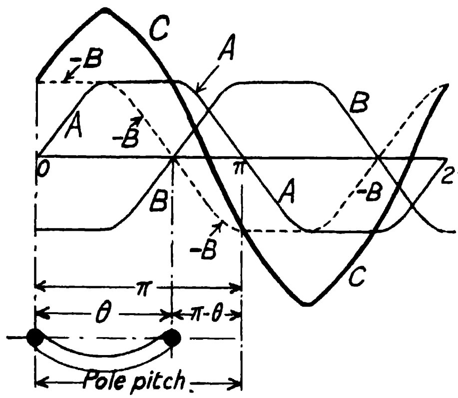


FIG. 161. WAVE-FORMS OF CONDUCTOR-E.M.FS. (A, B) AND COIL-E.M.F. (C) FOR FRACTIONAL-PITCH COIL OF ALTERNATOR

numerical values at the same instant. These E.M.Fs. may, therefore, be represented by two similar wave-forms displaced from each other by the angle $(\pi - \theta)$. The coil-E.M.F. will, therefore, be given by the sum of these wave-forms, as shown in Fig. 161.

If the conductor-E.M.Fs. are sine waves, the coil-E.M.F. will be of sine wave-form, but its maximum value will be less than twice that of each of the conductor-E.M.Fs. If, however, the conductor-E.M.Fs. are not sine waves, the wave-form of the coil-E.M.F. will differ from that of the former, and will approximate closer to a sine wave when the difference between the pole pitch and the coil pitch is small.

In the special case when the conductor-E.M.F. consists of a fundamental and a harmonic of the nth order, the coil-E.M.F. will be a sine wave if the coil pitch $= \pi - \pi/n = \pi(n-1)/n$, i.e. the coil pitch is π/n less than the full pitch. For example, if the third harmonic is present in the conductor-E.M.F., the coil-E.M.F. will

be sinusoidal when the coil pitch is $\frac{2}{3}\pi$ or 120°. Similarly, if the fifth harmonic is present in the conductor-E.M.F., the coil-E.M.F. will be sinusoidal when the coil pitch is $\frac{4}{5}\pi$ or 144°. A little investigation into these cases will show that the harmonic components of the E.M.Fs. in the two coil-sides have a phase difference of 180° and, therefore, cancel out in the coil-E.M.F.

Effect of Spread of Coil Side. Consider two full-pitch coils, each of one turn, with the coil-sides spaced $a°$ apart. If these coils are connected in series, the E.M.F. in the double coil will be represented by the sum of the wave-forms of the E.M.Fs. in the individual turns, i.e. by the sum of two similar wave-forms displaced

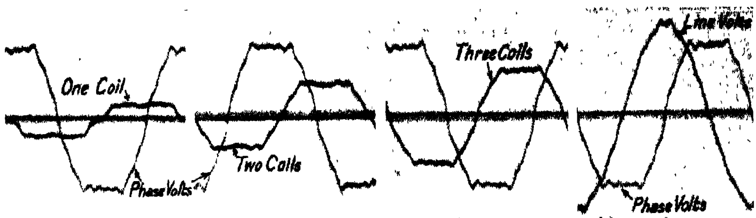


FIG. 162. OSCILLOGRAMS OF NO-LOAD E.M.F. OF THREE-PHASE ALTERNATOR, SHOWING EFFECT ON WAVE-FORM OF WINDING DISTRIBUTION AND INTER-CONNECTION (STAR) OF PHASES

The machine has four poles, and the stator winding is of the half-coiled type, with three coils per pair of poles per phase

For the purpose of obtaining the oscillograms of the coil-E.M.Fs., the ends of the individual coils of a coil-group of one phase were brought out to separate terminals, so that wave-forms could be obtained of the E.M.F. of (*a*) one coil, (*b*) two coils in series, (*c*) three coils in series

The oscillogram of the phase voltage was obtained from another phase of the winding with the full number of coils in circuit

$a°$ from each other. The conditions are, therefore, similar to those obtaining in a fractional-pitch coil having a pitch $(\pi - a)$.

Similarly, if three one-turn coils, with the coil sides displaced uniformly from one another, are connected in series, the E.M.F. in the triple coil will be represented by the sum of three similar wave-forms displaced uniformly from one another.

If the angle of displacement is small and the conductor-E.M.F. is non-sinusoidal, the resulting wave-form will more nearly approximate to a sine wave than that of the E.M.F. in the individual turns. This is illustrated in Fig. 162 by the reproductions from actual oscillograms of the coil-E.M.Fs. of an alternator.

Winding Distribution Factor. If, in the preceding case, the E.M.Fs. in the individual turns are sinusoidal, the E.M.F. of the coil-group can easily be calculated, since the R.M.S. values of the former may be added vectorially. Thus, for the case of a two-coil

group, the vector diagram for which is shown in Fig. 163, if the E.M.F. in each turn or coil is denoted by E_c and that in the coil-group by E, then $E = 2E_c \cos \frac{1}{2}\alpha$.

Now $2E_c$ is the arithmetic sum of the coil E.M.Fs. Hence the ratio $E/2E_c$ (i.e. vector sum of E.M.Fs./arithmetic sum of E.M.Fs.)

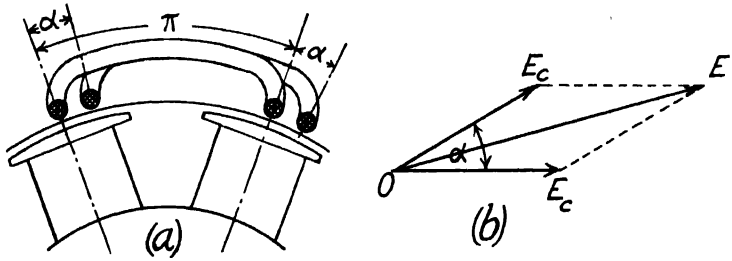
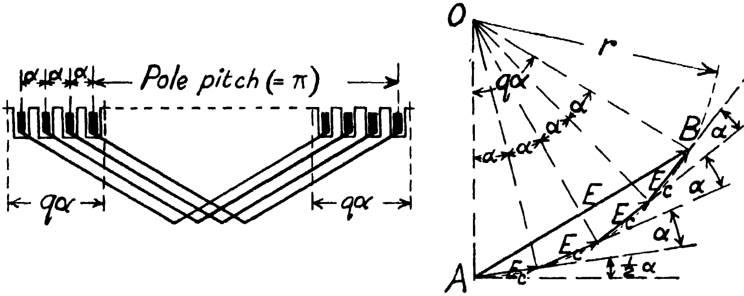


FIG. 163. SPACE DIAGRAM (*a*) FOR TWO FULL-PITCH COILS, AND VECTOR DIAGRAM (*b*) OF E.M.Fs.

$= \cos \frac{1}{2}\alpha$, which is called the *winding distribution factor* or *breadth coefficient*, and is denoted by K_b.

For a three-coil group, $K_b = \frac{1}{3}(1 + 2 \cos \alpha)$, and for a four-coil group, $K_b = \frac{1}{2}(\cos \frac{1}{2}\alpha + \cos \frac{3}{2}\alpha)$, where α is the angle between



(*a*) (*b*)

FIG. 164. SPACE DIAGRAM (*a*) OF DISTRIBUTED WINDING, AND VECTOR DIAGRAM (*b*) OF E.M.Fs.

adjacent turns or coil-sides of the individual coils. These values can readily be checked by drawing the vector diagrams.

If a vector polygon and its circumscribing arc are drawn, as in Fig. 164, for a group of q coils (q = number of slots per pole per phase), each side of the polygon can be expressed in terms of the radius, r, of this arc. Thus $E_c = 2r \sin \frac{1}{2}\alpha$. The chord, which represents the coil-group E.M.F., E, is given by $E = 2r \sin \frac{1}{2}q\alpha$. Whence,

$$K_b = E/qE_c = \sin \tfrac{1}{2}q\alpha / q \sin \tfrac{1}{2}\alpha \quad . \quad . \quad . \quad . \quad (84)$$

In the special case when the conductors of a coil-group side are packed closely together so as to be uniformly distributed over an angle β (which is called the *phase spread*, or, shortly, the *spread*, of the winding), the vector polygon for the individual E.M.Fs. can be considered to be a sector of a circle subtending an angle β. Hence, for this case,

$$K_b = \text{chord/arc} = \sin \tfrac{1}{2}\beta/\tfrac{1}{2}\beta \qquad . \quad . \quad . \quad . \qquad (85)$$

Analytical Deduction of Winding Distribution Factor. Since the E.M.Fs. in the individual coils are equal in magnitude but have a mutual phase difference of a, they can be expressed in terms of the E.M.F. in coil No. 1, which may be considered as a reference quantity. Thus $\underset{\cdot}{E}_{c1} = \underset{\cdot}{E}_c\varepsilon^{j0}$; $\underset{\cdot}{E}_{c2} = E_c\varepsilon^{ja}$; $\underset{\cdot}{E}_{c3} = E\varepsilon^{j2a}$; . . .

Hence the coil-group E.M.F. is

$$\underset{\cdot}{E} = \underset{\cdot}{E}_{c1} + \underset{\cdot}{E}_{c2} + \underset{\cdot}{E}_{cb} + \ . \ . \ . = E_c(1 + \varepsilon^{ja} + \varepsilon^{j2a} + \ . \ . \ . \ )$$
$$= E_c(\varepsilon^{jqa} - 1)/(\varepsilon^{ja} - 1)$$
$$= E_c[(\cos qa - 1) + j \sin qa]/[(\cos a - 1) + j \sin a].$$

Whence $E = E_c\sqrt{\{[(\cos qa - 1)^2 + \sin^2 qa]/[(\cos a - 1)^2 + \sin^2 a]\}}$

$$= E_c(\sin \tfrac{1}{2}qa)/\sin \tfrac{1}{2}a$$

and $K_b = E/qE_c = \sin \tfrac{1}{2}qa/q \sin \tfrac{1}{2}a$ . . .

In the special case of a uniformly-distributed winding with a spread of β, a approaches zero, and the value of K_b cannot be evaluated from this expression. Since, in this case, $\beta = qa$, the denominator ($q \sin \tfrac{1}{2}a$) can be written in the form $\beta \sin \tfrac{1}{2}a/a$; and if both numerator and denominator are differentiated with respect to a, we can determine the values in the limit when $a = 0$.

Thus $d(\beta \sin \tfrac{1}{2}a)/da \qquad = \tfrac{1}{2}\beta \cos \tfrac{1}{2}a$, and $da/da = 1$.

Hence in the limit when $a = 0$, $\beta \sin \tfrac{1}{2}a/a = \tfrac{1}{2}\beta$.

Therefore, for a uniformly-distributed winding,

$$K_b = \sin \tfrac{1}{2}\beta/\tfrac{1}{2}\beta$$

Coil Pitch Factor and Winding Distribution Factor for Non-sinusoidal E.M.Fs. In Chapter XIV it is shown that a non-sinusoidal E.M.F. can be represented by a series of sine-wave E.M.Fs. having frequencies 1, 3, 5, 7, . . . times the frequency of the non-sinusoidal E.M.F. (which is called the fundamental frequency). The triple-frequency component is called the third harmonic; the quintuple-frequency component is called the fifth harmonic; and so on.

With the usual (symmetrical) construction of the field magnets, only odd harmonics can occur in the wave-form of the flux distribution when this is not a sine wave, and, therefore, in this case the wave-form of the conductor-E.M.F. may be represented by $e = E_{1m} \sin \omega t + E_{3m} \sin 3\omega t + E_{5m} \sin 5\omega t + \ . \ . \ .$ Hence a time-angle, ωt, taken with reference to the fundamental becomes $n\omega t$ when taken with reference to the nth harmonic. Therefore a coil having a pitch of two-thirds of the pole pitch, or 120°, is, so far as the third harmonic of the flux wave-form is concerned, equivalent to a coil having a pitch of $3 \times 120 = 360°$; and so far as the fifth harmonic of the flux wave-form is concerned, this coil is equivalent to one having a pitch of $5 \times 120 = 600°$.

Thus the *coil pitch factor* for any particular harmonic—e.g. the nth—is given by

$$K_{pn} = \sin n\tfrac{1}{2}\theta$$

where θ is the pitch of the coil.

Similarly, the *winding distribution factor* for the nth harmonic is given by

$$K_{bn} = \sin \tfrac{1}{2}nq\alpha/q \sin \tfrac{1}{2}n\alpha$$

Winding Distribution Factors for Three-phase and Single-phase Machines. In a uniformly distributed winding for a three-phase alternator, the maximum value of β is 60°, and for a single-phase alternator the maximum value of β is 180°. The corresponding values of K_b are 0·955 ($= \sin 30°/\frac{1}{6}\pi = 3/\pi$) and 0·637 ($= 2/\pi$) respectively. These values show that if the whole of the stator periphery is utilized in the two cases, the utilization factor will be much lower for the single-phase alternator than for the three-phase machine.

Values of the winding distribution factor for three-phase and single-phase machines for sinusoidal E.M.Fs. and different numbers of slots per pole are as follows—

	Number of slots per pole .	1	2	3	4	6	9	12	15	18	∝
Three-phase Windings	No. of slots per pole per phase (q) .	—	—	1	—	2	3	4	5	6	∝
	Spacing of slots, electrical degrees (α)	-	—	—	—	30	20	15	12	10	—
	Winding distribution factor (K_b) .	—	—	1·0		0·966	0·96	0·958	0 957	0·956	0·955
Single-phase Windings	No. of slots per pole per phase (q) .	1	2	3	4	6	9	—	-	∝	∝
	Spacing of slots, electrical degrees (α)	—	20	20	20	20	20	—	—	—*	—†
	Winding distribution factor (K_b) .	1·0	0·985	0·96	0 925	0·832	0 64			0·637	0·827

* 180° spread. † 120° spread.

Relative Outputs of Uniformly-distributed Three-phase and Single-phase Windings. Consider a given stator wound (1) as a single-phase alternator, (2) as a three-phase alternator, the flux per pole, the frequency, and the total number of conductors, z, being the same in each case. Then the E.M.F. of the single-phase machine will be proportional to $\frac{1}{2}zK_{bs}$, or to $\frac{1}{2}z \times 0{\cdot}637$, and the phase E.M.F. of the three-phase machine will be proportional to $\frac{1}{3}(\frac{1}{2}z)K_{bt}$, or to $\frac{1}{3} \times \frac{1}{2}z \times 0{\cdot}955$. If the permissible current per conductor is I in each case, the VA. output in the single-phase case will be proportional to $\frac{1}{2}zI \times 0{\cdot}637$, and that in the three-phase case will be proportional to $3 \times \frac{1}{3} \times \frac{1}{2}zI \times 0{\cdot}955$. Whence the outputs are in the ratio of 0·637 to 0·955 or 1·0 to 1·5, i.e. the output as a three-phase machine is 1·5 times that as a single-phase machine. In practice, owing to armature reaction and other effects, the ratio of outputs will be greater than this value.

General E.M.F. Equation. Let Φ denote the flux per pole; f, the frequency; N, the number of turns in series per phase ($= \frac{1}{2}$ $\times$ number of conductors in series per phase); K_f, the form factor;

K_p, the coil-pitch factor ($= \sin \frac{1}{2}\theta$, where θ is the coil pitch); K_b, the winding distribution factor ($= \sin \frac{1}{2}q\alpha/q \sin \frac{1}{2}\alpha$).

Then the R.M.S. value of the E.M.F. per phase is given by

$$E = 4K_f K_p K_b \Phi N f \times 10^{-8} \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Example. The stator of a three-phase, 50-cycle, 16-pole alternator has 96 slots, and each slot contains 12 conductors, connected in series. If the flux per pole is 5 megalines and is sinusoidally distributed, calculate the no-load E.M.F. if the phases are star-connected and the coils are of full pitch.

The total number of conductors $= 96 \times 12 = 1152$,
and the number of turns per phase $= \frac{1}{3} \times \frac{1}{2} \times 1152 = 192$.
The number of slots per pole $= 96/16 = 6$.

Hence each coil-group consists of two 12-turn coils, and the angle between adjacent coil-sides is $\frac{1}{6}\pi$ or 30°. Whence

$$K_b = \cos \tfrac{1}{2} \times 30^\circ = 0{\cdot}966.$$

Therefore the E.M.F. per phase

$$= 4 \times 1{\cdot}1 \times 1{\cdot}0 \times 0{\cdot}966 \times 5 \times 10^6 \times 192 \times 50 \times 10^{-8}$$
$$= 2060 \text{ V}.$$

and the terminal voltage $= \sqrt{3} \times 2060 = 3570$ V.

Effect of the Stator Slots and Teeth on the Wave-form of the E.M.F. In a salient-pole alternator with the poles shaped so as to give a sine-wave flux distribution with a smooth (i.e. unslotted) stator core, the E.M.F. generated in the conductors will be a sine wave. But if the conductors are located in slots, the E.M.F. wave-form will now show ripples and may be somewhat similar to that shown in Fig. 201 (p. 298).

The cause of these ripples is easily explained by considering the physical conditions under which the E.M.Fs. are produced. Thus with a slotted core the flux tends to concentrate in the teeth, and when the pole moves relatively to a slot, the lines carried by one tooth swing across the slot to the adjacent tooth, and cut the conductors in the slot at relatively high speed compared with the movement of the pole and flux as a whole. This "flux swinging," therefore, will produce E.M.Fs. in the conductors, but these E.M.Fs. will be of relatively low amplitude and high frequency compared with the E.M.F. produced by the main flux. Hence the former will be superimposed upon the latter in the form of ripples in the wave-form.

With machines having nearly closed slots and a relatively large number of slots per pole, the ripples will usually be small and may not be objectionable in practice. But in machines having open slots or relatively wide slot-openings, the ripples may be practically suppressed by skewing either the slots or the pole faces by an amount equal to the slot pitch. Alternatively, the stator may be

built with an *odd* number of slots per phase. With such an arrangement of slots and a suitable winding, the equivalent of a uniformly-distributed winding is obtained in so far as E.M.F. harmonics are concerned, and the ripple effects practically cancel out.

Stator Windings. Although closed-circuit lap and wave windings, such as are employed in direct-current machines, are applicable also to A.C. machines, they are not, however, employed for the

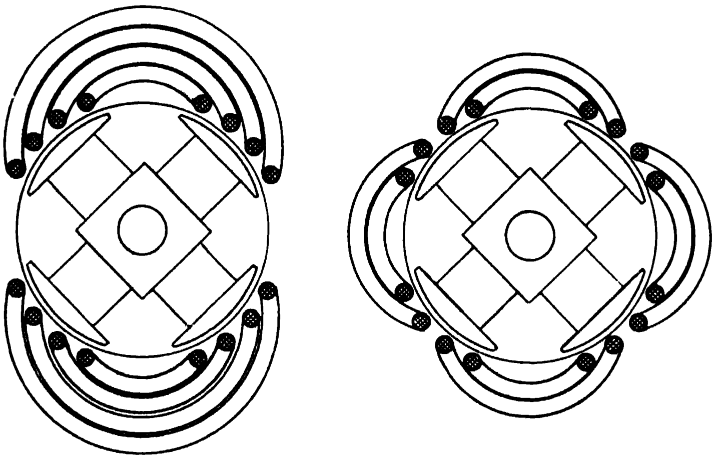


FIG. 165. DIAGRAMS SHOWING GROUPING OF COILS IN HALF-COILED (LEFT) AND WHOLE-COILED (RIGHT) WINDINGS FOR FOUR-POLE, SINGLE-PHASE ALTERNATOR

stator windings of alternators, which are invariably of the open-circuit type.

The conductors are located in slots, and although, for single-phase machines, windings with one or two slots per pole are possible—and, in fact, were employed in early single-phase alternators—such windings would not be permissible in modern machines, because of the large ripples which would occur in the wave-form of the E.M.F. due to flux swinging and pulsation. In modern three-phase alternators, the number of slots per pole varies from 6 to 36 according to the design, the smaller number being employed in medium-size machines having a relatively large number of poles, and the larger number for large turbo-alternators having two poles.

The coils formed by the conductors occupying a group of slots corresponding to one phase and one pair of poles may be arranged either in a single group or in two groups, as indicated diagrammatically in Fig. 165, which refers to single-phase windings. The

former arrangement is called *half-coiled*, or hemitropic, and the latter arrangement is called *whole-coiled*.

A whole-coiled winding possesses the advantages, over the corresponding half-coiled windings, of shorter end-connections and a lower leakage reactance; but three tiers or ranges are necessary for the end connections of a single-layer, three-phase winding with concentric or hairpin coils, whereas only two tiers are necessary for a half-coiled winding. This disadvantage, however, does not exist in double-layer windings.

In practice, whole-coiled three-phase windings (single-layer and double-layer types) are used principally for large turbo-alternators, as such windings are more suitable for these machines than half-coiled windings on account of the large number of slots per pole. For the ordinary type of salient-pole three-phase alternator, however, the winding is usually of the half-coiled type, a single-layer winding being employed for hand-wound and hairpin coils in semi-closed slots, and a double-layer or barrel winding for former-wound bar-conductor coils with diamond-shaped end connections.

Method of Connecting-up. In connecting-up the coil-groups of half-coiled and whole-coiled windings, it is important to remember that the connections must be such that a current passing from terminal to terminal of a given phase circulates in the *same* direction around every coil of a *half-coiled* winding (i.e. the coils would produce consequent poles if excited with direct current), and in *alternate* directions around the coils of a *whole-coiled* winding.

The method of connecting-up three-phase windings, half-coiled and whole-coiled, is shown in the development diagrams of Fig. 166. No difficulty will be experienced in working out the connections for a three-phase winding if it is remembered that the terminal connections must be taken from coil ends of the same polarity or sense which are two-thirds of a pole-pitch (i.e. 120°) apart relative to one another.

Voltage Drop in Stator Winding. When an alternator is loaded, the voltage drop in the stator winding is due to the effects of—

(1) resistance of the stator winding,
(2) inductance or leakage reactance of the stator winding,
(3) armature reaction.

The internal E.M.Fs. due to the effects of resistance and inductance (or leakage reactance) of the stator winding can be calculated when these quantities are known or have been determined.*

* For the method of calculating the inductance and leakage reactance of stator windings, consult *Papers on the Design of Alternating Current Machinery* by Hawkins, Smith, and Neville. (Pitman.)

If no armature reaction were present, the terminal E.M.F. could easily be determined from the vector diagram of internal E.M.Fs. (Fig. 167), which is similar to that for a transformer. In this diagram, OE represents the E.M.F. generated in the winding by the

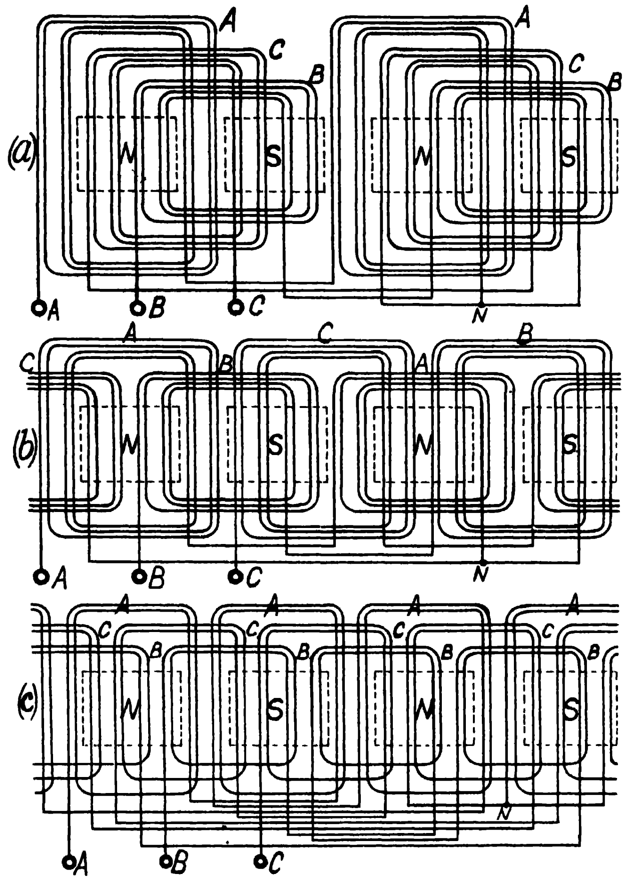


FIG. 166. DEVELOPMENT DIAGRAMS OF FOUR-POLE, THREE-PHASE WINDINGS (STAR CONNECTED)

(a) Half-coiled winding with end-connections in three ranges (suitable for split-frame machine); (b) Half-coiled winding with end-connections in two ranges; (c) Whole-coiled winding

main flux; OI, the current; ORI, the internal E.M.F. due to resistance; OXI, the internal E.M.F. due to leakage reactance, and OV, the terminal voltage.

Armature Reaction. The current in the stator winding produces

a magneto-motive force which, in a symmetrically-loaded polyphase machine, rotates synchronously with the field magnets, and is therefore stationary *relative* to the field poles.

In a single-phase alternator, however, the armature M.M.F. is stationary in space and alternates at the frequency of the current. But this alternating M.M.F. can be considered as the resultant of two equivalent rotating M.M.Fs., one (called the forward M.M.F.) rotating synchronously with the field magnets, and the other (called the backward M.M.F.) rotating at synchronous speed in the opposite direction to the field magnets.

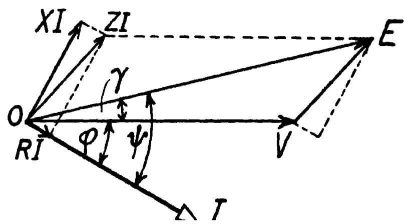


Fig. 167. Vector Diagram for Determining Internal E.M.F.

The forward M.M.F. is analogous to the armature-reaction M.M.F. in a polyphase alternator, i.e. this M.M.F. is stationary *relative* to the field magnets and, therefore, it reacts with the M.M.F. of the latter.

The backward M.M.F., however, is rotating, relative to the field magnets, at twice the normal speed of the machine; and, as this M.M.F. is unopposed, it induces in the pole shoes and adjacent parts (including also the field coils) E.M.Fs. having a frequency double the normal frequency of the machine. The resulting double-frequency eddy currents, if unrestricted, would cause losses and heating which would be detrimental to the performance of the machine. Hence, in practice, the pole shoes are fitted with damper (i.e. short-circuited) windings to provide a definite path for these currents.

Armature Reaction in Polyphase Machines. Since in a symmetrically-loaded polyphase alternator the M.M.F. due to the armature currents is stationary relative to the field magnets, the armature and field M.M.Fs. will react with each other in a manner somewhat similar to the armature reaction in a direct-current generator. Thus when the current is in phase with the generated E.M.F., the axes of the armature M.M.F. coincide with the neutral axes of the field magnets, and the armature M.M.F. has a *cross-magnetizing* effect on the field poles, corresponding to the cross-magnetizing effect in a direct-current machine when the brushes are in the neutral position.

Again, when the phase difference between the current and generated E.M.F. is 90°, the maximum value of current in a given

phase will occur when the axes of the field poles coincide with the magnetic axes of the coils of that phase. Hence, in these cases the magnetic axes of the armature M.M.F. will coincide with the poles, and the armature M.M.F. will have a *direct-magnetizing* effect, magnetizing the poles when the current is leading, and demagnetizing the poles when the current is lagging.

For a phase difference ψ, the armature will produce both direct- and cross-magnetizing effects, the direct-magnetizing effect being proportional to $\sin \psi$ and the cross-magnetizing effect being proportional to $\cos \psi$. Observe that the angle ψ is the phase difference between the generated E.M.F. and the current, and is therefore called the *internal* phase difference or the *internal* angle of lag or lead. It is determined from the vector diagram of Fig. 167.

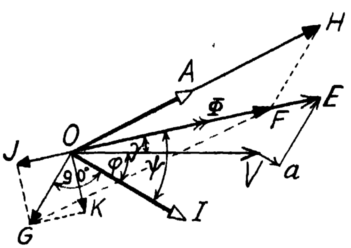


FIG. 168. VECTOR DIAGRAM FOR DETERMINING EXCITATION AMPERE-TURNS

Determination of Field Excitation. The resultant excitation on the field magnets which is necessary to obtain a given generated E.M.F. may be determined by compounding the M.M.F. due to armature reaction with the M.M.F. required to produce the requisite flux in the magnetic circuit. The vector diagram is shown in Fig. 168, in which the vector OE represents the generated E.M.F. corresponding to a terminal voltage OV, and a current OI; OF represents the M.M.F. necessary to produce the flux $O\Phi$; OG, the field M.M.F. equivalent to the armature M.M.F., and OH, the resultant M.M.F. to be produced by the field winding.

The vector OG is drawn with a phase difference of 90° lagging from the vector of the current, in order that the component vectors OJ, OK, representing the direct- and cross-magnetizing effects respectively of the armature M.M.F., may have their correct positions in relation to OF. For example, the component representing the direct-magnetizing M.M.F. must be proportional to $\sin \psi$, and must act in direct opposition to the field M.M.F. vector OF when ψ is lagging. Similarly, the component representing the cross-magnetizing M.M.F. must be proportional to $\cos \psi$ and must be perpendicular to the field M.M.F. vector OF.

The magnitude of OG represents the field M.M.F. *equivalent* to the armature M.M.F., both being expressed in ampere-turns per pole. This quantity may be conveniently determined from the *short-circuit characteristic* of the machine.

Short-circuit Characteristic. This characteristic shows the relationship between the excitation and the armature or stator current when the stator terminals are short-circuited and the machine is run at normal speed. The characteristic is a straight line for stator currents up to about twice full-load current, and in some machines this proportionality may extend to higher currents, but usually for these currents the characteristic bends over slightly towards the excitation axis.

The excitation necessary to pass full-load current through the short-circuited stator windings may be from about one-third to one-half of the excitation required to give normal voltage at full load at the rated power factor of the machine (usually 0·8, lagging).

Under the conditions of proportionality between the excitation and the short-circuit current, the field ampere-turns are expended almost entirely in neutralizing the armature ampere-turns, since under these (i.e. short-circuit) conditions the E.M.F. generated in the stator winding simply balances the impedance voltage, and therefore the flux in the machine has a very low value. Moreover, the phase difference between the generated E.M.F. and the current is almost 90° lagging, and therefore maximum current occurs when the axes of the poles coincide with the magnetic axes of the stator winding. Hence, with low-reactance machines the excitation (i.e. the field ampere-turns per pole) required to obtain full-load current will be a close approximation to the equivalent armature-reaction ampere-turns per pole (i.e. the maximum value of the armature-reaction ampere-turns per pole). But with high-reactance machines, the approximation will not be so close owing to the excitation required to produce the higher flux under these conditions.

Leakage Reactance and Inductance of Armature. This quantity cannot be determined by a direct method, such as by passing a current of normal frequency through the stator winding and measuring the applied voltage, as the results obtained from such a test will not represent the leakage reactance under normal working conditions, owing to the different flux densities and the different flux distribution in the two cases.

An approximation to the actual value of the leakage reactance may be obtained from the open-circuit characteristic and the load characteristic for zero power factor (which is a curve showing the relationship between voltage and excitation for full wattless load —i.e. full-load current—in the armature and a power factor of zero).

Alternatively, if the winding is star-connected and the neutral point is available, the leakage reactance may be determined by taking two short-circuit tests, one with only one phase of the

armature winding short-circuited, and the other with all (three) phases of the armature winding short-circuited. From the results of these tests the armature reaction and the leakage reactance may be calculated,* but the value of the leakage reactance so obtained is only an approximation to the true value on account of the magnetic saturation in the teeth not being normal.

Voltage Regulation. The voltage regulation (expressed as a percentage) of an alternator is defined as the percentage *rise* in voltage

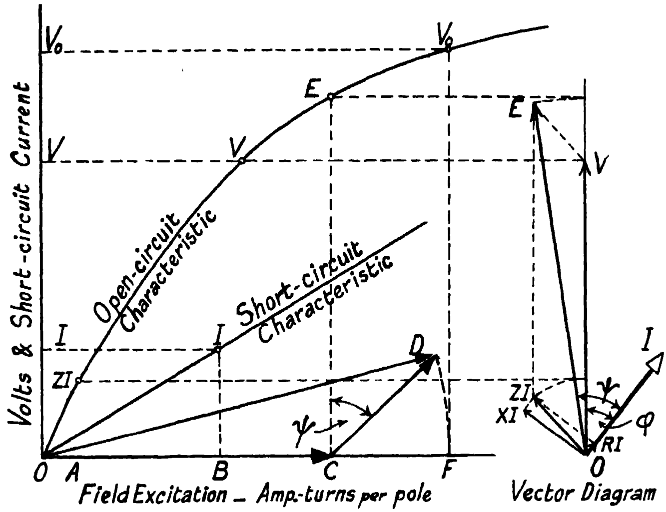


FIG. 169. GRAPHIC METHOD OF DETERMINING VOLTAGE REGULATION

above normal when full load is thrown off, the excitation remaining constant at the full-load value and the speed remaining constant at its normal value. Thus if V is the normal voltage at full load and V_0 is the no-load voltage when full load is thrown off under the above conditions, the voltage regulation is given by

$$100(V_0 - V)/V$$

Calculation of Voltage Regulation. To determine the voltage regulation without actually loading the machine, it is necessary to have a knowledge of (1) the open-circuit characteristic; (2) the resistance and the leakage reactance of the stator windings; (3) the equivalent field ampere-turns corresponding to the maximum armature-reaction ampere-turns.

* "Experimental Analysis of Armature Reaction." Paper by Dr. Gisbert Kapp. *Journ. I.E.E.*, Vol. 42, p. 703.

The procedure is as follows: (1) Calculate the voltage drops due to resistance and leakage reactance; (2) compound these voltages with the terminal voltage to obtain the internal or generated E.M.F., *OE*, Fig. 169; (3) determine from the open-circuit characteristic the corresponding excitation, *OC*; (4) determine from the vector diagram the internal phase angle ψ (i.e. the phase difference between the generated E.M.F. and the current); (5) set off from *C* the equivalent armature reaction ampere-turns at the angle ψ from a perpendicular erected at *C* (these ampere-turns are represented by *CD*), and add these ampere-turns to the excitation ampere-turns *OC*, thus obtaining the full-load excitation, *OF*; (6) determine the open-circuit voltage V_0 corresponding to this excitation. Whence the voltage regulation $= 100(V_0 - V)/V$.

When the calculation is made for a three-phase machine, the open-circuit characteristic is plotted in terms of the phase E.M.F., and all calculations of voltage drops, E.M.Fs., and voltages are made on the basis of phase quantities.

In cases where the armature-reaction ampere-turns are not given, it is necessary to have available the short-circuit characteristic. The equivalent maximum armature-reaction ampere-turns, *AB* (Fig. 169), are then obtained by subtracting from the excitation ampere-turns, *OB*, required to circulate full-load current the ampere-turns, *OA*, necessary to produce the flux to generate the requisite E.M.F. (i.e. the impedance voltage).

Synchronous Impedance. This quantity is defined as the ratio: (open-circuit E.M.F./short-circuit current), both corresponding to the same excitation. It is, however, a fictitious impedance, as armature reaction is considered as having an effect similar to that of inductance or reactance. Moreover, in modern alternators the synchronous impedance is not a constant quantity because the open-circuit characteristic is not a straight line. Thus, although formerly the quantity was used in calculations of voltage regulation and parallel working, nowadays such calculations are usually made by the above method, in which the effects of reactance and armature reaction are considered separately.

CHAPTER XII

THE POLYPHASE INDUCTION MOTOR

THE polyphase induction motor is a machine in which the torque is produced by the interaction of a *rotating* magnetic field—produced by a stationary member, called the *primary* or *stator*—and polyphase currents induced, by this field, in a rotatable member, called the *secondary* or *rotor*. The rotating magnetic field is an

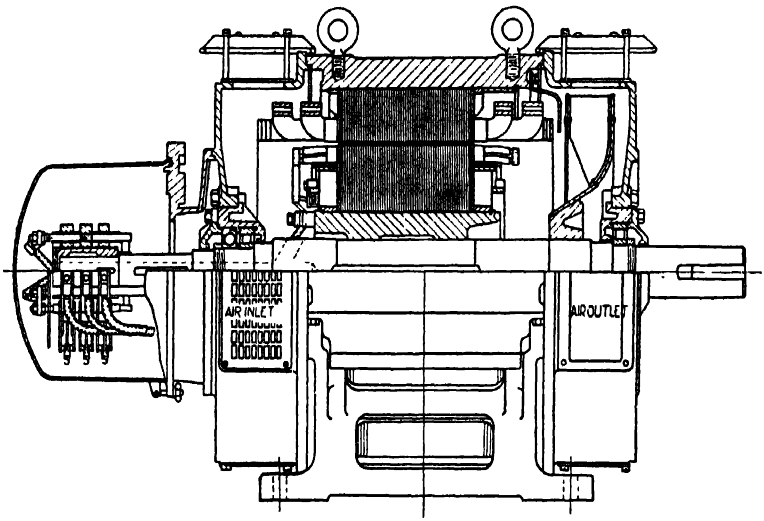


FIG. 170. LONGITUDINAL VIEWS (PART ELEVATION AND PART SECTION) OF THREE-PHASE SLIP-RING INDUCTION MOTOR
(*Metropolitan-Vickers Electrical Co.*)

essential feature of all induction motors, and it enables energy to be transferred from the primary to the secondary by electromagnetic induction—whence the name "induction" motor—so that no electrical inter-connections between primary and secondary are necessary.

In virtue of the rotating magnetic field and the polyphase currents induced in the secondary, the torque is steady and free from pulsation. Moreover, a large torque at starting can be obtained, if desired.

The Stator. The primary member, or stator, consists of a laminated core—built of annular or segmental silicon-steel laminations

—the inner periphery of which is slotted and wound with a polyphase winding suitable for the supply system. In motors for industrial purposes, the core laminations are fitted and clamped into a frame which is fitted with end shields and bearings for the shaft carrying the rotor. Fig. 170 shows longitudinal views of a typical industrial motor.

The slots are usually semi-closed, and the winding is usually of the half-coiled type, the number of poles depending upon the speed

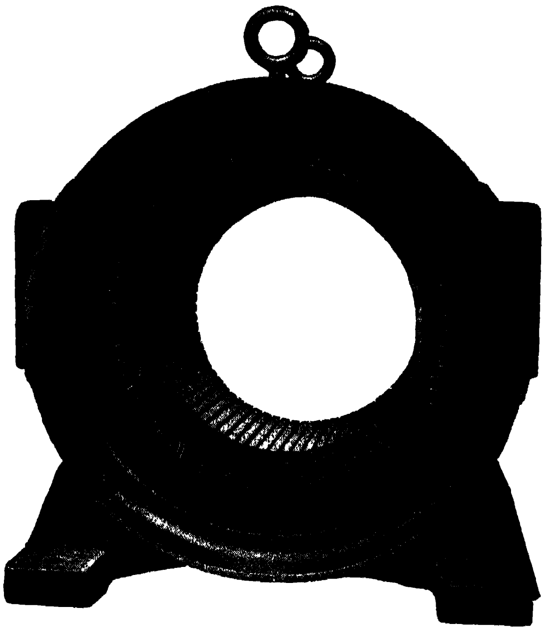


FIG. 171. WOUND STATOR OF THREE-PHASE INDUCTION MOTOR SHOWING BAR WINDING AND AXIAL VENTILATING HOLES

(*Metropolitan-Vickers Electrical Co.*)

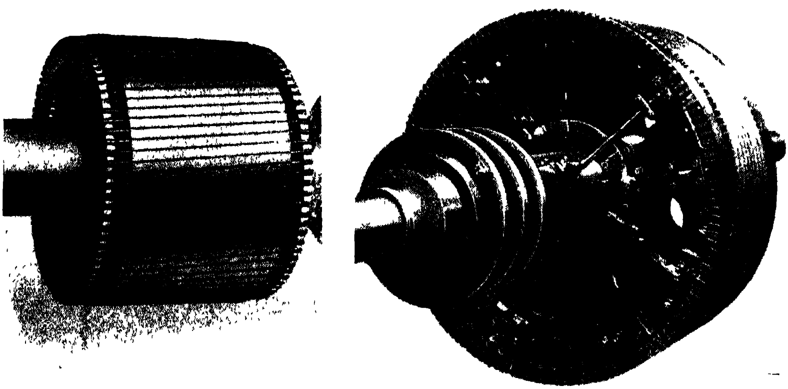
required and the frequency of the supply system. A typical stator is illustrated in Fig. 171.

The Rotor. The secondary member, or rotor, consists of a cylindrical laminated core mounted on a shaft. The outer periphery of the core is slotted for the rotor winding, which may consist either of a permanently short-circuited winding or a polyphase star-connected winding with the ends brought out and connected to three slip-rings mounted on the shaft.

A rotor having a permanently short-circuited winding is called a

squirrel-cage rotor, and one having a polyphase winding connected to slip-rings is called a *slip-ring rotor*. The object of the slip-rings is to enable resistances to be inserted temporarily in the rotor circuits for the purpose of controlling the torque and current at starting.

Rotor Windings. The permanently short-circuited form of rotor winding consists of solid conductors (one per slot) with the ends short-circuited by solid end-rings. Except for the smallest sizes, the conductors and end-rings are of copper, and the ends of the conductors are welded to the end-rings to obtain a joint of low



(a) (b)

FIG. 172. SQUIRREL-CAGE (a) AND SLIP-RING (b) ROTORS FOR INDUCTION MOTORS

(Metropolitan-Vickers Electrical Co. and British Thomson-Houston Co.)

electrical resistance. A typical rotor is illustrated in Fig. 172 (a). In the smallest sizes, however, which are manufactured in large quantities, the conductors and end-rings are of aluminium, which is cast by a special process in the completely-assembled core.

The polyphase star-connected rotor winding—usually a three-phase winding—has the same number of poles as the stator winding, but usually fewer turns per phase. The number of turns per phase is arranged to give a maximum voltage between slip-rings (which occurs when the slip-rings are open-circuited with the rotor stationary and the stator excited) of about 100 V. or less for small motors, increasing, with the size of the motor, to values of about 1000 V. or more. The choice of rotor voltage is at the discretion of the designer, and usually involves a compromise between a robust winding on the one hand, and, on the other hand, rotor currents

which will not involve too costly slip-ring brush-gear and starting rheostats. A typical rotor is illustrated in Fig. 172 (*b*).

Action—Rotor Circuits Open. Consider a slip-ring motor with the rotor stationary, the slip-rings open-circuited, and the stator winding excited from the supply system. The magneto-motive force due to the magnetizing currents in the stator winding produces a rotating field which cuts the turns of both stator and rotor windings. Hence, E.M.Fs. of supply frequency are induced in the rotor winding, and the action is similar to that of a transformer, except that in the present case the field is rotating instead of alternating.

But due to the air-gap between the stator and rotor (which causes magnetic leakage), the flux cutting the rotor conductors will be slightly less than that cutting the stator conductors, and in consequence the ratio of the induced E.M.Fs. per phase will not be exactly equal to that of the turns per phase. Moreover, on account of the relatively high reluctance of the air-gap, the magnetizing current will be relatively large compared with that for a corresponding transformer.

If a voltmeter is connected across a pair of slip-rings and the rotor is slowly turned by hand, the reading of the voltmeter will be practically constant for all relative positions of rotor and stator, the slight deviations from the constant value being due to variations of the leakage and reluctance of the magnetic circuit. But if the phase difference of this voltage relative to the supply voltage is measured, the value obtained will vary with the position of the rotor and will undergo a total variation of $p\pi$ radians for one revolution of the rotor.

Hence, under these conditions the motor acts as a polyphase phase-shifting transformer and has practical applications for this purpose (e.g. in the testing of supply meters), and also as a booster transformer for regulating the voltage in three-phase circuits. When used in this form the apparatus is called an *induction regulator*.

Action—Rotor Circuits Closed. For this case we may consider a motor with either a squirrel-cage rotor or a slip-ring rotor, with the slip-rings short-circuited or alternatively connected to external resistances.

Currents. When the stator is excited, the E.M.Fs. induced in the rotor winding will now produce currents in each of the phases,* the

* The number of phases in a squirrel-cage rotor winding is equal to the number of conductors per pair of poles. Such a winding is equivalent to a multiple-circuit polyphase half-turn winding, the number of parallel circuits being equal to the number of pairs of poles. Each phase consists of one conductor and the portions of the end-rings corresponding to two pole-pitches.

magnitude of the current being determined by the phase E.M.F. and impedance. The phase difference between the current and E.M.F. in each phase is determined by the ratio (reactance/resistance).

If the rotor is held stationary, the conditions, so far as rotor and stator currents and M.M.Fs. are concerned, are similar to those in a transformer with short-circuited secondary, but with the important differences that (1) the stator, or primary, current is relatively small in comparison with the short-circuit current (at normal voltage) of a corresponding transformer; (2) the ratio of stator and rotor currents is not equal to the ratio of the turns per phase in the stator and rotor windings. These differences are, of course, due to the larger magnetic leakage and leakage reactance of the induction motor.

Torque—Rotor Stationary. Since the rotor currents are produced by a *rotating* field, the direction of the currents in relation to the field will be such as to produce a torque, the magnitude of which will be proportional to rotor current $\times$ flux $\times$ cosine of the phase difference between current and flux, i.e.

$$\text{torque} \propto \Phi I_2 \cos \varphi_2,$$

where I_2 denotes the rotor current, Φ the flux, and φ_2 the phase difference between current and flux.

Thus for a given flux and current the torque is proportional to $\cos \varphi_2$, i.e. the power factor of the rotor circuit. A highly inductive low-resistance rotor circuit is therefore very undesirable for the production of torque and is, in consequence, always avoided in practice.

Torque—Rotor Running. If the rotor is free to rotate and the torque produced exceeds the load torque, the rotor will run up to speed in the same direction as the rotating field. During this running-up period the relative speed at which the flux cuts the rotor conductors progressively decreases, which results in a corresponding decrease in both the rotor E.M.F. and the rotor frequency. The speed will become steady when the torque produced is equal to the retarding torque, and at this speed the E.M.F. induced in the rotor winding must be just sufficient to produce the current to give this torque.

At no-load the speed of the rotor will be nearly equal to that of the rotating field (which is called the *synchronous speed*) and, in revolutions per second, is equal to

$$\text{supply frequency/number of pairs of poles.}$$

If load is applied gradually, the speed will drop progressively, the change in speed being proportional to the change in torque. But with industrial motors, the percentage drop in speed from no load to full load is very small, being about 2 per cent in large motors and from 4 to 6 per cent in small motors. Thus an induction motor is practically a constant-speed machine, and its speed-torque characteristic resembles that of a shunt-wound direct-current motor.

If the load is increased beyond about twice full-load torque, the speed will drop rapidly and at about $2\frac{1}{2}$ times full-load torque the motor will break down, i.e. the torque exerted will be incapable of driving the load and the rotor will come to rest.

Rotor E.M.F., Slip, and Frequency. The E.M.F. (E_2) induced in each phase of the rotor winding at standstill is calculated, by means of equation (86), p. 239, in the same manner as the E.M.F. of an alternator. Thus,

$$E_2 = 4K_f K_p K_b \Phi N_2 f \times 10^{-8},$$

where Φ is the flux per pole, which is linked with the rotor conductors; N_2, the number of turns per phase; f, the supply frequency K_f, the form factor, and K_p, K_b, the pitch factor and winding distribution factor respectively for the rotor winding.

At standstill the flux cuts the rotor conductors at a speed of n_s revolutions per second [$n_s = f/\frac{1}{2}p$, where p = number of poles]. Hence, when the rotor is running at a speed of n r.p.s., the flux cuts the rotor conductors at a speed of $n_s - n$ r.p.s. Whence the E.M.F. induced in the rotor at this speed (n) is given by

$$E_{2n} = E_2(n_s - n)/n_s = sE_2 \quad . \quad . \quad . \quad . \quad . \quad (86)$$

where $s = (n_s - n)/n_s$, and is called the fractional slip, or, shortly, the *slip*.

The frequency of the induced E.M.F. (E_{2n}) at the speed n is

$$f_2 = \tfrac{1}{2}p\,(n_s - n) = \tfrac{1}{2}\,pn_s\frac{(n_s - n)}{n_s} = fs \quad . \quad . \quad . \quad (87)$$

Hence, if, at full load, the slip is 5 per cent, the frequency of the E.M.Fs. and currents in the rotor will be 0·05 × supply frequency; or, if the motor is running on a 50-cycle supply system, $2\frac{1}{2}$ cycles per second.

Rotor Current and Power Factor. The impedance per phase of the rotor circuits at standstill = $\sqrt{(R_2{}^2 + X_2{}^2)}$, where R_2 denotes the resistance per phase and X_2 the *standstill* reactance per phase. Observe that X_2 corresponds to the supply frequency, f, since $s = 1{\cdot}0$.

Hence the rotor current (per phase) at standstill is

$$I_2 = E_2/\sqrt{(R_2^2 + X_2^2)}$$

When the rotor is running with a slip s, the reactance becomes sX_2, and the impedance becomes $\sqrt{(R_2^2 + s^2X_2^2)}$. Hence the current is now

$$I_{2n} = E_{2n}/\sqrt{(R_2^2 + s^2X_2^2)} = sE_2/\sqrt{(R_2^2 + s^2X_2^2)}$$

The power factor of the rotor circuits at standstill is $\cos\varphi_2 = R_2/\sqrt{(R_2^2 + X_2^2)}$, which, when the slip is s, becomes $\cos\varphi_{2n} = R_2/\sqrt{(R_2^2 + s^2X_2^2)}$.

Since at the normal running speed of the motor, the slip, s, is very small, the power factor of the rotor circuits at this speed will be almost unity.

Starting Torque. At standstill the torque ($\mathfrak{T}_0$) is proportional to $\Phi I_2 \cos\varphi_2$. Substituting for I_2 and $\cos\varphi_2$ and introducing a constant of proportionality, K, we have

$$\mathfrak{T}_0 = K\Phi I_2 \cos\varphi_2 = \frac{K\Phi E_2 R_2}{R_2^2 + X_2^2} = \frac{K\Phi E_2}{X_2}\left(\frac{R_2/X_2}{1 + R_2^2/X_2^2}\right)$$

$$= \frac{K_1\Phi^2}{X_2}\,\frac{(R_2/X_2)}{(1 + R_2^2/X_2^2)} \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where $K_1\Phi = KE_2$.

Observe that: (1) when the ratio R_2/X_2 is fixed, the starting torque is inversely proportional to the reactance of the rotor circuit; (2) for a given value of X_2 in a given machine, the starting torque will increase as R_2 increases until $R_2 = X_2$,* when further increase of R_2 will give a decreased torque; (3) for fixed values of R_2 and X_2, the starting torque is proportional to the *square* of the flux.

Now, the flux is approximately proportional to the voltage impressed on the stator winding. Hence, if this voltage is reduced for the purpose of reducing the starting current, the starting torque will be reduced in proportion to the *square* of the reduction in voltage (e.g. if the voltage is reduced to one-half of normal, the starting torque will only be one-quarter of that corresponding to normal voltage).

* This relationship is obtained by differentiating (88) with respect to R_2, equating to zero and solving for R_2. Equating the differential coefficient to zero, we have

$$\frac{1}{X_2}\left(1 + \left(\frac{R_2}{X_2}\right)^2\right) - \frac{2R_2}{X_2^2}\cdot\frac{R_2}{X_2} = 0,$$

Whence $R_2/X_2 = 1{\cdot}0$, i.e. $R_2 = X_2$.

Herein, then, is one of the advantages of a slip-ring rotor over a squirrel-cage rotor, as, at starting, external resistances may be connected to the slip-rings to reduce the starting current. Thus the motor may be started with normal flux and, if the resistance per

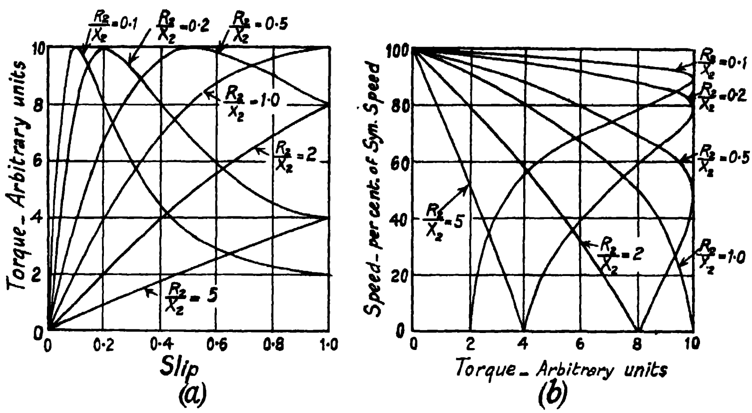


FIG. 173. TORQUE-SLIP (*a*) AND SPEED-TORQUE (*b*) CURVES FOR SLIP-RING INDUCTION MOTOR

phase is made equal in value to the standstill reactance of the rotor, maximum torque will be obtained.

Running Torque. When the rotor is running with a slip s corresponding to a rotor current I_{2n}, the torque, assuming the flux to be unchanged, will be

$$= K\Phi I_{2n}\cos\varphi_{2n} = \frac{K\Phi sE_2R_2}{R_2^2+s^2X_2^2} = \frac{K\Phi E_2}{X_2}\cdot\frac{sR_2/X_2}{R_2^2/X_2^2} \qquad (89)$$

The torque now varies with the slip and, if R_2 is variable, the maximum torque may be obtained at any desired slip. Thus the slip corresponding to maximum torque is given by $s = R_2/X_2$, i.e. maximum torque occurs at the slip for which the rotor reactance is equal to the rotor resistance. For example, if $R_2 = 0{\cdot}1X_2$, the slip corresponding to maximum torque is 0·1 or 10 per cent; if $R_2 = 0{\cdot}5X_2$, the slip required for maximum torque is 0·5, or 50 per cent; if $R_2 = X_2$, maximum torque occurs at standstill ($s = 1{\cdot}0$).

The curves of Fig. 173 (*a*) show the manner in which the torque varies with the slip for a given machine when the resistance of the rotor circuit is adjusted to give definite values for the ratio R_2/X_2.

Speed Control. If the torque-slip characteristics of Fig. 173 (*a*) are converted into torque-speed characteristics (Fig. 173 (*b*)), we

observe that it is possible to control the speed of a slip-ring induction motor at a given torque by inserting resistances in the rotor circuits.

But this feature of controllable speed by means of adjustable external resistances connected in the rotor circuit has its disadvantages. Thus (1) the slope of the speed-torque characteristic is increased, so that the method is suitable only for drives requiring approximately constant torque at all speeds; (2) the efficiency is reduced almost in proportion to the reduction in speed; (3) a large amount of heat may have to be dissipated in the external resistances.

In practice, therefore, speed control, by means of resistances connected in the rotor circuits, is employed only when a *temporary* reduction in speed is desired, i.e. the motor is to run at the reduced speed for short periods only.

When speed control at high efficiency and with "shunt" characteristics is desired, it must be obtained either by pole-changing (i.e. by so arranging the stator winding that two or more numbers of poles may be obtained by changing certain external connections) or by means of a commutator machine connected to the rotor circuits of the induction motor. In the former case, speed control at high efficiency is limited to definite speeds only (e.g. 2, 3, or 4, according to the possible numbers of poles); but, in the latter case, continuous adjustment of speed, at high efficiency, is possible over the whole speed range (which may be 1 : 3 or more). The function of the commutator machine (which may form part of the induction motor itself) is to convert into mechanical energy the energy which would be dissipated as heat in the rheostatic method of speed control.

Power Developed by Rotor—Mechanical Output. If $\mathfrak{T}$ is the torque developed by the rotor and n is the speed (r.p.s.), the gross mechanical power developed is $\mathfrak{T}n$. Now for a three-phase motor, $\mathfrak{T} = K(p\Phi)(3I_2N_2)\cos\varphi_2$, and since $\Phi = K_1E_2/pN_2n_s$, we have $\mathfrak{T} = K_2(3E_2I_2\cos\varphi_2/n_s)$, where $K_2 = KK_1$.

Whence $\mathfrak{T}n/K_2 = P_M = 3E_2I_2\cos\varphi_2 n_s(1-s)/n_s$

$$= 3E_2I_2(1-s)\cos\varphi_2 \qquad (90)$$

Therefore $3E_2I_2(1-s)\cos\varphi_2$ may be considered as representing the mechanical power developed by the rotor.

Rotor I²R Loss. When the rotor is running at a slip s, the current in the rotor winding (per phase) is equal to $sE_2/\sqrt{(R_2^2 + s^2X_2^2)}$, and the total I^2R loss in the rotor circuits (assuming a three-phase winding) is

$$\begin{aligned} P_R &= 3R_2[sE_2/\sqrt{(R_2^2 + s^2X_2^2)}]^2 \\ &= 3(sE_2)[sE_2/\sqrt{(R_2^2 + s^2X_2^2)}][R_2/\sqrt{(R_2^2 + s^2X_2^2)}] \\ &= 3sE_2I_2\cos\varphi_2 \qquad (91) \end{aligned}$$

Whence

$$P_R/P_M = s/(1-s)$$

i.e. $\quad P_R = P_M/(1/s - 1)$ . . . . (92)

Electrical Efficiency of Rotor. The electrical input power (P_2) to to the rotor is

$$P_2 = P_M + P_R = 3E_2I_2\cos\varphi_2 \quad . \quad . \quad . \quad . \quad (93)$$

Hence, rotor efficiency $= P_M/P_2 = 1 - s$ . . . . . (94)

Also $\quad P_R/P_2 = s$

i.e. $\quad P_R = sP_2$ . . . . (95)

Relationship Between Starting Torque and Full-load Torque Let $\mathfrak{T}_f$, s_f denote the torque and slip respectively, at full load' and $\mathfrak{T}_s$ the torque at standstill. Then, from equation (89), $\mathfrak{T}_f = K\Phi I_{2f}\cos\varphi_{2f}$. Multiplying and dividing the right-hand side by s_f/E_2, substituting $R_2/\sqrt{(R_2^2 + s_f^2X_2^2)}$ for $\cos\varphi_{2f}$, and replacing $s_fE_2/\sqrt{(R_2^2 + s_f^2X_2^2)}$ by I_{2f}, we obtain

$$\mathfrak{T}_f = K\Phi I_{2f}^2R_2/s_fE_2$$

$$= K_1I_{2f}^2R_2/s_f,$$

where $\quad K_1 = K\Phi/E_2$;

and I_{2f}, $\cos\varphi_{2f}$, denote the full-load rotor current and power factor respectively.

Similarly, $\mathfrak{T}_s = K_1I_{2s}^2R_2$, where I_{2s} is the rotor current at standstill.

Whence $\quad \mathfrak{T}_s/\mathfrak{T}_f = (I_{2s}/I_{2f})^2s_f$

or $\quad \mathfrak{T}_s = \mathfrak{T}_f(I_{2s}/I_{2f})^2s_f$ . . . . . . (96)

Now if the magnetizing current is ignored, the ratio of the rotor currents is the same as that of the stator currents, so that, approximately,

$$\mathfrak{T}_s = \mathfrak{T}_f(I_{1s}/I_{1f})^2s_f \quad . \quad . \quad . \quad . \quad . \quad (96a)$$

Methods of Starting Squirrel-cage Motors

Four methods are available: (1) Switching the motor direct on to the supply; (2) inserting resistances in the stator windings; (3) regrouping the stator windings (Y for starting, Δ for running); (4) applying reduced voltage to the stator windings by means of a transformer. The methods are called: (1) direct starting; (2) primary resistance starting; (3) star-delta starting; (4) transformer starting.

Direct Starting. This method involves starting currents of the order of six to eight times full-load current, and is, therefore, usually restricted to small motors of about 2 to 3 h.p. With these

motors, the starting torque is about twice full-load torque. Hence the starting period is of only a few seconds' duration, and the starting current is not harmful to the motor.

Resistance Starting. In this method the starting current is reduced by inserting resistances in the stator circuits. But, as equation (96) shows, the starting torque is reduced in proportion to the *square* of the reduction in starting current. For example, if with direct starting the starting current is six times full-load current and the starting torque is twice full-load torque, then if the current is reduced to twice full-load current by inserting resistances in the stator circuits, the starting torque will be $(2/6)^2 \times 2$ full-load torque $= 0{\cdot}22 \times$ full-load torque. The method is, therefore, only suitable for small motors which can be started light.

Star-delta Starting. This method requires the stator winding to be delta-connected for normal running and to be provided with six terminals (i.e. the phases must not be interconnected). At starting, the windings are connected in star, and when the motor has run up to speed, the connections are changed to delta, the change-over being made rapidly by means of a suitable switch.

The starting current will be one-third of that (i.e. the *line* current) for direct starting, and the starting torque will also be one-third of that obtained with direct starting. For example, if these (direct starting) values are equal to six times full-load current and twice full-load torque respectively, then with star-delta starting the starting current will be $\frac{1}{3} \times 6 = 2 \times$ full-load current, and the starting torque will be $\frac{1}{3} \times 2 = 0{\cdot}66 \times$ full-load torque.

Proof. Let I_s denote the (line) starting current with direct starting (i.e. delta connections). Then the direct starting current per phase $= I_s/\sqrt{3}$. Hence when the windings are star-connected, the starting current for the same line voltage will be $(I_s/\sqrt{3})/\sqrt{3} = \frac{1}{3}I_s$.

If I_f denotes the full-load line current (delta connections), the current per phase $= I_f/\sqrt{3}$, and the ratio - [starting current per phase (Y)/full-load current per phase (Δ)] $= (\frac{1}{3}I_s)/(I_f/\sqrt{3}) = (1/\sqrt{3})\,(I_s/I_f)$.

Hence applying equation (96*a*), we have

$$\text{Starting torque} = \text{Full-load torque} \times \left(\frac{\text{starting current per phase}}{\text{full-load current per phase}}\right)^2$$

$$= \tfrac{1}{3}\,(I_s/I_f)^2 \times \text{full-load torque.}$$

$$= \tfrac{1}{3} \times \text{starting torque with direct starting.}$$

Transformer Starting. The limitations of star-delta starting are—

(1) A six terminal motor is necessary, the stator windings of which are designed for delta connections.

(2) The starting torque has a fixed value relative to that with direct switching, being about $\frac{2}{3} \times$ full-load torque for small motors to about $\frac{1}{3} \times$ full-load torque for medium size motors. These limitations do not occur with the transformer method of starting.

Since low ratios of transformation are required, an *auto-transformer* is employed. In some cases a three-phase auto-transformer is used, and in other cases two single-phase V-connected transformers are used. In all cases, however, three or four tappings are provided per phase, in order that a suitable starting voltage may be selected for the particular starting conditions. When three tappings are provided, the starting voltages which can be obtained are approximately 55, 65, 75 per cent of the line voltage.

If K is the ratio of transformation of the auto-transformer, the starting current taken from the supply system is approximately $(1/K)^2 \times$ direct starting current, and the starting torque is approximately $(1/K)^2 \times$ direct starting torque.

Proof. Let I_s denote the (line) starting current with direct starting, and I_f the full-load current. Then, with the transformer, if V denotes the line voltage, the voltage applied to the motor $= V/K$, and the current taken by the motor $= I_s(V/K)/V = I_s/K$. Hence, ignoring magnetizing current and losses in the transformer, the line current $= (1/K)\,(I_s/K) = I_s/K^2$.

Hence applying equation (96*a*), we have

$$\begin{aligned}\text{Starting torque} &= \text{Full-load torque} \times \left(\frac{\text{motor-starting current}}{\text{full-load current}}\right)^2\\ &= \text{Full-load torque} \times \left(\frac{I_s/K}{I_f}\right)^2\\ &= (1/K)^2 \times \text{Starting torque with direct starting.}\end{aligned}$$

Equivalent Circuit and Circle Diagram

Equivalent Circuit. The equivalent circuit of the polyphase induction motor is obtained in a similar manner to that for a corresponding transformer, *phase quantities* only being considered. At standstill the conditions are the direct equivalent to those of a transformer, but when the rotor is running, the frequency in the rotor circuit differs from that in the stator circuit. To obtain the equivalent conditions in this case, we must bring the rotor to rest and insert resistances in the rotor circuits to obtain the same rotor current and power factor as when running. Thus the rotor current at slip s is $I_{2n} = sE_2/\sqrt{(R_2^2 + s^2X_2^2)}$. Hence for this current to be the same at standstill (when the voltage is E_2 and the reactance is X_2), we must have $sE_2/\sqrt{(R_2^2 + s^2X_2^2)} = E_2/\sqrt{[(R_2 + R)^2 + X_2^2]}$. Whence $R = R_2(1 - s)/s$.

The equivalent circuit is therefore represented by a series-parallel circuit of the type shown in Fig. 174, which for approximate purposes can be reduced to the simpler type of circuit shown in Fig. 175. Due, however, to the relatively large magnetizing current of the motor compared with that of a transformer, the circuit of Fig. 175 cannot be reduced to a simpler form as was done for the transformer.

Circle Diagram. The deduction of the no-load and short-circuit and the load diagrams for the circuit shown in Fig. 174 is given in Chapter XXII. For the present we shall concern ourselves only with the method of constructing the diagram from no-load and short-circuit tests, and the use of the diagram for predetermining the performance of the motor.

The no-load test is taken with the motor running light at normal voltage and frequency, the slip-rings, if any, being short-circuited.

The short-circuit test is taken with the slip-rings short-circuited, but with a low voltage (at normal frequency) applied to the stator

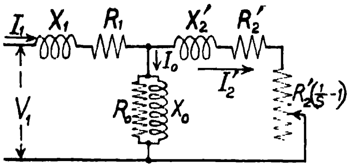


Fig. 174

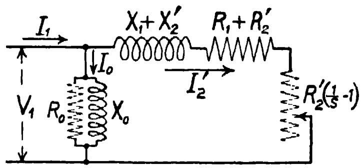


Fig. 175

Equivalent Circuit Diagrams for Induction Motor

to give approximately full-load current. In both tests the current, voltage, and power are measured.

The stator resistance is measured and, if the machine has slip-rings, the rotor resistance is also measured. The values obtained directly from the readings of the instruments will give the resistances between terminals or slip-rings, as the case may be. These values must be converted to "per phase" values. Since slip-ring rotor windings are always star-connected,* the rotor resistance per phase will be one-half of the value obtained from the instrument readings. The stator windings, however, may be connected either in star or delta, and if the method of connection is not known, the stator I^2R loss per phase can be calculated as $\frac{1}{2}I^2R'$, where I is the *line* current and R' is the resistance as measured between any pair of terminals.†

Construction for Determining Centre of Circle. In Fig. 176, the

* In some small rotors a two-phase L-connected winding is employed. This fact is readily ascertained by measuring the resistances between successive slip-rings.

† If R = resistance per phase, and R' = resistance between any pair of terminals, then $R' = 2R$ for the star connection and $\frac{2}{3}R$ for the delta connection. Hence if I = line current and I_{ph} = phase current,

$$I^2R \text{ loss per phase for star connection} = \tfrac{1}{2}I^2R', \text{ and}$$

$$I^2R \text{ loss per phase for delta connection} = I_{ph}^2(\tfrac{3}{2}R') = \tfrac{1}{2}I^2R'.$$

reference vector OE represents the terminal voltage, OI_0 the no-load current, and OI_s the short-circuit current at normal voltage.*

Join I_0I_s, bisect, and draw a perpendicular PQ. From I_0 draw a parallel to OE, intercepting OI_s at A. Bisect I_0A, and draw BQ at right angles to cut PQ at Q, which is the centre of the circle, the

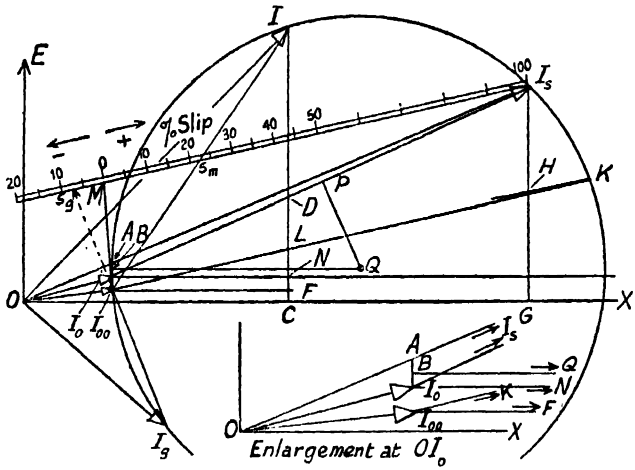


Fig. 176. Circle Diagram for Induction Motor

circumference of which passes through the no-load point, I_0, and the short-circuit point, I_s.

Datum Lines. The horizontal axis, OX, is the datum line for the power input.

The line, I_0, I_s, joining the no-load and short-circuit points is the datum line for power output.

The datum line for the torque is obtained as follows—

(1) Draw the vector OI_{00} representing the current input at synchronous speed. [This is obtained by separating out the core loss from the no-load losses, the ordinate at I_{00} representing the core loss.] The point I_{00}, therefore, corresponds to zero torque, and the torque datum line passes through this point.

(2) Draw the ordinate I_sG at the short-circuit point. Divide this at H, such that I_sH : HG : : rotor I^2R loss at short-circuit : stator I^2R loss at short-circuit. Join H to O and produce to cut the

* The short-circuit current (I_s) at normal voltage is obtained from the measured values of current (I_s') and voltage (V') on the short-circuit test. Thus $I_s = I_s'(V/V')$, where V = normal voltage.

circumference at K (which is a point corresponding to a slip of infinity and zero torque). Join $I_{00}K$, which is, therefore, the torque datum line.

Scales. The current scale (e.g. 1 cm. $= p$ amp.) is fixed when the diagram is drawn and the vectors drawn from O to the circumference represent line currents. The equivalent rotor current is represented by vectors drawn from I_{00} to the circumference. A vector such as $I_{00}I$ represents the equivalent rotor current per phase corresponding to the line current OI.

The power scale is 1 cm. $= \sqrt{3}pE$ watts, where E is the *line* voltage.

The torque scale is obtained by determining the torque ordinate corresponding to full-load output and using this as a unit.

The slip scale is obtained by (1) drawing from I_s a parallel, I_sM, to the torque datum line; (2) drawing a tangent from the point I_{00} to cut this line at M; (3) dividing the portion MI_s into 100 parts with the zero at M. The per cent slip corresponding to the current OI is given by the scale reading at which the rotor-current vector cuts the slip scale.

Performance Summary. Current input, OI; power input, IC; power output, ID; torque, IL; slip, s_m; total losses, DC[I^2R losses (total), DN; friction and windage loss, NF; core loss, FC]; efficiency, ID/IC; power factor, cos $\angle EOI$; starting torque, rotor short-circuited, $I.H$.

Operation of Polyphase Induction Motor as Generator. An induction motor connected to a supply system and driven mechanically at speeds above synchronism (i.e. the slip is negative) operates as a generator, the power (within the limit of the maximum output) increasing as the slip increases. The frequency is determined by the supply system to which the induction machine is connected and which supplies the magnetizing current. The performance as generator may be predetermined from the circle diagram constructed from the no-load and short-circuit currents of the machine operating as a motor. The portion of the circumference below the point I_{00} (synchronous speed) and the horizontal axis OX is the locus of the stator current vector; the horizontal axis is the output datum line; the line $I_{00}K$ is the torque datum line; and the line I_0I_s is the input datum line. The slip scale is extended to the left of the zero point, and the slip is given by the scale reading at which the line, produced, joining I_{00} and the extremity of the current vector OI_g cuts the slip scale.

CHAPTER XIII

THREE-PHASE TRANSFORMERS

WHEN a transformer is required for a three-phase system, either three single-phase transformers, suitably interconnected, or a single three-phase transformer, may be employed. The latter arrangement is more economical than the former, and is, therefore, always adopted in practice, unless special conditions require the use of three single-phase transformers.

Arrangement of Core and Windings. Three-phase transformers are of the "core" type. The primary and secondary windings of each phase are arranged as concentric coils on separate cores which are united magnetically by yokes. The drawings in Figs. 177, 178 show the general arrangement of the core and windings for the smaller sizes (50 to 500 kVA.) of transformers for distribution circuits. The drawings show also the mounting of the transformer in the tank, the cooling tubes, and the terminals.

The cores of very large transformers, however, are usually built with five limbs. The primary and secondary windings are arranged on the three inner limbs, and the two outer limbs are without windings. This construction is employed in order to obtain a lower overall height for the yokes, together with a better balance between the reluctances of the several magnetic paths, than is possible with a three-limb core. The diagrams in Fig. 179 show the flux distribution in the magnetic circuits of three-limb and five-limb transformers at the instant corresponding to maximum flux in one core. In the transformer with three limbs, the maximum flux has to be carried by portions of the yokes as well as the core, but in the transformer with five limbs the yokes carry only one-half of the maximum core flux. Hence, for equal fluxes and flux densities in the two cases, the yokes of the five-limb transformer will only be one-half of the height of those of the three-limb transformer.

Windings. The core type of magnetic circuit lends itself naturally to the use of circular coils for the windings. This (circular) type of coil is not only cheaper to wind and insulate than the rectangular type, but is mechanically stronger than the latter, and its shape will not distort under the large electro-mechanical forces which may occur with short circuits.

The *low-voltage winding* is wound, in one or more layers, directly on a cylinder of insulating material, which is arranged adjacent to the core when the transformer is assembled (see Fig. 178).

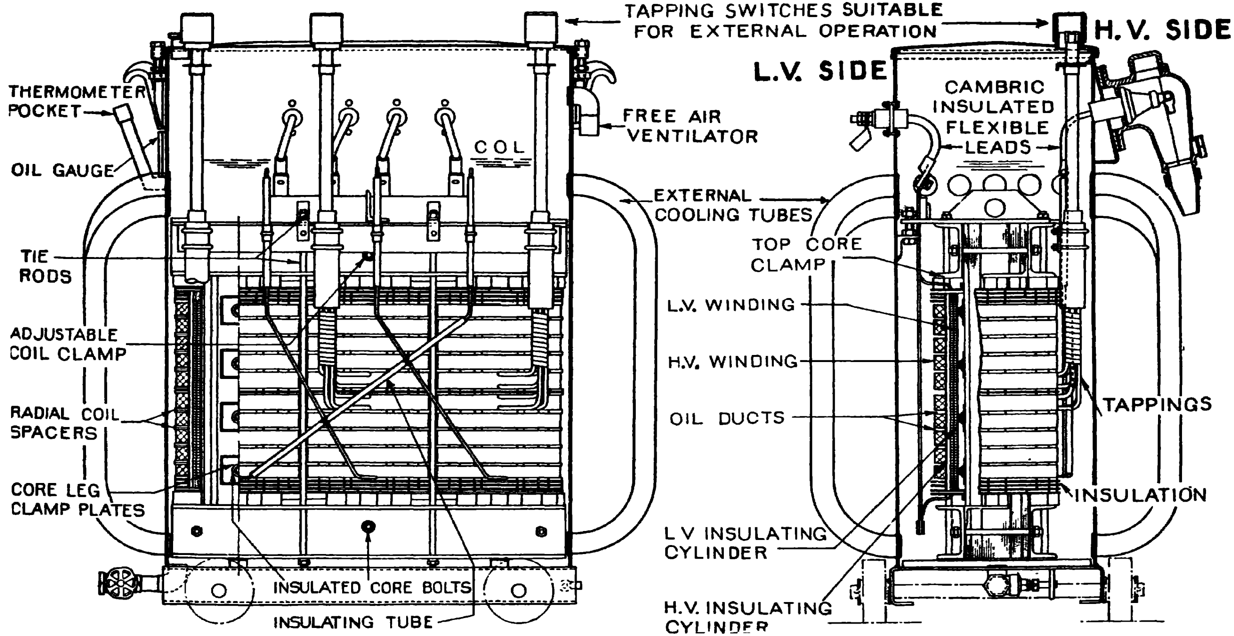


FIG. 177 FIG. 178

LONGITUDINAL AND TRANSVERSE SECTIONAL VIEWS OF SELF-COOLED OIL-IMMERSED, THREE-PHASE TRANSFORMER

(British Thomson-Houston Co.)

The *high-voltage winding* is subdivided into a number of coils of relatively small height (see Fig. 178). When practicable, the individual coils are wound directly on a cylinder of insulating material, and radial spacers of insulating material are placed between the coils to provide ducts for the cooling oil.

The windings on each core occupy symmetrical positions relatively to each other, so that the ampere-turns of each are distributed over the same length of the core. When tappings are required,

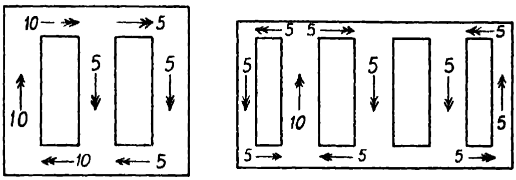


FIG. 179. ALTERNATIVE ARRANGEMENTS OF MAGNETIC CIRCUITS OF THREE-PHASE TRANSFORMERS (CORE-TYPE)

The arrows and numbers in the cores and yokes represent the directions and magnitudes of the fluxes at a particular instant

they are arranged on the coils occupying central positions on the core (see Fig. 177).

Interconnection of Windings. Transformers for supplying three-phase loads may have the secondary windings interconnected either in star or delta, but when the loads are supplied on the four-wire system only the star connection is permissible. Alternatively, the interconnected-star or zigzag connection may be used in cases where the ordinary star connection is undesirable (e.g. transformers for supplying three-phase mercury-arc rectifiers and three-phase rotary converters having a three-wire direct-current load).

The primary windings may also be interconnected either in star or delta, but in transformers for extra-high-voltage (66 kV. to 220 kV.) circuits the star connection is always employed, because the windings of each limb have then to be designed for only 57·7 ($=100/\sqrt{3}$) per cent of the terminal or line voltage. In consequence, fewer turns and less insulation are required than if the windings were designed for the full-line voltage. Moreover, if the neutral point of the winding is earthed, the insulation on the high-voltage windings can be graded and still further economies obtained in construction. The graded insulation, however, cannot be applied to the individual turns, as the voltage per turn is constant throughout the winding.*

* The insulation on the turns adjacent to the high-voltage line terminals has to be specially reinforced to withstand surges due to switching and line disturbances.

Another advantage of a star-connected high-voltage winding, with earthed neutral point, is that the transmission system may be better protected against earth faults than is possible with an unearthed system. Also the voltage to earth at all parts of the system is limited to the phase voltage.

Rules for Connecting-up. In making the star or delta connection, the rules given on pp. 177, 179 must be rigorously followed. The matter, however, is not so simple as that of connecting up the

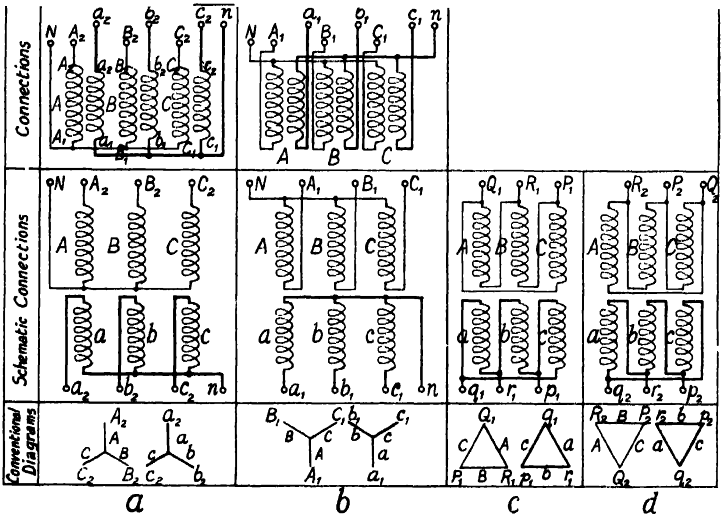


FIG. 180. CONNECTION DIAGRAMS SHOWING ALTERNATIVE METHODS OF MAKING STAR AND DELTA CONNECTIONS IN THREE-PHASE TRANSFORMERS

The terminal markings and the lettering of the windings are in accordance with British Standard Specifications

In the diagrams marked "Connections," the primary and secondary windings are shown in the relative positions which they occupy on the cores. In the "Schematic Connections," these windings are shown separated in order that the interconnections may be more easily traced

windings of a three-phase alternator as, with three-phase transformers, four combinations are possible for the primary and secondary windings (considering only the ordinary star and delta connections), and two further combinations (involving phase sequence) are possible for the connections to the supply system.

Connecting-up Combinations for Star and Delta Connections. Of the four combinations of connections for the windings, two refer to the methods of making the star connection and two to the delta connection. These methods are shown diagrammatically in Fig. 180, and in the diagrams all the coils are assumed to be similar in so far as direction of winding and assembly on the core are concerned.

In the diagrams (*a*), (*b*), which refer to the star connection, the neutral-point connection in one case (*a*) is made by connecting together all the lower ends (e.g. the starting or inner ends of the coils), and in the other case (*b*) this connection is made by connecting together all the upper ends (e.g. the finishing or outer ends of the coils).

In the diagrams (*c*), (*d*), which refer to the delta connection, in one case, (*c*), the delta is obtained by connecting the lower end of *A* to the upper end of *B*, and so on; and, in the other case, (*d*), the delta is obtained by connecting the upper end of *A* to the lower end of *B*, and so on. To provide a ready means of distinguishing and identifying these alternative methods of connection, it is convenient to use the letters Z and N, as the slope of the inclined portion of the letter then indicates the slope of the intermediate connections in the conventional diagram.

Combinations of Connections for a Star-star* Transformer. The neutral-point connections of each winding may be made according to either of the methods (*a*) or (*b*), Fig. 180. Although four combinations are theoretically possible, two of these give polarities which are identical with those of the other two combinations, so that we need only consider the combinations which give opposite polarities. Thus one combination is obtained by making the neutral-point connections at the same ends of the coils of both primary and secondary windings, and the other combination is obtained by making the neutral-point connections for the secondary winding at the opposite coil-ends to those used in making this connection for the primary winding. These combinations may be represented conventionally by ⅄⅄ and ⅄Y.

The connection and vector diagrams are shown in Fig. 181. Observe that in the connection diagrams a given coil or terminal of the primary windings is connected to the same line wire. Hence the phase sequence for the secondary voltages is the same for the two combinations, as will be seen from the vector diagrams.

If the connections between two of the primary terminals and the line wires are reversed, the phase sequence of the secondary voltages will be reversed, as is shown by the vector diagram (*f*), Fig. 181.

Combinations of Connections for a Delta-delta Transformer. In this case, as in the star-star case, the combinations of connections reduce to two, viz. (1) similar interconnections (N or Z) for both

* The designation star-star denotes that both primary and secondary windings are star-connected. The first word of the designation refers to the interconnection of the primary windings and the second refers to the interconnection of the secondary windings.

windings; (2) dissimilar interconnections, e.g. N or Z for the primary, Z or N for the secondary. These combinations may be represented conventionally by △△ and △▽. The connection and vector diagrams are shown in Fig. 182.

The vector diagrams should be compared with those for a star-star transformer. This comparison will show that the diagrams for

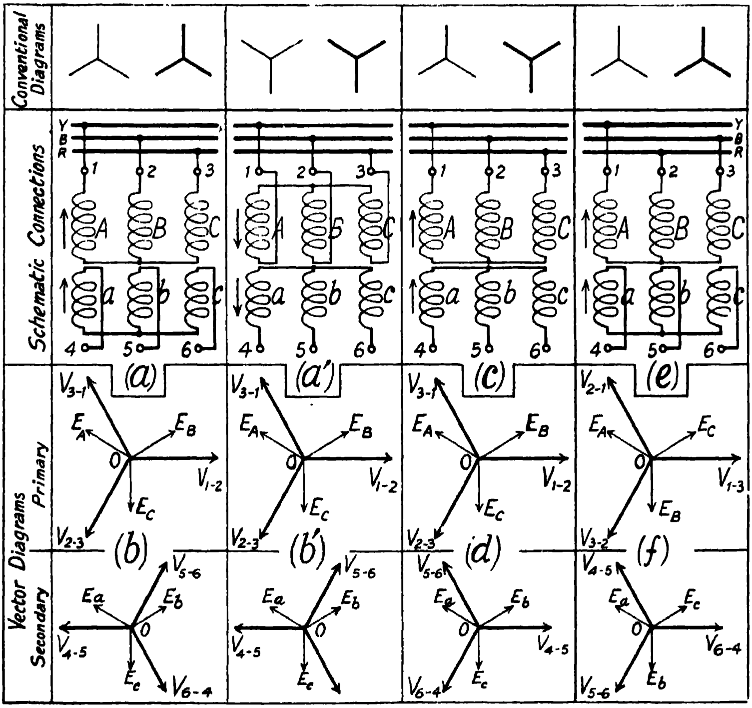


FIG. 181. CONNECTIONS AND VECTOR DIAGRAMS FOR THREE-PHASE TRANSFORMER

Star/Star Connections

the ⅄⅄ and △△ combinations are similar, and those for the ⅄Y and △▽ combinations are also similar, but the latter differ from the former, as there is a phase difference of 180° between the vector groups of the secondary voltages. Hence parallel operation is impossible between ⅄⅄ and ⅄Y, or △△ and △▽ transformers.

Combinations of Connections for a Star-delta Transformer. Four combinations are possible for the windings, viz. (1) ⅄△, (2) ⅄▽, (3) Y△, (4) Y▽. With the same connections between primary terminals and line wires in all cases, however, the vector diagrams are of two

forms, similar diagrams being obtained for the ⅄△ and Y▽ combinations, and similar diagrams for the ⅄▽ and Y△ combinations, as shown in Fig. 183.

The two groups of vectors representing the secondary voltages, however, have a phase difference of 180° relative to each other, and phase differences of 30° (leading in one case and lagging in the

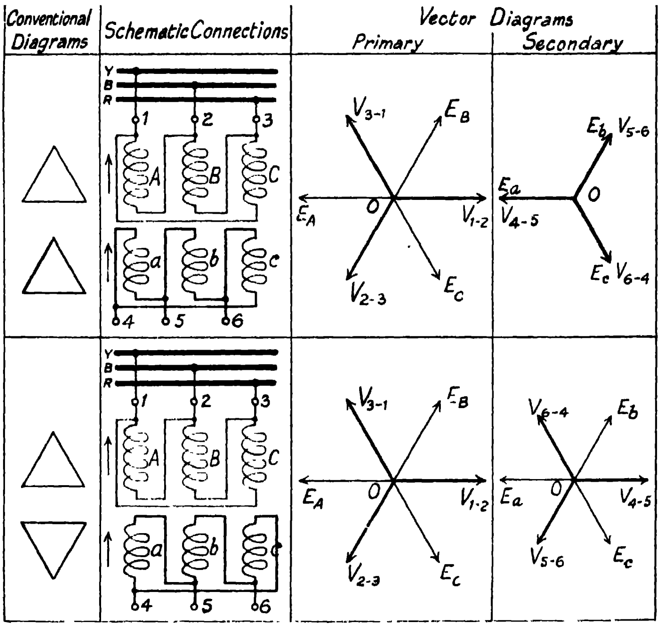


FIG. 182. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE TRANSFORMER
Delta/Delta Connections

other) relative to the vector group (Fig. 181) for a normal star-star connection. The 30°-phase displacement between the star-delta and star-star (or delta-delta) vector diagram is a characteristic feature of the star-delta connection, and it applies also to the delta-star connection.

Combinations of Connections for a Delta-star Transformer. Four combinations are possible for the windings, viz. (1) △⅄, (2) △Y, (3) ▽⅄, (4) ▽Y. But, as in the star-delta case, with the same connections between primary terminals and line wires, the vector diagrams are of two forms, as shown in Fig. 184. If, however, the two sets of vector diagrams are compared, it will be found that the

form of vector diagram for the ⅄△ combination of the star-delta transformer agrees with that for the △Y combination of the delta-star transformer. Similarly, the form of vector diagram for the Y△ combination of the star-delta transformer agrees with that for the △Y combination of the delta-star transformer.

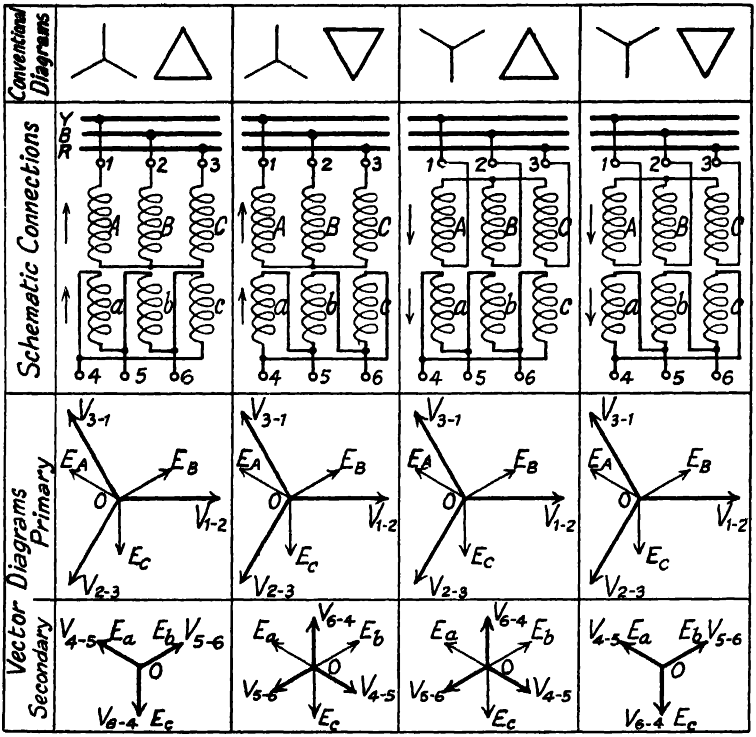


FIG. 183. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE TRANSFORMER
Star/Delta Connections

Thus, as will be shown later, parallel operation of star-delta and delta-star transformers is possible.*

Interconnected-star or Zigzag Connection. This connection, which has already been considered in detail on p. 191, is used for the *secondary* winding of a three-phase transformer in conjunction with either a star or a delta connection for the primary winding. But

* In order to obtain parallel operation when the primary connections are arranged as shown in Figs. 183, 184, the secondary terminals in Fig. 183 should be marked in the order 6, 4, 5, instead of 4, 5, 6.

on account of the reduced kVA. rating of such a winding compared with a normal star-connected winding, the use of the zigzag connection is restricted to special cases where the ordinary star connection cannot be employed.

Thus (1) in a three-phase mercury-arc rectifier installation the

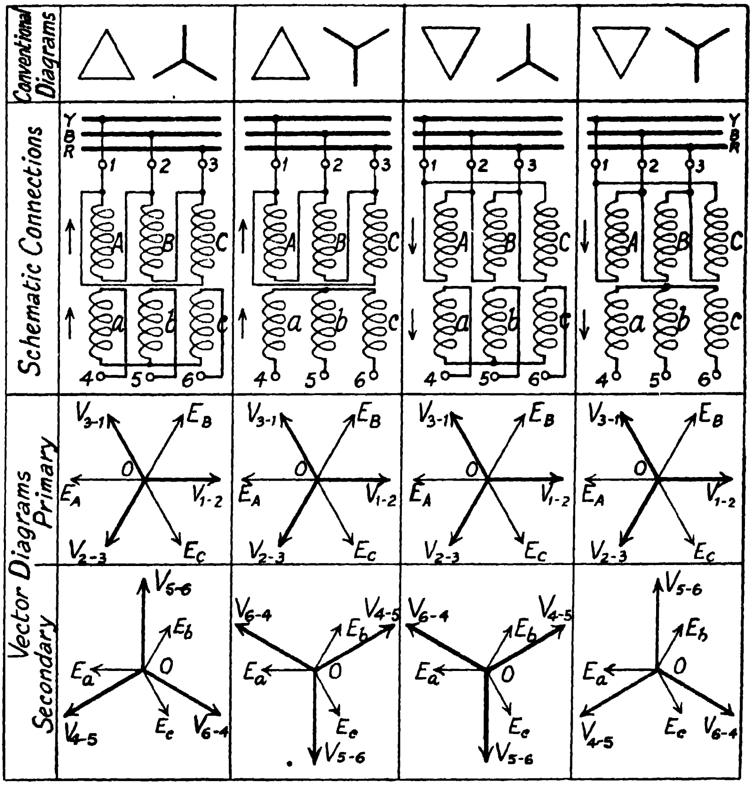


FIG. 184. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE TRANSFORMER

Delta/Star Connections

direct current returns to the neutral point of the transformer, and the zigzag connection of the secondary winding is necessary in order that no magnetization of the core may result from the circulation of this current.

(2) In a three-phase rotary converter supplying a three-wire direct-current system, the middle wire of this system is connected

to the neutral point of the transformer, in order that the out-of-balance current may return, via the secondary windings and slip-rings, to the armature winding. The zigzag connection is, therefore, required in this case.

(3) In A.C. multi-operator welding equipments, the individual

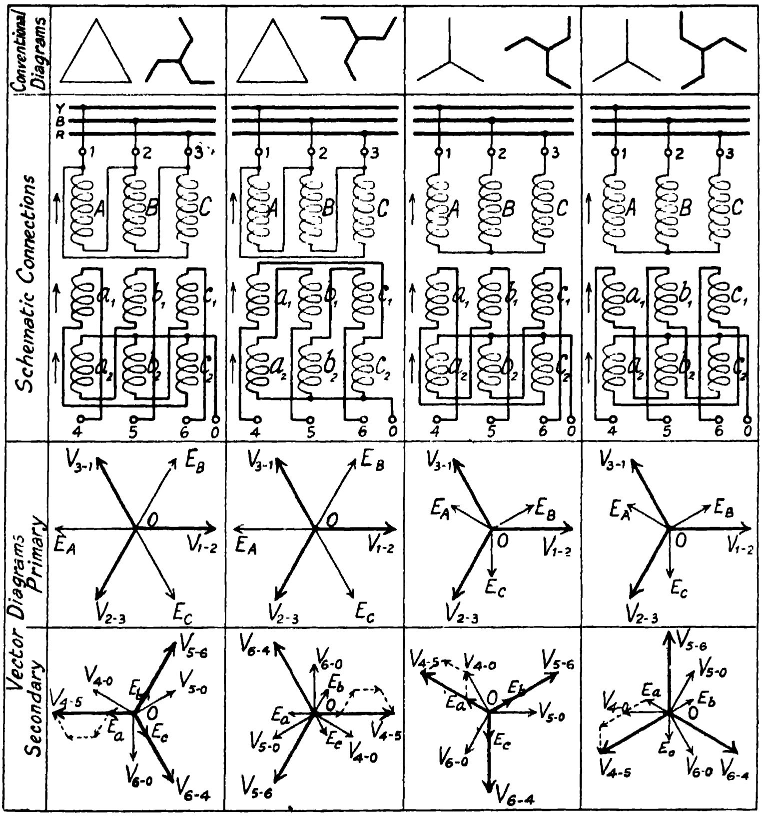


FIG. 185. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE TRANSFORMER

Delta/Zigzag (Interconnected-star) and Star/Zigzag (Interconnected-star) Connections

(single-phase) welders are supplied from a three-phase transformer on the four-wire system. The delta-zigzag connection results in less unbalance in the three-phase system than the alternative delta-star connection.

(4) When a third harmonic is present in the flux wave-form, the

corresponding E.M.F. may be eliminated from the phase voltage of the secondary by the zigzag connection.

Combination of Interconnections for Zigzag-connected Transformers. The zigzag connection requires each phase of the secondary winding to be wound in two equal sections. An important feature in arranging the interconnections is that a section belonging to one phase, e.g. A, is connected, with *reversed polarity*, in series with a section belonging to an adjacent phase, e.g. B or C. Although a number of combinations are possible, those which can be employed in practice must be chosen so that the resulting vector diagrams for the primary and secondary terminal voltages agree with those for one of the ordinary methods of interconnection (e.g. star-star, delta-star, etc.). When this limitation is imposed and the interconnection of the primary winding is taken into account, the number of combinations reduce to four.

These combinations are shown in the connection diagrams of Fig. 185. The combinations (*a*) and (*b*) are used with a normal delta-connected primary winding (Fig. 180 (*c*)), and give vector diagrams for the terminal voltages agreeing with those for the $\triangle\triangle$ and $\triangle\triangledown$ combinations respectively (Fig. 182.)

The combinations (*c*) and (*d*) are used with a normal star-connected primary winding, and give vector diagrams for the terminal voltages agreeing with those for the delta-star combinations (Fig. 184). It will be observed that in these diagrams the secondary voltages have phase differences of 30° (leading in one case and lagging in the other) relative to those of a normal star-star transformer.

Marking of Terminals. It is apparent from the variety of methods of interconnecting the windings that a definite scheme of arranging the internal connections and marking the terminals must be employed by *all* manufacturers if confusion is to be avoided when transformers are installed and connected to the supply system and the load. Such a scheme has been standardized by the British Standards Institution (see British Standard Specification No. 171) and is shown in all the diagrams of Figs. 180–185. To facilitate the installation and connecting-up of transformers for parallel working, a vector diagram of the primary and secondary voltages is given on the name-plate of each transformer.

Parallel Operation. The conditions for the parallel operation of single-phase transformers (p. 165) apply also to three-phase transformers, but with the following additions—

(1) The voltage ratios must refer to the *terminal* voltages (hence such ratios may not be equal to the ratio of the numbers of turns per phase).

(2) The phase displacement, if any, between the vector diagrams representing the primary and secondary voltages must be the same for all transformers which are to be connected in parallel.

(3) The phase sequence, or phase rotation, of the secondary voltages, when the primaries are excited from the same bus-bars, must be the same for all transformers.

Condition (2), which requires that there shall be phase displacement between the vector diagrams representing primary and secondary voltages, limits parallel operation to certain combinations of connections. These may be placed into four groups, as indicated in Table VI, and parallel operation is possible between any transformers belonging to the *same* group, provided that the other conditions are satisfied.

TABLE VI

GROUP CLASSIFICATION OF THREE-PHASE TRANSFORMERS WITH WINDINGS CONNECTED ACCORDING TO B.S.S. No. 171

Group No.	Phase Displacement	Interconnection of Windings		Group No.	Phase Displacement	Interconnection of Windings	
		Primary	Secondary			Primary	Secondary
1 1	0° 0°	⅄ △	⅄ △	3 3	− 30° − 30°	△ ⅄	≺ ◁
2 2	180° 180°	⅄ △	Y ▽	4 4	+ 30° + 30°	△ ⅄	≻ ▷

Note. The interconnected-star or zigzag combinations are not included, but may be obtained from B.S.S. No. 171.

Tests for Polarity, Phase Displacement, and Phase Sequence. Although transformers with terminals marked according to British Standard Specification may safely be connected in parallel without checking the polarity, cases may arise in practice where such a test, together with tests for phase displacement and phase sequence, are desirable.

The test for polarity is made by exciting both primary windings from the same bus-bars, connecting together temporarily two corresponding secondary terminals, say, a_1, a_1', and connecting a voltmeter successively to the other pairs of corresponding secondary terminals. If the voltmeter reads zero, the polarities are correct for parallel operation. If the voltmeter reads double the normal secondary voltage, the polarities are incorrect, and parallel operation

will be possible by a reversal of polarity (assuming this to be possible) of one transformer. Voltmeter readings of other magnitudes will indicate either incorrect phase sequence on one transformer or a phase displacement between the vector diagrams representing the secondary voltages.

The test for phase sequence is made by one of the methods described in Chapter XVIII (p. 431).

Calculation of Loadings of Transformers Operating in Parallel. When the load connected to the secondary bus-bars is balanced, the current supplied by each transformer can be calculated by applying the same principles as were employed in the case of single-phase transformers operating in parallel. The calculations are, of course, made for one phase only, and the value of equivalent impedance to be used in the calculations is the *equivalent impedances per phase* referred to the secondary winding. When the separate impedances of the windings are given, the impedance of the primary winding must be converted into an equivalent impedance referred to the secondary winding. In calculating this quantity for star-delta and delta-star transformers, it must be remembered that the voltage ratios for these transformers refer to the *terminal* voltages and are not equal to the "turn ratios." For example, if V_1, V_2, denote the primary and secondary terminal voltages, the "turn ratio" for a star-delta transformer is $(V_1/\sqrt{3})/V_2 = V_1/\sqrt{3}V_2$, and the "turn ratio" for a delta-star transformer is $V_1/(V_2/\sqrt{3}) = \sqrt{3}V_1/V_2$.

If the equivalent impedance per phase is reduced to its appropriate value for a star-connected secondary winding, the equivalent circuit per phase for each transformer then becomes a simple $R-L$ series circuit; and the vector diagram, drawn for phase quantities, is similar to that (Fig. 95) for the parallel operation of single-phase transformers.

Example 1. Two three-phase transformers, A and B, of equal kVA. ratings, are connected in parallel to share a load of 500 kVA. at 0·8 power factor lagging. The equivalent delta impedances referred to the secondary terminals are $2 + j5$ ohms for A, and $2 + j4$ ohms for B. Calculate the loading of each transformer.

The equations (53), (56), deduced for single-phase transformers, are applicable to this case, and since only the ratio of the equivalent impedances is required for the calculation, the given (delta) values may be used directly without converting them into "star" values.

Thus, $\quad \mathrm{k\dot{V}A_A} = \mathrm{k\dot{V}A}/(1 + \dot{Z}_A/\dot{Z}_B)$

$\quad \mathrm{k\dot{V}A_B} = \mathrm{k\dot{V}A}/(1 + \dot{Z}_B/\dot{Z}_A)$

Now, $\quad \dot{Z}_A/\dot{Z}_B = (2 + j5)/(2 + j4) = 1{\cdot}2 + j0{\cdot}1$

$\quad \dot{Z}_B/\dot{Z}_A = 0{\cdot}827 - j0{\cdot}069$

Hence $\quad \mathrm{k\dot{V}A_A} = 500(0{\cdot}8 - j0{\cdot}6)/(2{\cdot}2 + j0{\cdot}1)$

$\quad = 175 - j144$

and $\quad kVA_B = 500(0{\cdot}8 - j0{\cdot}6)/(1{\cdot}827 - j0{\cdot}069)$

$$= 225 - j156$$

Whence

$$kVA_A = \sqrt{(175^2 + 144^2)} = 226{\cdot}8$$
$$\cos\varphi_A = 175/226{\cdot}8 = 0{\cdot}772$$
$$kVA_B = \sqrt{(225^2 + 156^2)} = 273{\cdot}2$$
$$\cos\varphi_B = 225/273{\cdot}2 = 0{\cdot}823.$$

Example 2. Two delta-star, 11,000/400 V. transformers, A and B, of equal kVA. ratings, are connected in parallel to supply a load of 1000 kVA. at 0·8 power factor lagging. The impedances per phase are: A—primary $7 + j21$ ohms, secondary $0{\cdot}003 + j0{\cdot}0075$ ohm; B—primary $8 + j19$ ohms, secondary $0{\cdot}0035 + j0{\cdot}007$ ohm. Calculate the loading of each transformer.

The voltage ratio 11,000/400 refers to the terminal voltages. The secondary phase voltage $= 400/\sqrt{3} = 230$ V., and therefore the ratio of turns is approximately 11,000/230.

The equivalent impedances per phase referred to the secondary are—

$$Z_A = 0{\cdot}003 + j0{\cdot}0075 + (230/11{,}000)^2\,(7 + j21)$$
$$= 0{\cdot}00606 + j0{\cdot}01668$$
$$Z_B = 0{\cdot}0035 + j0{\cdot}007 + (230/11{,}000)^2\,(8 + j19)$$
$$= 0{\cdot}007 + j0{\cdot}0153$$

Hence

$$Z_A/Z_B = 1{\cdot}05 + j0{\cdot}085$$
$$Z_B/Z_A = 0{\cdot}947 - j0{\cdot}0764$$
$$kVA_A = 1000(0{\cdot}8 - j0{\cdot}6)/(2{\cdot}05 + j0{\cdot}085)$$
$$= 377 - j290$$
$$kVA_B = 1000(0{\cdot}8 - j0{\cdot}6)/(1{\cdot}947 - j0{\cdot}0764)$$
$$= 423 - j310$$

Whence

$$kVA_A = \sqrt{(377^2 + 290^2)} = 476$$
$$\cos\varphi_A = 377/476 = 0{\cdot}792$$
$$kVA_B = \sqrt{(423^2 + 310^2)} = 524$$
$$\cos\varphi_B = 423/524 = 0{\cdot}808$$

Parallel Operation of Delta- and V- (Open-delta) connected Transformers. The open-delta or V connection, which has already been considered on p. 192, is obtained from the delta connection of three single-phase transformers by removing one transformer. It is useful as an *emergency* connection, and is not employed for normal operation because the output from the V-group is only $1/\sqrt{3}$ ($= 0{\cdot}577$) of that from the original Δ-group.

When a group of (two) vee-connected transformers is connected in parallel with a group of (three) delta-connected transformers—all of identical ratings—the total kVA. available for the load is approximately only 80 per cent of the total kVA. rating of the groups if none of the transformers is be overloaded. Thus five 100 kVA. transformers (identical in all respects) would be required

for a load of 400 kVA. One transformer would be fully loaded, and each of the other four would be operating at about three-quarters of full load. The method of calculating the loadings is deduced in the following section.

Calculation of Loadings of V- and $\triangle$-connected Transformers Operating in Parallel. The secondary circuits of the transformers (of which A, B, C form the delta group and D, E the V-group) and the load are shown in Fig. 186.

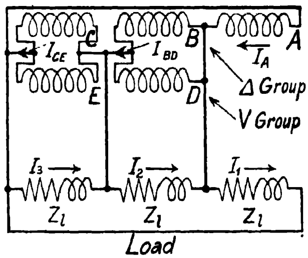


FIG. 186. CIRCUIT DIAGRAM (SECONDARY CIRCUITS ONLY) FOR PARALLEL OPERATION OF $\triangle$- AND V-CONNECTED TRANSFORMERS

Let Z_l denote the impedance of each phase of the balanced load; I_1, I_2, I_3, the load currents; I_A, the current in transformer A; I_{BD}, the sum of the currents in transformers B and D; I_{CE}, the sum of the currents in transformers C and E; Z, the equivalent impedance of each transformer referred to the secondary; E_A, E_B, E_C, the no-load voltages.

Then, from Fig. 186, we have

$I_1 + I_2 + I_3 = 0$; $I_A - I_{CE} = I_1 - I_3$; $I_{BD} - I_A = I_2 - I_1$; $I_{BD} - I_{CE} = I_2 - I_3$

Hence $I_{BD} = I_A - I_1 + I_2$; $I_{CE} = I_A - 2I_1 - I_2$.

Ignoring exciting currents, we have

$$E_A = I_1 Z_l + I_A Z \qquad (97)$$

$$E_B = I_2 Z_l + \tfrac{1}{2} Z I_{BD} = I_2 Z_l + \tfrac{1}{2} Z(I_A - I_1 + I_2) \qquad (97a)$$

$$E_C = I_3 Z_l + \tfrac{1}{2} Z I_{CE} = -(I_1 + I_2) Z_l + \tfrac{1}{2} Z(I_A - 2I_1 - I_2) \qquad (97b)$$

Hence $E_B + E_C = I_A Z - I_1(Z_l + \tfrac{3}{2} Z) = -E_A$, since $E_A + E_B + E_C = 0$.

Subtracting this equation from (97), we have

$$2E_A = I_1(2Z_l + \tfrac{3}{2} Z), \text{ or } I_1 = 2E_A/(2Z_l + \tfrac{3}{2} Z).$$

Substituting this value in (97) and solving for I_A, we obtain

$$I_A = (E_A - I_1 Z_l)/Z = E_A[3Z/(4Z_l + 3Z)] \qquad (98)$$

Similarly, by expressing E_B and E_C in symbolic form, we obtain

$$I_{BD} = E_A \frac{-(6Z_l + 3Z) + j\sqrt{3}(4Z_l + 3Z)}{(4Z_l + 3Z)(2Z_l + Z)} \qquad (99)$$

$$I_{CE} = E_A \frac{-(6Z_l + 3Z) - j\sqrt{3}(4Z_l + 3Z)}{(4Z_l + 3Z)(2Z_l + Z)} \qquad (100)$$

If, as an approximation, Z is ignored in comparison with Z_l, the expressions for the currents in the transformers reduce to

$$I_A = \tfrac{3}{4}(E_A/Z_l);\ I_{BD} = \tfrac{1}{4}(-3 + j2\sqrt{3})(E_A/Z_l);$$
$$I_{CE} = \tfrac{1}{4}(-3 - j2\sqrt{3})(E_A/Z_l)$$

Whence $I_A = \tfrac{3}{4}E/Z_l$; $I_{BD} = \tfrac{1}{4}\sqrt{21}(E/Z_l)$; $I_{CE} = \tfrac{1}{4}\sqrt{21}(E/Z_l)$.

Therefore the currents in the individual transformers A, B, C are in the ratio of $3 : \frac{1}{2}\sqrt{21} : \frac{1}{2}\sqrt{21}$ or $1 : 0{\cdot}763 : 0{\cdot}763$. Likewise, the currents in transformers D and E are only 0·763 of the current in A.

Phase Transformation

One of the advantages of the three-phase system is that other polyphase systems can be obtained by means of a transformer with suitable windings. *Six-, nine-, and twelve-phase systems can be easily obtained from the ordinary* type of three-phase transformer by providing suitable secondary windings.

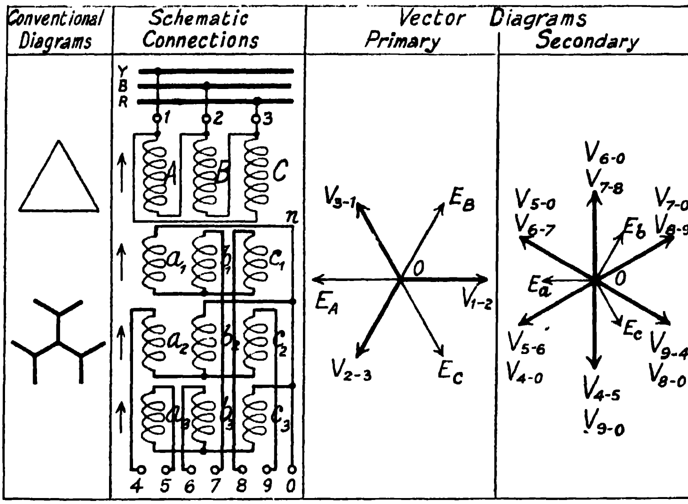


FIG. 187. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE/SIX-PHASE TRANSFORMER

Delta/Triple-star Connections

Two-phase and four-phase systems can be obtained by means of either two single-phase transformers or a three-phase transformer.

Three-phase to Six-phase Transformation. We have already shown in Chapter IX how a six-phase system may be obtained from a three-phase system. Thus referring to p. 205, we can obtain a six-phase supply by providing two secondary windings (each having the same number of turns) per phase, connecting these to form two three-phase star-connected systems using the combinations Y⅄, and then connecting the two neutral points together.

In cases where this simple six-phase star connection is undesirable (such as when the third harmonic, which is known to be present in the secondary, E.M.F's. is to be eliminated from the phase voltage of the six-phase system), the triple-star connection is employed. This connection requires three equal secondary windings (each having the same number of turns) per phase, and the method of interconnecting the sections is somewhat similar to that employed for the zigzag connection (p. 191). Fig. 187 shows the connection and vector diagrams for the present case. The remarks, on p. 191, concerning ratings, apply also to the present case.

The six-phase star connection (sometimes called the double-star connection) and the triple-star connection are used chiefly in transformers supplying

mercury-arc, and other forms of six-phase, rectifiers requiring a star-connected transformer for the anode supply system, the neutral point forming the negative pole of the output (direct-current) circuit.

For supplying rotary converters, or other mesh-connected loads, alternative connections such as the diametrical and the double-delta (p. 208) are employed. These are discussed fully in Chapter IX.

Three-phase to Twelve-phase Transformation. The connections of a transformer for supplying a twelve-phase system can be readily devised by an extension of the principles employed in three-phase/six-phase transformation.

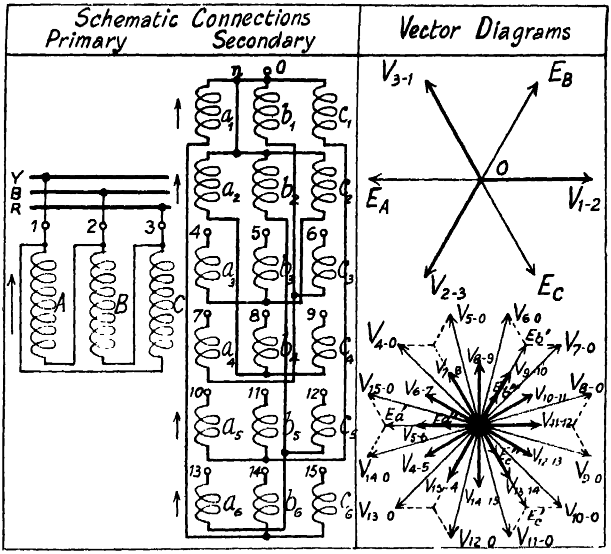


FIG. 188. CONNECTION AND VECTOR DIAGRAMS FOR THREE-PHASE/TWELVE-PHASE TRANSFORMER

Delta/Star (interconnected) Connections

The dotted lines in the vector diagram indicate which phase E.M.Fs. are compounded to obtain the voltages between each of the line terminals and the neutral point (0).

A relatively large number of combinations (requiring from two to six sections of secondary winding per limb) are possible for a mesh-connected load (e.g. a rotary converter), but for a twelve-anode mercury-arc rectifier only the combinations which have a common neutral point can be employed.

One example for the latter purpose is shown in Fig. 188. Six sections are required on each limb: four sections are each wound to give a voltage of 81·6 per cent of the phase voltage of the twelve-phase system, and the remaining two sections are each wound to give a voltage of 29·9 per cent of the phase voltage of the twelve-phase system. These numerical values are readily obtained from the vector diagram.

Four combinations suitable for a mesh-connected load are shown in Fig. 189.

Method *A* requires triple secondary windings, two sets of which are similar and are interconnected according to the double-delta method. The remaining set of windings may be provided with mid-point tappings, which, when interconnected, form a neutral point to the mesh-connnected load.

Method *B* requires quadruple secondary windings: two sets are connected

in double delta, and the remaining two sets are connected in mesh. The windings forming the double-delta group are similar, and the windings forming the mesh-connected group are also similar, but the voltages of the two groups are unequal.

Methods C and D require only double secondary windings, the sections of which are not inter-connected except through the load.

The relative voltages to be supplied by each section of the secondary

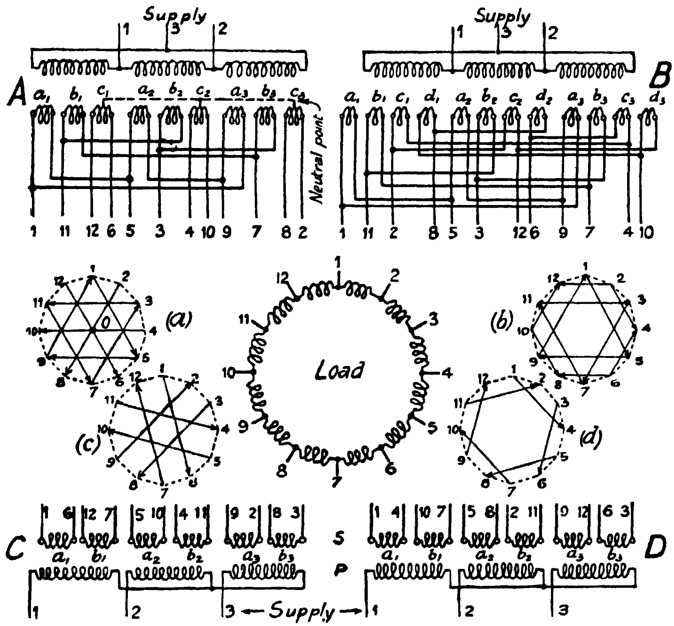


FIG. 189. FOUR METHODS OF SUPPLYING A TWELVE-PHASE LOAD FROM A THREE-PHASE TRANSFORMER

windings for the several methods of connection can be easily determined from the geometry of the vector diagrams (a, b, c, d) of E.M.Fs.

Three-phase to Two-phase Transformation. The method employing two single-phase transformers is shown in Fig. 190.*

Both transformers have similar magnetic circuits and may have similar secondary windings, in which case the primary winding of one transformer, B, must have only 0·866 ($= \frac{1}{2}\sqrt{3}$) of the turns of the other transformer, A, and the latter must have its primary winding tapped at the mid-point. One end of the 86·6 per cent winding is connected to the mid-point of the primary winding of transformer, A, and the other end of the primary winding of B, as well as both ends of the primary winding of A, are connected to the three-phase line wires. The ends of the secondary windings are connected to the two-phase line wires.

Assuming the two-phase side to be unloaded, and the magnetizing currents, which are supplied from the three-phase side, to be balanced and sinusoidal, the resultant ampere-turns in transformer A are equal to the vector difference of the ampere-turns due to the currents in the half-sections of its primary

* This connection is due to Prof. C. F. Scott, and is usually called the "Scott" connection.

winding. Hence, since these currents have a phase difference of 120°, the resultant ampere-turns are given by $F_A = \sqrt{3}.I_0 \times \frac{1}{2} N_1 = 0{\cdot}866 I_0 N_1$, where I_0 is the magnetizing current and N_1 the number of turns in the primary winding.

Similarly, the ampere-turns in transformer B are given by $F_B = 0{\cdot}866 I_0 N_1$, since the primary winding of this transformer has only 86·6 of the number of turns in the primary winding of transformer A. Moreover, the ampere-turns, F_B, have a phase difference of 90° with respect to F_A.

Hence the fluxes in the two transformers are equal and have a phase difference of 90°. Therefore the E.M.Fs., E_1, E_{11}, induced in the secondary windings are equal and have a phase difference of 90°.

Consider now the E.M.Fs. induced in the primary windings by these fluxes. Since the fluxes are equal, the E.M.Fs. will be proportional to the number of turns in each winding, and therefore the E.M.F., E_B, induced in B, will only be 86·6 per cent of that E_A, induced in the two half-sections of A. Moreover, these E.M.Fs. have a phase difference of 90°.

Taking the positive directions of these internal or induced E.M.Fs. as those marked by the arrows in Fig. 190, and neglecting the resistance and reactance of the primary winding, the resultant internal E.M.Fs. between the terminals 1, 2, 3, of the three-phase side are

$$\underset{\cdot}{E}_{1\text{-}2} = \underset{\cdot}{E}_B - \tfrac{1}{2}\underset{\cdot}{E}_A, \qquad \underset{\cdot}{E}_{2\text{-}3} = \underset{\cdot}{E}_A, \qquad \underset{\cdot}{E}_{3\text{-}1} = -\tfrac{1}{2}\underset{\cdot}{E}_A - \underset{\cdot}{E}_B$$

Expressing these quantities symbolically, and taking E_B as the quantity of reference, we have

$$\begin{aligned}
\underset{\cdot}{E}_B &= E_B(1 + j0) = E_A(0{\cdot}866 + j0)\,; \; \underset{\cdot}{E}_A = E_A(0 - j1) \\
\underset{\cdot}{E}_{1\text{-}2} &= \underset{\cdot}{E}_B - \tfrac{1}{2}\underset{\cdot}{E}_A = E_A(0{\cdot}866 + j0{\cdot}5) \\
\underset{\cdot}{E}_{2\text{-}3} &= E_A(0 - j1) \\
\underset{\cdot}{E}_{3\text{-}1} &= -\tfrac{1}{2}\underset{\cdot}{E}_A - \underset{\cdot}{E}_B = E_A(-\,0{\cdot}866 - j0{\cdot}5)
\end{aligned}$$

Whence the magnitudes of these E.M.Fs. are given by

$$\begin{aligned}
E_{1\text{-}2} &= E_A\sqrt{(0{\cdot}5^2 + 0{\cdot}866^2)} = E_A, \\
E_{2\text{-}3} &= E_A, \\
E_{3\text{-}1} &= E_A\sqrt{(0{\cdot}5^2 + 0{\cdot}866^2)} = E_A
\end{aligned}$$

and their phase differences with respect to E_B are given by

$$\begin{aligned}
\varphi_{1\text{-}2} &= \tan^{-1}(-\,0{\cdot}5/\,0{\cdot}866) = 30°, \\
\varphi_{2\text{-}3} &= \tan^{-1} - 1/0 = -\,90°, \\
\varphi_{3\text{-}1} &= \tan^{-1}(-\,0{\cdot}5/\,0{\cdot}866) = 150°.
\end{aligned}$$

Therefore the internal E.M Fs. between the terminals 1, 2, 3, of the three-phase side are equal to one another and have a mutual phase difference of 120°. Thus the symmetry and balance of the three-phase supply system are not affected.

A vector diagram showing the induced E.M.Fs. and fluxes is given in Fig. 190 (c), in which the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\text{-}1}$ represent the line voltages of the three-phase system; OI_{01}, OI_{02}, OI_{03}, the magnetizing currents (which are lagging 90° with respect to the corresponding phase voltages of the three-phase system); OF_A, OF_B, the ampere-turns supplied by the magnetizing currents; $O\Phi_A$, $O\Phi_B$, the fluxes due to the magnetizing ampere-turns; OE_{A1}, OE_{A2}, and OE_{B1}, OE_{B2}, the E.M.Fs. induced in the primary and secondary windings of the two transformers.

If the fluxes are to remain constant when the two-phase side is loaded—which will be the case if the three phase supply voltage is constant and the resistance and reactance voltage drops in the primary windings are negligible—the magnetizing ampere-turns must remain constant, and therefore the

ampere-turns due to the load currents in the secondary windings must be balanced by an equivalent number of ampere-turns in the primary windings, the vector sum of the ampere-turns in primary and secondary windings being, in all cases, equal to the magnetizing ampere-turns.

The conditions for balanced loads are represented in the vector diagram (*d*) Fig. 190, in which the magnetizing ampere-turns are represented by the vectors OF_A, OF_B; the load currents by OI_I, OI_{II}; the secondary ampere-turns by OF_{A2}, OF_{B2}; and the primary ampere-turns by OF_{A1}, OF_{B1}, which

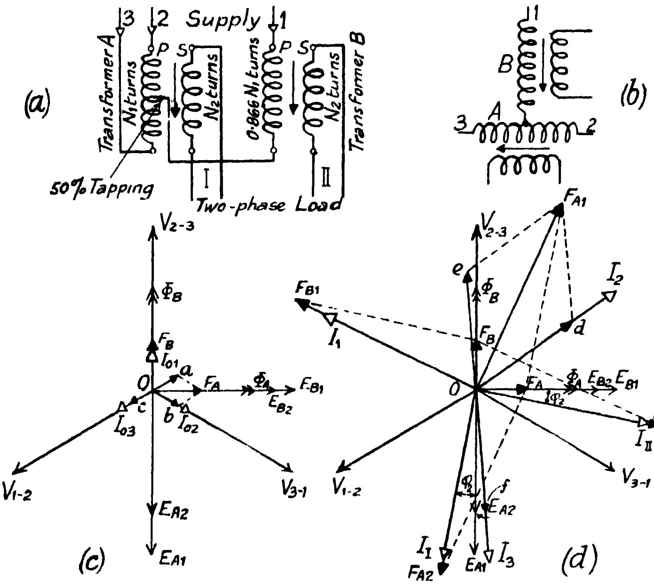


FIG. 190. CIRCUIT AND VECTOR DIAGRAMS FOR SCOTT'S METHOD OF SUPPLYING A TWO-PHASE LOAD FROM A THREE-PHASE SYSTEM

(*a*) Transformer Connections; (*b*) Conventional Circuit Diagram; (*c*) Vector Diagram for No-load; (*d*) Vector Diagram for Balanced Loads.

The effects of losses and magnetic leakage are ignored in the vector diagrams.

have a phase difference of 90°. The primary current of transformer *B* is, therefore, represented by the vector OI_1.

The primary ampere-turns of transformer *A* are due to the currents in the half-sections of this winding, and these currents are supplied from lines 2 and 3 of the three-phase system. Hence if OF_{A1} is resolved into vectors *Oe*, *Od*, 60° apart and each equal to $\frac{1}{2}OF_{A1}$, then *Od* will represent the ampere-turns due to the current in line 2, and *Oe* reversed, i.e *Of* will represent those due to the current in line 3. Since the angle between OF_{A1} and OF_{B1} is 90°, the vectors *Oe*, *Of* will have a mutual phase difference of 120° with respect to the vector OF_{B1}. Hence if these ampere-turn vectors are converted into the current vectors OI_2, OI_3, then each of these vectors will be found to be equal in magnitude to the vector OI_1.

Therefore, with balanced loads on the two-phase side the currents on the three-phase side will also be balanced.

In order to maintain these balanced conditions in transformers of the

"core" type, the coils forming the two half-sections of the primary winding of transformer A must be interlaced, or sandwiched, in order that each part of the magnetic circuit may be acted upon equally by the joint M.M.Fs. due to the two phases of the three-phase system which supplies this transformer.

Moreover, if the magnetizing ampere-turns are neglected, and if $N_2 = N_1 = N$, say—i.e. the voltage across each phase of the two-phase side is equal to the voltage between the line wires of the three-phase side, the effects of resistance and reactance of the windings being neglected—and I_1 is the current in the primary windings, we have

$$I_2N = 0{\cdot}866I_1N$$

whence,
$$I_1 = I_2/0{\cdot}866 = I_2(2/\sqrt{3}) = 1{\cdot}15I_2.$$

This, then, is the numerical relationship between the currents in the primary and secondary windings when the loads are balanced and the magnetizing current is ignored.

Wave-form Distortion in Three-phase Transformers*

In Chapter XV it is shown that when iron is magnetized by alternating current, and the flux density is carried beyond the straight portion of the magnetization curve, the wave-form of the magnetizing current differs from that of the flux. In general, to obtain a sine wave of flux, the magnetizing current must contain third, fifth, and higher harmonics, the third being the most prominent.

Star-star Connected Single-phase Transformers. If three single-phase transformers are star-connected on both primary and secondary sides, and are excited from a three-wire three-phase system giving a sine wave voltage, no third harmonic can be present in the magnetizing current because of the star connection. The flux wave-form, therefore, cannot be a sine wave but must be flat-topped. Similarly, the wave-form of the induced E.M.Fs., due to the flat-topped flux wave-form, will be peaked, indicating the presence of third and other harmonics. Hence, third harmonics will be present in the phase E.M.Fs. of both primary and secondary windings, but will not appear in the wave-forms of the terminal voltages. Therefore voltages of triple frequency exist between the neutral point of each winding and the natural neutral point of each system (e.g. the neutral point of the (star-connected) generator or a group of balanced star-connected resistances connected to the line wires as shown on p. 306).

If, then, such a star-star transformer is used as a step-up transformer to supply a high-voltage transmission system and the neutral point of the secondary (i.e. high-voltage) winding is earthed, the capacitance of the system will cause triple-frequency currents (and also currents of higher frequency if any multiples of the third harmonic, e.g. ninth, fifteenth, etc., are present in the phase E.M.Fs.) to circulate between the line wires and the earthed neutral point. Such currents are liable to cause trouble with telephone and similar circuits.

Star-star Connected Single-phase Transformers with Tertiary Windings. If each of the above transformers is provided with an additional secondary winding of low impedance (called a *tertiary* winding) and these windings are delta-connected, the triple frequency E.M.Fs. will produce a circulating current of triple frequency which will, therefore, supply the third harmonic component for the magnetizing current. In consequence, the flux will be approximately sinusoidal, and no triple-frequency E.M.Fs. will appear in the phase E.M.Fs. of the star-connected windings.

Oscillograms. Practical confirmation of the above phenomena may be obtained by means of an oscillograph, and examples of oscillograms are shown in Figs. 191, 192.

* Chapters XIV and XV should be studied before reading this section.

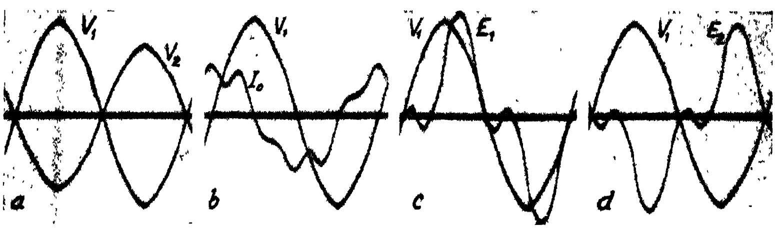


FIG. 191. OSCILLOGRAMS OF NO-LOAD VOLTAGES AND CURRENTS IN THREE SINGLE-PHASE TRANSFORMERS

Connected Star/Star, supplied at Constant Voltage, from a Three-phase System

(*a*) Supply voltage (V_1) and secondary line voltage (V_2); (*b*) Supply voltage (V_1) and exciting current (I_0); (*c*) Supply voltage (V_1) and primary phase voltage (E_1); (*d*) Supply voltage and secondary phase voltage (E_2). (*Note.* The line and phase voltages are shown to different scales.)

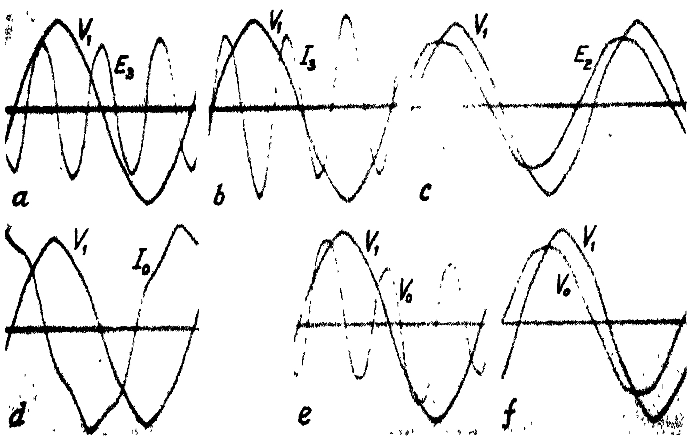


FIG. 192. OSCILLOGRAMS OF VOLTAGES AND CURRENTS IN THREE SINGLE-PHASE, THREE-WINDING TRANSFORMERS

Supplied, at Constant Voltage, from a Three-phase System
Primary and Secondary Windings Star-connected
Tertiary Winding arranged for Delta Connection

(*a*) Supply voltage (V_1) and voltage (E_3) in opened delta of tertiary winding; (*b*) Supply voltage (V_1) and current (I_3) in closed (delta-connected) tertiary winding; (*c*) Supply voltage (V_1) and no-load secondary phase voltage (E_2) tertiary windings closed; (*d*) Supply voltage (V_1) and exciting current (I_0), tertiary windings closed [*Note.* Fig. 191 (*b*) shows the exciting current with opened tertiary windings]; (*e*), (*f*) Supply voltage (V_1) and voltage (V_0) between neutral point of secondary winding and neutral point of a balanced star-connected load supplied by secondary, with tertiary windings open (*e*) and closed (*f*).

The oscillograms of Fig. 191 refer to three ordinary star-star connected transformers, and show the wave-forms of (*a*) the line voltages, (*b*) the exciting current, (*c*) the primary phase voltage, (*d*) the secondary phase voltage. The triple-frequency component in the phase voltages is quite apparent.

The oscillograms of Fig. 192 were obtained on the same transformers, with the addition of a tertiary winding on each, these windings being arranged for

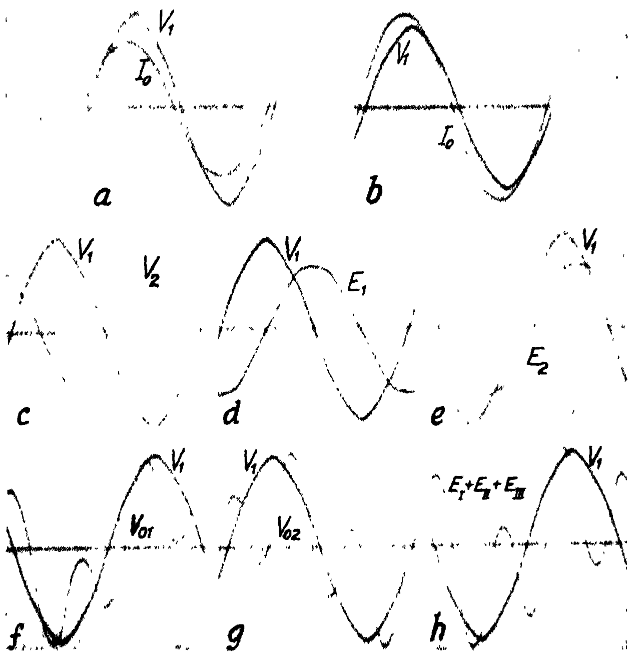


FIG. 193. OSCILLOGRAMS OF VOLTAGES AND NO-LOAD CURRENTS IN CORE-TYPE THREE-PHASE TRANSFORMER SUPPLIED AT CONSTANT VOLTAGE

(*a*) Supply voltage (V_1) and exciting current (I_0), star/star connections; (*b*) Supply voltage (V_1) and exciting current (I_0), star/delta connections; (*c*) Supply voltage (V_1) and secondary line voltage (V_2), star/star connections; (*d*) Supply voltage (V_1) and primary phase voltage (E_1), star/star connections, no load; (*e*) Supply voltage (V_1) and no-load secondary phase voltage (E_2), star/star connections; (*f*) Supply voltage (V_1) and voltage (V_{01}) between neutral points of primary winding and balanced star connected load connected to supply system, star/star connections, balanced load connected to secondary; (*g*) Supply voltage (V_1) and voltage (V_{02}) between neutral points of secondary winding and balanced star-connected load supplied by secondary; star/star connections; (*h*) Supply voltage (V_1) and resultant voltage ($E_I + E_{II} + E_{III}$) in open delta-connected secondary winding, star-connected primary.

delta connection. They show the wave-forms of (*a*) the resultant E.M.F. in the open-delta tertiary winding; (*b*) the current in the closed tertiary windings; (*c*) the secondary phase voltage; (*d*) the exciting current; (*e*) the voltage between the neutral point of the secondary windings and the neutral point of a balanced star-connected load on the secondary, the tertiary windings being open; (*f*) the voltage between these neutral points for the same load currents, but with the tertiary windings closed.

Delta-delta Connected Single-phase Transformers. In this case the delta-connected windings provide a closed circuit for the circulation of triple-frequency currents. Therefore a sine wave of flux is obtained with sinusoidal applied voltage.

Delta-star and Star-delta Connected Single-phase Transformers. These connections are particularly suitable for transmission circuits; the delta-star connection being used at the generator end of the line for stepping up the voltage, and the star-delta connection being used at the load end of the line for stepping down the voltage. In both cases the third harmonic component of the magnetizing current can circulate in the delta windings, and therefore no E.M.Fs. of triple frequency will appear in the phase voltages. If, however, fifth, seventh, etc., harmonics are present in the primary voltage of a star-delta transformer, the wave-form of the secondary voltage will differ from that of the primary voltage, e.g. if these harmonics cause the primary voltage to be dimpled the secondary voltage will be peaked, and vice versa.

Star-star Three-phase Transformer. In a three-phase transformer the M.M.Fs. of the individual phases act upon interlinked magnetic circuits instead of isolated magnetic circuits, when a bank of single-phase transformers is employed. Hence, with the usual three-limbed core, the fluxes in the cores will be practically free from third harmonic components, because of the high reluctances of their magnetic paths. Thus, since any third harmonic components of the core fluxes will be in phase with one another, their return paths must be through the air space between the yokes. Therefore the wave-form of the phase-voltage voltages will be practically free from third harmonic components.

Fig. 193 shows oscillograms taken on such a transformer.

CHAPTER XIV

COMMERCIAL AND NON-SINUSOIDAL WAVE-FORMS

In the preceding chapters sinusoidal currents and voltages have been considered almost exclusively to enable the fundamental principles of alternating-current circuits to be deduced in a simple manner and to allow graphical methods to be applied to the solution of problems. Although the sine wave is the ideal wave-form and is closely approached in modern alternators operating at no-load, the load conditions in generators and commercial circuits frequently cause considerable deviations from the sine wave. It is necessary, therefore, to consider some of the causes of wave distortion and the manner in which the relationship between current and E.M.F. is affected by this distortion.

Equation to a Complex Wave. By the application of Fourier's theorem, any single-valued* periodic function can be completely expressed by a series of simple harmonic functions (i.e. sine curves) having frequencies which are multiples of that of the complex function. These simple harmonic functions are called the *harmonics* of the complex function ; the function which has the same frequency as the complex function is called the *first harmonic*, or the *fundamental* ; that of double frequency, the *second harmonic* ; that of triple frequency, the *third harmonic*, and so on. For example, in the case of a complex wave the fundamental (e_1) may be represented by

$$e_1 = E_{1m} \sin(\omega t + \alpha_1) ;$$

the second harmonic by

$$e_2 = E_{2m} \sin(2\omega t + \alpha_2) ;$$

the third harmonic by

$$e_3 = E_{3m} \sin(3\omega t + \alpha_3) ; \quad \text{etc.,}$$

and the complex wave may be represented by the equation

$$e = E_{1m} \sin(\omega t + \alpha_1) + E_{2m} \sin(2\omega t + \alpha_2) + E_{3m} \sin(3\omega t + \alpha_3) + \ldots$$

where E_{1m}, E_{2m}, E_{3m}, . . . denote the maximum values, or amplitudes, of the first, second, and third harmonics respectively, and α_1, α_2, α_3 . . . denote the phase differences with respect to the

* A single-valued function is one in which the dependent variable has only one value for each value of the independent variable.

complex wave (i.e. the angles between the zero value of the complex wave and the corresponding zero values of the harmonic waves).

The number of terms in the series depends on the shape of the complex wave. Under certain conditions the number of terms may be indefinite, but under other conditions only a few terms may be involved. Again, the series may contain both even and odd harmonics, or only, alternatively, odd or even harmonics.

Shape of Complex Wave Containing only Even Harmonics. A complex wave containing only even harmonics is unsymmetrical,

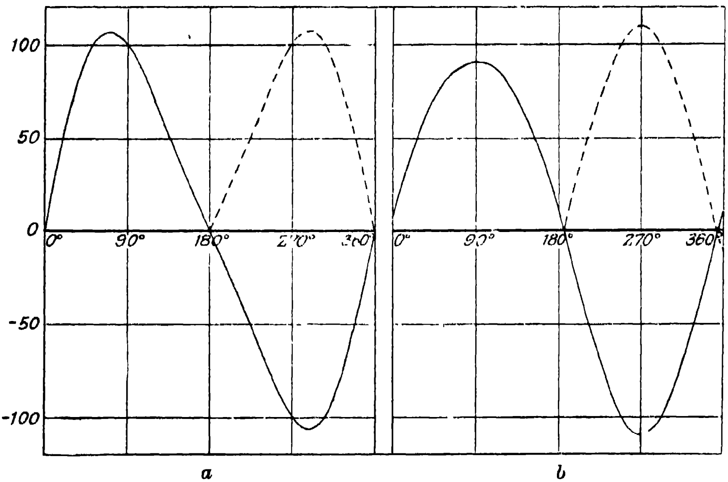


FIG. 194. WAVE-FORMS CONTAINING A FUNDAMENTAL AND A SECOND HARMONIC

(*a*) Second harmonic in phase with fundamental; amplitude 20 per cent of that of fundamental. (*b*) Second harmonic leading fundamental by 90°; amplitude 10 per cent of that of fundamental.

i.e. the shape of the curve when rising positively from a zero value differs from that when rising negatively from another zero value. For example, in Fig 194, which shows complex wave-forms containing a fundamental and a second harmonic, if the negative half-cycle be reversed in sign and plotted, as shown dotted, above the horizontal axis, the dissimilarity in the shape of the two half-waves is emphasized. Observe that the two half-waves may have dissimilar shapes, as shown in the wave-form (*b*)

The analytical proof of the asymmetry of a complex wave containing only even harmonics is as follows—

Let the ordinate at any abscissa, ωt, in the positive half-wave be given by

$$e_1 = E_{1m} \sin(\omega t + a_1) + E_{2m} \sin(2\omega t + a_2) + E_{4m} \sin(4\omega t + a_4) + \ldots$$

The corresponding ordinate in the negative half-wave is obtained by substituting $(\omega t + \pi)$ for ωt in the preceding equation, and is therefore given by

$$\begin{aligned} e_2 &= E_{1m}\sin(\omega t + \pi + a_1) + E_{2m}\sin[2(\omega t + \pi) + a_2] \\ &\qquad + E_{4m}\sin[4(\omega t + \pi) + a_4] + \ldots \\ &= -E_{1m}\sin(\omega t + a_1) + E_{2m}\sin(2\omega t + a_2) \\ &\qquad + E_{4m}\sin(4\omega t + a_4) + \ldots \\ &= -[E_{1m}\sin(\omega t + a_1) - E_{2m}\sin(2\omega t + a_2) \\ &\qquad - E_{4m}\sin(4\omega t + a_4) - \ldots] \end{aligned}$$

Hence the ordinate at abscissa $(\omega t + \pi)$ is not equal to the ordinate at abscissa ωt.

Shape of Complex Wave Containing only Odd Harmonics. A complex wave containing only odd harmonics is always symmetrical, the negative half-wave being an exact reproduction (with the reversed sign) of the positive half-wave. Examples are shown in Figs. 1, 3, . . .

The majority of the waves met with in alternating-current engineering are of this type, because of the symmetrical construction of the field magnets and the armature coils of alternating-current generators. Even harmonics, however, may also occur (in addition to the odd harmonics) under certain conditions of loading, and may also be produced when certain classes of apparatus (e.g. arc lamps, and electromagnetic apparatus working with an unsymmetrical magnetization curve or loop) are connected to the circuit.

The analytical proof of the symmetry of a complex wave containing only odd harmonics is as follows—

Let the ordinate at abscissa ωt in the positive half-wave be given by

$$\begin{aligned} e_1 &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots \end{aligned}$$

Then the corresponding ordinate in the negative half-wave is given by

$$\begin{aligned} e_2 &= E_{1m}\sin(\omega t + \pi + a_1) + E_{3m}\sin[3(\omega t + \pi) + a_3] \\ &\qquad + E_{5m}\sin[5(\omega t + \pi) + a_5] + \ldots \\ &= -E_{1m}\sin(\omega t + a_1) - E_{3m}\sin(3\omega t + a_3) \\ &\qquad - E_{5m}\sin(5\omega t + a_5) - \ldots \\ &= -[E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots] \\ &= -e_1. \end{aligned}$$

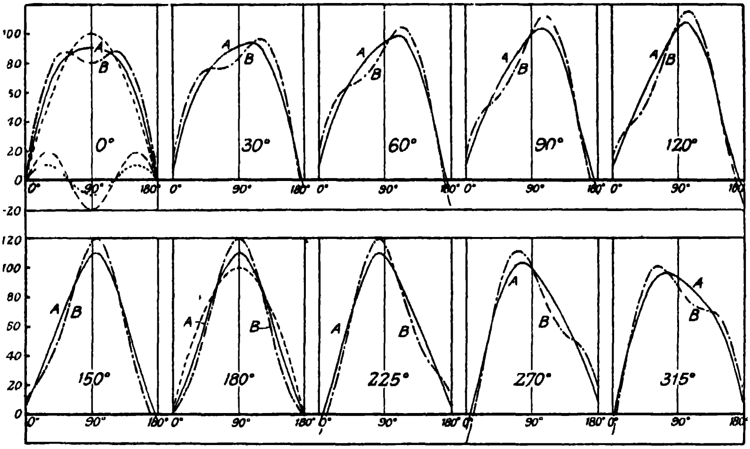


FIG. 195. WAVE-FORMS CONTAINING A FUNDAMENTAL AND A THIRD HARMONIC

A. Amplitude of third harmonic 10 per cent of that of fundamental.
B. Amplitude of third harmonic 20 per cent of that of fundamental.
Phase difference between fundamental and harmonic is indicated in diagrams.
The additional dotted curves in the first diagram show the fundamental and third harmonics separately.

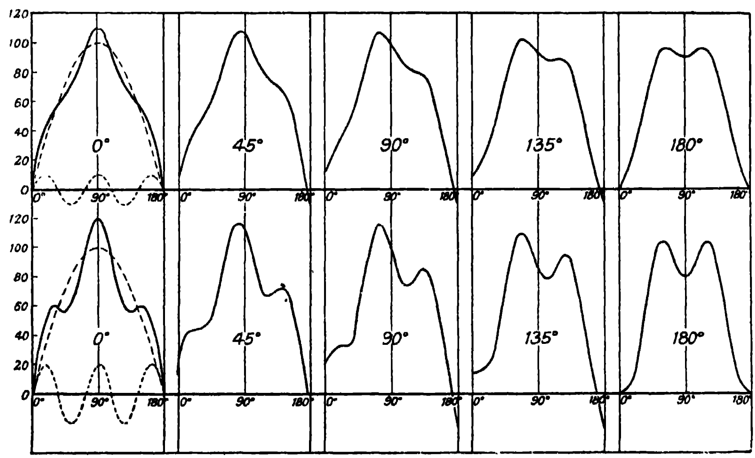


FIG. 196. WAVE-FORMS CONTAINING A FUNDAMENTAL AND A FIFTH HARMONIC

Amplitude of fifth harmonic is 10 per cent of that of fundamental for upper set of wave-forms and 20 per cent for lower set of wave-forms. Phase difference between fundamental and harmonic is indicated in the diagrams.

Effect of Phase Positions of Given Harmonic on Shape of Complex Wave. The deviation of a complete wave from a sine wave depends not only on the relative magnitude and order of the several harmonics, but also on their phase with respect to the complex wave. Fig. 195 illustrates the effect of superimposing a third harmonic of given amplitude, but of varying phase, on a given fundamental sine wave. Fig. 196 illustrates the effect produced

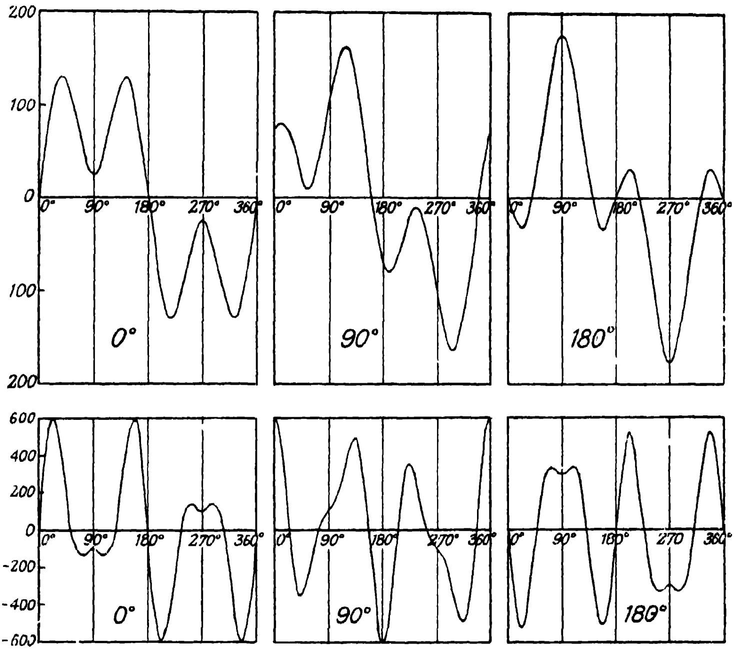


FIG. 197. WAVE-FORMS CONTAINING FUNDAMENTAL AND PRONOUNCED THIRD HARMONICS

Amplitude of third harmonic is 75 per cent of fundamental for upper set of wave-forms and 600 per cent for lower set of wave-forms. Phase difference between fundamental and harmonic is indicated in the diagrams.

under similar conditions by a fifth, and Fig. 197 illustrates the effect produced when the amplitude of the fundamental is small in comparison with the amplitudes of the harmonics.*

These illustrations show clearly that the combination of a fundamental with only one or two harmonics may produce a very great variety of wave-forms. In many cases the order of the harmonic

* Complex curves similar to Fig. 197 occur in alternators under sustained short circuit.

can be ascertained by inspection, but in general a systematic analysis, as discussed later (p. 310), is required to determine the character of the several harmonics. For the present it will be desirable to investigate the manner in which the harmonics affect the relationship between current and E.M.F. in the simpler types of circuits.

CURRENT WAVE-FORMS IN SINGLE-PHASE CIRCUITS SUPPLIED WITH NON-SINUSOIDAL E.M.F.

General. The current wave-form in a circuit, for which the constants (i.e. resistance, inductance, capacity) are invariable, is, in general, of different shape to that of the impressed E.M.F., and only in special cases are the two wave-forms similar. We have shown in Chapters III and IV that for circuits containing constant resistance, inductance, or capacity, the current due to a sinusoidal impressed E.M.F. is of the same frequency as the latter. Hence if a number of sinusoidal E.M.Fs. of different frequencies be impressed upon the circuit each E.M.F. will produce a current component of its own frequency, quite independently of the others. The instantaneous value of the current in the circuit is therefore the algebraic sum of the instantaneous currents due to the several E.M.Fs. This principle of superposition enables us to determine readily the current in such circuits when the equation to the impressed E.M.F. is known.

Relation Between Impressed E.M.F. and Current for Circuits Containing Resistance. Let the impressed E.M.F. be represented by

$$e = E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(\omega t + a_3) + E_{5m}\sin(\omega t + a_5).$$

Then, if R is the resistance of the circuit, the current (i_1) due to the fundamental (e_1) is given by

$$i_1 = \frac{e_1}{R} = \frac{E_{1m}}{R}\sin(\omega t + a_1);$$

that (i_3) due to the third harmonic (e_3) is given by

$$i_3 = \frac{e_3}{R} = \frac{E_{3m}}{R}\sin(3\omega t + a_3);$$

that and (i_5) due to the fifth harmonic (e_5) is given by

$$i_5 = \frac{e_5}{R} = \frac{E_{5m}}{R}\sin(5\omega t + a_5).$$

Hence the current (i) in the circuit is given by

$$\begin{aligned} i &= i_1 + i_2 + i_3 \\ &= \frac{E_{1m}}{R}\sin(\omega t + a_1) + \frac{E_{3m}}{R}\sin(3\omega t + a_3) \\ &\qquad + \frac{E_{5m}}{R}\sin(5\omega t + a_5) \qquad (101) \end{aligned}$$

Thus the wave-form of the current is similar to that of the impressed E.M.F.

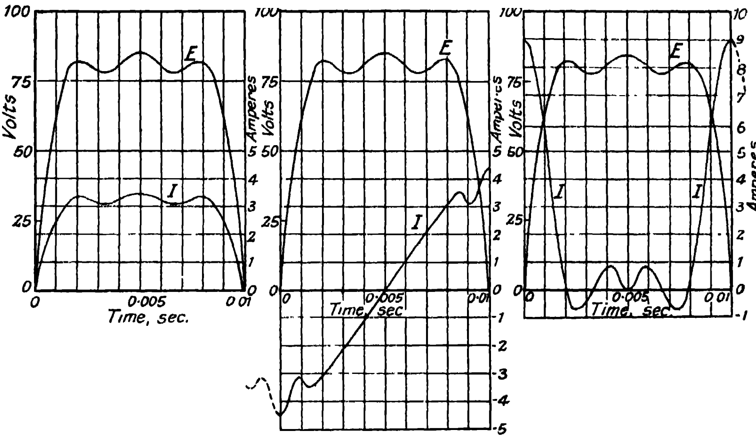


FIG. 198 FIG. 199 FIG. 200

WAVE-FORMS OF E.M.F. AND CURRENT FOR SIMPLE CIRCUITS CONTAINING RESISTANCE (FIG. 198), INDUCTANCE (FIG. 199), AND CAPACITANCE (FIG. 200)

Example. Let a non-inductive resistance of 25 ohms be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

The current in the circuit is given by

$$\begin{aligned} i &= \frac{100}{25}\sin 314t + \frac{25}{25}\sin 942t + \frac{10}{25}\sin 1570t \\ &= 4 \sin 314t + \sin 942t + 0{\cdot}4 \sin 1570t \end{aligned}$$

The current curve is shown in Fig. 198. The impressed E.M.F. curve is also shown and a comparison of the two will show that the wave-forms are of similar shape.

Relation Between Impressed E.M.F. and Current for Circuits Containing Inductance. Consider a purely inductive circuit of

inductance L and negligible resistance. Let the impressed E.M.F. be represented by the equation

$$e = E_{1m} \sin(\omega t + \alpha_1) + E_{3m} \sin(3\omega t + \alpha_3) + E_{5m} \sin(5\omega t + \alpha_5).$$

Then the current (i_1) due to the first harmonic (e_1) is

$$i_1 = \frac{e_1}{X_1} = \frac{E_{1m}}{\omega L} \sin\,(\omega t + \alpha_1 - \tfrac{1}{2}\pi)\,;$$

that (i_3) due to the third harmonic (e_3) is

$$i_3 = \frac{e_3}{X_3} = \frac{E_{3m}}{3\omega L} \sin\,(3\omega t + \alpha_3 - \tfrac{1}{2}\pi)\,;$$

and that (i_5) due to the fifth harmonic (e_5) is

$$i_5 = \frac{e_5}{X_5} = \frac{E_{5m}}{5\omega L} \sin(\omega t + \alpha_5 - \tfrac{1}{2}\pi),$$

where $X_1 (= \omega L)$, $X_3 (= 3\omega L)$, and $X_5 (= 5\omega L)$ are the reactances due to the first, third, and fifth harmonics respectively.

Hence the current (i) in the circuit is given by

$$\begin{aligned} i &= i_1 + i_3 + i_5 \\ &= \frac{E_{1m}}{\omega L} \sin(\omega t + \alpha_1 - \tfrac{1}{2}\pi) + \frac{E_{3m}}{3\omega L} \sin(3\omega t + \alpha_3 - \tfrac{1}{2}\pi) \\ &\qquad + \frac{E_{5m}}{5\omega L} \sin(5\omega t + \alpha_5 - \tfrac{1}{2}\pi) \qquad . \qquad . \qquad (102) \end{aligned}$$

Thus each component of the current has a phase difference of 90° (lagging) with respect to E.M.F. harmonic to which it is due, and, therefore, the wave-form of the current differs from that of the impressed E.M.F. It will be observed, however, that the reactance due to a given harmonic is directly proportional to the order of that harmonic; hence the current components due to the higher harmonics will be very much smaller than those in the case of a circuit containing only resistance. Accordingly, *in an inductive circuit supplied with a non-sinusoidal E.M.F. the current wave-form shows less distortion than the E.M.F. wave-form, and the current in such a circuit more nearly approaches a sine curve than does the current in a circuit containing resistance.*

Example. Let an inductive coil, of inductance 0·08 henry and negligible resistance, be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

Then the current (i) in the circuit is given by

$$i = \frac{100}{314 \times 0{\cdot}08} \sin(314t - \tfrac{1}{2}\pi) + \frac{25}{942 \times 0{\cdot}08} \sin(942t - \tfrac{1}{2}\pi) + \frac{10}{1570 \times 0{\cdot}08} \sin(1570t - \tfrac{1}{2}\pi)$$

$$= 4 \sin(314t - \tfrac{1}{2}\pi) + 0{\cdot}33 \sin(942t - \tfrac{1}{2}\pi) + 0{\cdot}08 \sin(1570t - \tfrac{1}{2}\pi)$$

The current curve is shown in Fig. 199. On comparing this with the current curve in Fig. 198 a marked difference in wave shape will be noted, the curve of Fig. 199 showing considerably less distortion than the current curve in Fig. 198.

Relation Between Impressed E.M.F. and Current for Series Circuits Containing Resistance and Inductance. Let the impressed E.M.F. be represented by the equation

$$e = E_{1m} \sin(\omega t + a_1) + E_{3m} \sin(3\omega t + a_3) + E_{5m} \sin(5\omega t + a_5)$$

Then the current (i) in the circuit is given by

$$i = \frac{E_{1m}}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t + a_1 - \varphi_1) + \frac{E_{3m}}{\sqrt{(R^2 + 9\omega^2 L^2)}} \sin(3\omega t + a_3 - \varphi_3) + \frac{E_{5m}}{\sqrt{(R^2 + 25\omega^2 L^2)}} \sin(5\omega t + a_5 - \varphi_5) \quad . \quad (103)$$

where $\varphi_1 (= \tan^{-1} \omega L/R)$, $\varphi_3 (= \tan^{-1} 3\omega L/R)$, $\varphi_5 (= \tan^{-1} 5\omega L/R)$ are the phase differences between the E.M.Fs. and currents due to the respective harmonics.

Observe that these phase differences have different magnitudes.

Relation Between Impressed E.M.F. and Current for Circuits Containing Capacitance. If an E.M.F. represented by the equation

$$e = E_{1m} \sin(\omega t + a_1) + E_{3m} \sin(3\omega t + a_3) + E_{5m} \sin(5\omega t + a_5)$$

be applied to a condenser of capacitance C farads, the charging current will be given by the equation

$$i = \omega C E_{1m} \sin(\omega t + a_1 + \tfrac{1}{2}\pi) + 3\omega C E_{3m} \sin(3\omega t + a_3 + \tfrac{1}{2}\pi) + 5\omega C E_{5m} \sin(5\omega t + a_5 + \tfrac{1}{2}\pi) \quad . \quad . \quad . \quad . \quad (104)$$

Therefore, in this case, the amplitudes of the currents due to the higher harmonics are increased and the current wave will show more distortion than the E.M.F. wave. Thus the effect of capacitance on wave distortion is exactly the reverse to that of inductance.

Example. Let a condenser having a capacitance of 127·5 microfarads be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

The charging current (i) in the condenser is given by

$$i = 314 \times 127{\cdot}5 \times 10^{-6} \times 100 \sin(314t + \tfrac{1}{2}\pi) + 942 \times 127{\cdot}5 \times 10^{-6} \times 25 \sin(942t + \tfrac{1}{2}\pi) + 1570 \times 127{\cdot}5 \times 10^{-6} \times 10 \sin(1570t + \tfrac{1}{2}\pi)$$

$$= 4 \sin(314t + \tfrac{1}{2}\pi) + 3 \sin(942t + \tfrac{1}{2}\pi) + 2 \sin(1570t + \tfrac{1}{2}\pi)$$

The current curve is shown in Fig. 200. This curve should be compared with the current curve in Fig. 199, as the capacitive reactance and the inductive reactance for these examples have been chosen so as to give the same maximum value of the fundamental in each case.

Relation Between Impressed E.M.F. and Current for Series Circuits Containing Resistance, Inductance, and Capacitance. In the general case of a series-connected circuit containing resistance, inductance, and capacitance, the current resulting from an impressed E.M.F. of complex wave-form (which is represented by the equation $e = E_{1m}\sin(\omega t + a_1) + \ldots$) is given by

$$i = \frac{E_{1m}}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\sin(\omega t + a_1 - \varphi_1)$$
$$+ \frac{E_{3m}}{\sqrt{[R^2 + (3\omega L - 1/3\omega C)^2]}}\sin(3\omega t + a_3 - \varphi_3)$$
$$+ \frac{E_{5m}}{\sqrt{[R^2 + (5\omega L - 1/5\omega C)^2]}}\sin(5\omega t + a_5 - \varphi_5) \quad . \qquad (104)$$

where

$$\varphi_1 = \tan^{-1}\left(\frac{\omega L}{R} - \frac{1}{\omega CR}\right); \quad \varphi_3 = \tan^{-1}\left(\frac{3\omega L}{R} - \frac{1}{3\omega CR}\right);$$
$$\varphi_5 = \tan^{-1}\left(\frac{5\omega L}{R} - \frac{1}{5\omega CR}\right)$$

are the phase differences between the E.M.Fs. and currents for the respective harmonics.

The amplitude (I_{nm}) of any harmonic, say the nth, is equal to

$$I_{nm} = \frac{E_{nm}}{\sqrt{[R^2 + (n\omega L - 1/n\omega C)^2]}}$$

and its phase difference (φ_n), with respect to the E.M.F. producing it, is

$$\varphi_n = \tan^{-1}\left(\frac{n\omega L}{R} - \frac{1}{n\omega CR}\right)$$

Resonance Due to Harmonics. When the angle (φ_n) is zero (i.e. when $n\omega L = 1/n\omega C$) resonance occurs with respect to this particular harmonic. Under resonance conditions considerable voltages may be produced at the terminals of the condenser and the inductive resistance, although the amplitude of the E.M.F. due to this harmonic may be relatively small. For example, if the E.M.F. wave of a 50-cycle alternator contains a 13th harmonic which has an amplitude equal to 1 per cent of the fundamental, and this alternator is connected to a series circuit containing an

inductive resistance ($R = 5$ O., $L = 0{\cdot}12$ H.) and a condenser ($C = 0{\cdot}5\mu$ F.), resonance occurs with the 13th harmonic (since $n\omega L = 13 \times 2\pi \times 50 \times 0{\cdot}12 = 490$, and $1/n\omega C = 10^6/(13 \times 2\pi \times 50 \times 0{\cdot}5) = 490$). The maximum value ($I_{13m}$) of the current due to this harmonic is given by

$$I_{13m} = \frac{E_{13m}}{R} = \frac{E_{1m}}{100R} = \frac{E_{1m}}{100 \times 5} = 0{\cdot}002E_{1m},$$

and the voltage across the terminals of the condenser due to this current is equal to

$$0{\cdot}002E_{1m} \times 10^{12}/(13 \times 2\pi \times 50 \times 0{\cdot}5)^2 = 0{\cdot}98E_{1m}.$$

Similarly the voltage at the terminals of the inductive resistance is equal to

$$0{\cdot}002E_{1m}\sqrt{[5^2 + (13 \times 2\pi \times 50 \times 0{\cdot}12)^2]} = 0{\cdot}98E_{1m}.$$

The actual voltages may be much higher and will depend on the relative amplitudes of the fundamental and other harmonics.

Pressure-rises due to these causes may occur in practice under certain conditions. For instance, when an alternator is connected to unloaded cables, the capacitance of the latter is in series with the inductance of the former, and the conditions may be favourable for obtaining resonance with a particular harmonic.

Experimental Method of Ascertaining Presence of Any Particular Harmonic in E.M.F. Wave. The property of resonance may be utilized to ascertain the order of the harmonics in a complex E.M.F. wave. For this purpose an oscillograph, a variable inductance, and a condenser of variable capacitance are required. The inductance, condenser, a variable non-inductive resistance, and a fixed non-inductive resistance, or shunt, for the oscillograph are connected in series, and the combination is connected to the source of E.M.F. to be tested ; the oscillograph being connected to show the wave-form of the voltage across the fixed non-inductive resistance. The inductance and capacitance are adjusted successively to values which will give resonance conditions for the 1st, 3rd, 5th, 7th, etc., harmonics, and a record of the wave-form is obtained by the oscillograph. From the shape of the latter the presence, or absence, of a given harmonic can be detected by inspection.

For example, if a particular harmonic is present, the wave-form will show this harmonic with a relatively large amplitude, as under resonance conditions any other harmonics will have only a very small amplitude. Thus if the 5th harmonic is present, the current wave-form under resonance conditions will show a wave of quintuple frequency superimposed upon a wave of fundamental frequency.

Fig. 201 shows a typical set of wave-forms obtained by this method. An examination of these shows that the E.M.F. wave-form contains 5th, 7th, 11th, 13th, 15th, and 17th harmonics.

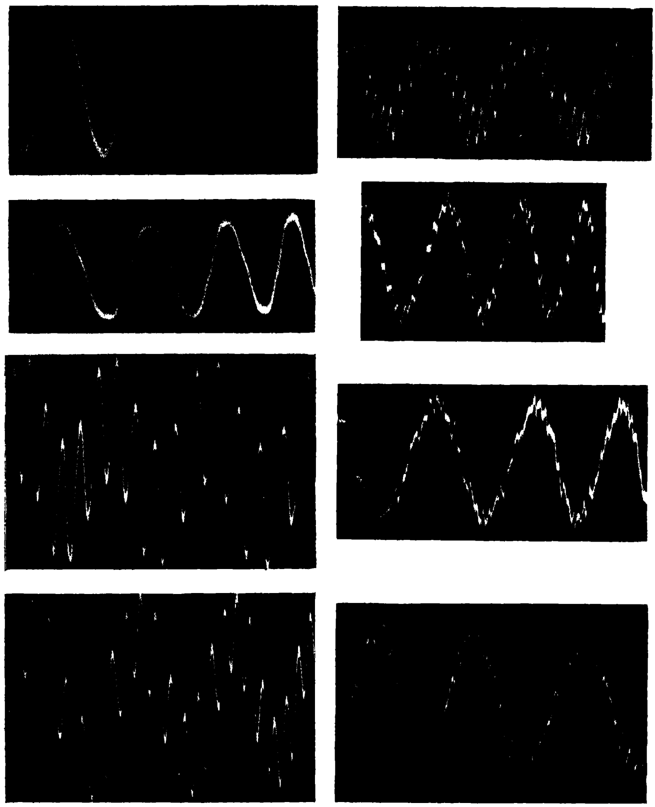


FIG. 201. OSCILLOGRAMS OF CURRENT AND VOLTAGE

Oscillograms showing (*a*) the E.M.F. wave-form of a given alternator, and the wave-forms of current (*b*) to *h* in a special resonant circuit supplied by this alternator, the constants of the circuits being adjusted to give, successively, resonance conditions for the 3rd (wave-form *b*), 5th (*c*), 7th (*d*), 11th (*e*), 13th (*f*), 15th (*g*), and 17th (*h*) harmonics.

NOTE. The following data give the values of inductance (L) and capacitance (C) in the resonant circuit for the several current wave-forms; also the values of the non-inductive shunt (R) to which the oscillograph vibrator was connected.

Wave-form . .	*b*	*c*	*d*	*e*	*f*	*g*	*h*
L (henry) . .	0·059	0·059	0·059	0·059	0·059	0·059	0·059
C (μ F.) . . .	18·9	6·9	3·52	1·4	1·0	0·76	0·59
R (ohms) . . .	0·5	0·833	2·22	5	15	20	35

R.M.S. Value of a Complex Wave. Let the wave be represented by $e = E_{1m}\sin(\omega t + \alpha_1) + E_{3m}\sin(3\omega t + \alpha_3) + E_{5m}\sin(\omega t + \alpha_5) + \ldots$. Then if the R.M.S. value of this wave be denoted by E, we have

$$E^2 = \frac{1}{\pi}\int_0^\pi \{E_{1m}\sin(\omega t + \alpha_1) + E_{3m}\sin(3\omega t + \alpha_3) + E_{5m}\sin(5\omega t + \alpha_5) + \ldots\}^2 d\omega t$$

$$= \frac{1}{\pi}\int_0^\pi \{E_{1m}^2\sin^2(\omega t + \alpha_1) + E_{3m}^2\sin^2(3\omega t + \alpha_3) + E_{5m}^2\sin^2(5\omega t + \alpha_5) + \ldots + 2E_{1m}E_{3m}\sin(\omega t + \alpha_1)\sin(3\omega t + \alpha_3) + 2E_{1m}E_{5m}\sin(\omega t + \alpha_1)\sin(5\omega t + \alpha_5) + \ldots\}d\omega t.$$

Now, $\int_0^\pi \sin^2(n\omega t + \alpha_n)d\omega t = \frac{1}{2}\pi$;

and $\int_0^\pi \sin(\omega t + \alpha_1)\sin(n\omega t + \alpha_n)d\omega t = 0$;

provided that n is an integer greater than unity.

Hence when the above integral is evaluated, all terms involving the product of quantities having different frequencies become zero, so that

$$E^2 = \frac{1}{\pi}\{(E_{1m}^2 \times \tfrac{1}{2}\pi) + (E_{3m}^2 \times \tfrac{1}{2}\pi) + (E_{5m}^2 \times \tfrac{1}{2}\pi) + \ldots\}$$

$$= \tfrac{1}{2}(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)$$

and $E = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}$ . . (105)

Examples. The R.M.S. values of the E.M.F. and current waves of Figs. 198–200 can now be obtained.

(1) *E.M.F. wave*, Fig. 198.

$$e = 100\sin 314t + 25\sin 942t + 10\sin 1570t$$

which may be written—

$$e = 100\sin\omega t + 25\sin 3\omega t + 10\sin 5\omega t,$$

where $\omega = 314$.

Hence,

$$E_{1m} = 100, \quad E_{3m} = 25, \quad E_{5m} = 10$$

Therefore,

$$E = 0{\cdot}707\sqrt{(100^2 + 25^2 + 10^2)} = 73{\cdot}2.$$

R.M.S. value of fundamental $(E_1) = 70{\cdot}7$.

(2) *Current wave*, Fig. 198.

$$i = 4\sin 314\omega t + \sin 925t + 0{\cdot}4\sin 1570t$$

Therefore,

$$I = 0{\cdot}707\sqrt{(4^2 + 1^2 + 0{\cdot}4^2)} = 2{\cdot}93.$$

R.M.S. value of fundamental $= 0{\cdot}707\sqrt{4^2} = 2{\cdot}83$.

(3) *Current wave*, Fig. 199.

$$i = 4\sin(314t - \tfrac{1}{2}\pi) + 0\cdot33\sin(942t - \tfrac{1}{2}\pi) + 0\cdot08\sin(1570t - \tfrac{1}{2}\pi)$$
$$I = 0\cdot707\sqrt{(4^2 + 0\cdot33^2 + 0\cdot08^2)} = 2\cdot84$$

R.M.S. value of fundamental $= 0\cdot707\sqrt{4^2} = 2\cdot83$.

(4) *Current wave*, Fig. 200.

$$i = 4\sin(314t + \tfrac{1}{2}\pi) + 3\sin(942t + \tfrac{1}{2}\pi) + 2\sin(1570t + \tfrac{1}{2}\pi)$$
$$I = 0\cdot707\sqrt{(4^2 + 3^2 + 2^2)} = 3\cdot8$$

R.M.S. value of fundamental $= 0\cdot707\sqrt{4^2} = 2\cdot83$.

(5) An electromotive force,

$$e = 2000\sin\omega t + 400\sin 3\omega t + 100\sin 5\omega t,$$

is connected to a circuit consisting of a resistance of 10 ohms, a variable inductance, and a condenser of 30μ F. capacitance, arranged in series with a hot-wire ammeter. Find the value of the inductance which will give resonance with the triple frequency component of the pressure, and estimate the readings on the ammeter and on a hot-wire voltmeter connected across the supply when resonance exists. $\omega = 300$. (*L.U.*)

For resonance to occur at triple frequency we must have

$$3\omega L = 1/3\omega C.$$

Substituting numerical values for ω and C, we obtain

$$L = 10^6/(9 \times 300^2 \times 30) = 0\cdot0411 \text{ H}.$$

The R.M.S. value of the current is given by

$$I = 0\cdot707\sqrt{\left[\left(\frac{E_{1m}}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\right)^2 + \left(\frac{E_{3m}}{\sqrt{[R^2 + (3\omega L - 1/3\omega C)^2]}}\right)^2 + \left(\frac{E_{5m}}{\sqrt{[R^2 + (5\omega L - 1/5\omega C)^2]}}\right)^2\right]}$$

where E_{1m}, E_{3m}, E_{5m}, denote the maximum values of the fundamental, triple, and quintuple frequency components of the pressure. Substituting numerical values for these and the other quantities, and noting that, for the given conditions, $3\omega L - 1/3\omega C = 0$, we obtain

$$I = 0\cdot707\sqrt{\left[\left(\frac{2000}{\sqrt{[10^2 + (300 \times 0\cdot0411 - 10^6/(300 \times 30))^2]}}\right)^2 + \left(\frac{400}{10}\right)^2 + \left(\frac{100}{\sqrt{[10^2 + (5 \times 300 \times 0\cdot0411 - 10^6/(5 \times 300 \times 30))^2]}}\right)^2\right]}$$
$$= 0\cdot707\sqrt{(20\cdot15^2 + 40^2 + 2\cdot45^2)}$$
$$= 31\cdot75 \text{ A}.$$

The R.M.S. value of the supply pressure is given by

$$E = 0\cdot707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2)} = 0\cdot707\sqrt{(2000^2 + 400^2 + 100^2)}$$
$$= 1445 \text{ V}.$$

Effect of Wave Distortion on Measurements of Inductance and Capacitance. Since alternating-current ammeters and voltmeters are calibrated to read R.M.S. values of current and pressure, respectively, the expression (105) deduced on p. 299 for the R.M.S. value of a complex wave enables the readings of the instruments connected in a circuit to be calculated when the "constants" of the circuit and the wave-forms of current and pressure are known. Conversely,

the "constants" of the circuit may be calculated from the instrument readings, but corrections have, in general, to be applied to take into account the shapes of the current and pressure waves. In the special case of circuits containing pure resistance, however, the wave-forms of current and impressed E.M.F. are similar, and therefore the ratio of the R.M.S. values of these quantities will be constant whether they have a sinusoidal or non-sinusoidal wave-form. Hence the resistance of a non-inductive alternating-current circuit may be determined by means of the ammeter and voltmeter method without a knowledge of the wave-form.* In all other cases a knowledge of the E.M.F. wave-form is necessary to obtain correct results. Thus, consider a purely inductive circuit having an inductance of L henries, and let E and I denote, respectively, the R.M.S. values of impressed E.M.F. and current as read on instruments connected in the circuit. Then

$$E = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}$$

$$I = 0{\cdot}707\sqrt{\left[\left(\frac{E_{1m}}{\omega L}\right)^2 + \left(\frac{E_{3m}}{3\omega L}\right)^2 + \left(\frac{E_{5m}}{5\omega L}\right)^2 + \ldots\right]}$$

$$= \frac{0{\cdot}707}{\omega L}\sqrt{(E_{1m}^2 + \tfrac{1}{9}E_{3m}^2 + \tfrac{1}{25}E_{5m}^2 + \ldots)}$$

Whence

$$L = \frac{0{\cdot}707}{\omega I}\sqrt{(E_{1m}^2 + \tfrac{1}{9}E_{3m}^2 + \tfrac{1}{25}E_{5m}^2 + \ldots)}$$

This expression involves a knowledge of the absolute values of the amplitudes of the several harmonics in the E.M.F. wave. In practice it is more convenient to deal with the relative values of these amplitudes, and accordingly the expression for L is modified as follows—

Multiply and divide the right-hand side by E, writing the denominator in the form $[0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}]$, thus

$$L = \left(\frac{0{\cdot}707}{\omega I}\sqrt{(E_{1m}^2 + \tfrac{1}{9}E_{3m}^2 + \tfrac{1}{25}E_{5m}^2 + \ldots)}\right)\left(\frac{E}{0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}}\right)$$

$$= \frac{E}{\omega I}\sqrt{\left(\frac{(E_{1m}^2 + \tfrac{1}{9}E_{3m}^2 + \tfrac{1}{25}E_{5m}^2 + \ldots)}{E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots}\right)},$$

* The value of the resistance obtained by this method may differ slightly from that obtained by a test using direct current, owing to possible eddy-currents in the conductors and the non-uniform distribution of current over their cross-section when the testing current is alternating.

$$= \frac{E}{\omega I}\sqrt{\left(\frac{1 + \frac{1}{9}(E_{3m}/E_{1m})^2 + \frac{1}{25}(E_{5m}/E_{1m})^2 + \ldots}{1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_{1m})^2 + \ldots)}\right)} \quad (106)$$

Now $E/\omega I$ is the apparent value of the inductance obtained from the uncorrected values of the instrument readings. Hence the quantity under the radical is the correction factor by which the apparent inductance must be multiplied to contain the true inductance.

Similarly, in the case of a circuit containing pure capacitance ($= C$ farads), let E and I denote the R.M.S. values of impressed E.M.F. and current as read on instruments connected in the circuit. Then

$$E = 0{\cdot}707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + \ldots)}$$

$$I = 0{\cdot}707\sqrt{[(\omega CE_{1m})^2 + (3\omega CE_{3m})^2 + (5\omega CE_{5m})^2 + \ldots}$$

$$= 0{\cdot}707\omega C\sqrt{(E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + \ldots)}$$

Whence

$$C = \frac{I}{0{\cdot}707\omega\sqrt{(E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + \ldots)}}$$

$$= \left(\frac{I}{0{\cdot}707\omega E\sqrt{(E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + \ldots}}\right)$$
$$(0{\cdot}707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + \ldots)}$$

$$= \frac{I}{\omega E}\sqrt{\left(\frac{E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + \ldots}{E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + \ldots}\right)}$$

$$= \frac{I}{\omega E}\sqrt{\left(\frac{1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_{1m})^2 + \ldots}{1 + 9(E_{3m}/E_{1m})^2 + 25(E_{5m}/E_{1m})^2 + \ldots}\right)} \quad (107)$$

Observe that the expression for the correction factor differs from that for the preceding case.

For example, if the E.M.F. wave contains 3rd and 5th harmonics, having amplitudes of 15 per cent and 10 per cent, respectively, of the fundamental, the correction factor for measurements of inductance is equal to

$$\sqrt{\left(\frac{1 + \frac{1}{9}(E_{3m}/E_{1m})^2 + \frac{1}{25}(E_{5m}/E_{1m})^2}{(1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_{1m})^2}\right)}$$

$$= \sqrt{\left(\frac{(1 + \frac{1}{9}(\frac{15}{100})^2 + \frac{1}{25}(\frac{10}{100})}{1 + (\frac{15}{100})^2 + (\frac{10}{100})^2}\right)}$$

$$= 0{\cdot}985$$

Hence the apparent value of the inductance, as obtained from the uncorrected readings of the instruments, is about $1\frac{1}{2}$ per cent

higher than the true value. Thus the error introduced by not taking into account the wave-form is relatively small.

Similarly, the correction factor for measurements of capacitance with this E.M.F. wave-form is equal to

$$\sqrt{\left(\frac{1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_{1m})^2}{1 + 9\,(E_{3m}/E_{1m})^2 + 25(E_{5m}/E_{1m})^2}\right)}$$
$$= \sqrt{\left(\frac{1 + (\frac{15}{100})^2 + (\frac{10}{100})^2}{1 + 9\,(\frac{15}{100})^2 + 25(\frac{10}{100})^2}\right)}$$
$$= 0{\cdot}844$$

Hence the apparent value of the capacitance, as obtained from the uncorrected values of the instrument readings, is 15·6 per cent higher than the true value of the capacitance. In this case the error introduced by not taking into account wave distortion is very large.

Example. A current of 50 frequency containing first, third, and fifth harmonics of crest values 100, 15, 12 amperes, respectively, is sent through an ammeter and an inductive coil of negligibly small losses. A voltmeter connected to the terminals shows 75 volts. What will be the current indicated on the ammeter, and what is the exact value of the inductance of the coil in henries ? (C. and G.)

The R.M.S. current (I) is given by

$$I = 0{\cdot}707\sqrt{(I_{1m}^2 + I_{3m}^2 + I_{5m}^2)}$$
$$= 0{\cdot}707\sqrt{(100^2 + 15^2 + 12^2)}$$
$$= 0{\cdot}707 \times 101{\cdot}9 = 72 \text{ A.}$$

Let L = inductance of coil in henries and let E_{1m}, E_{3m}, E_{5m}, represent the crest values of the first, third, and fifth harmonics of the E.M.F. wave. Then the R.M.S. value (75 volts) indicated by the voltmeter is given by

$$E = 75 = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2)}$$

But $I_{1m} = E_{1m}/\omega L$; $I_{3m} = E_{3m}/3\omega L$; $I_{5m} = E_{5m}/5\omega L$.

Hence
$$E = 0{\cdot}707\sqrt{[(I_{1m}\omega L)^2 + (3I_{3m}\omega L)^2 + (5I_{5m}\omega L)^2]}$$
$$= 0{\cdot}707\omega L\sqrt{(I_{1m}^2 + 9I_{3m}^2 + 25I_{5m}^2)}$$
$$= 0{\cdot}707\omega L I_{1m}\sqrt{[1 + 9(I_{3m}/I_{1m})^2 + 25(I_{5m}/I_{1m})^2]}$$

Substituting the numerical values given above, we have

$$75 = 0{\cdot}707 \times 2\pi \times 50 \times L \times 100\sqrt{[1 \times 9(0{\cdot}15)^2 + 25(0{\cdot}12)^2]}$$
$$\therefore \quad L = 0{\cdot}0027 \text{ H.}$$

[NOTE. Apparent value of inductance = 75/(314 × 72) = 0·00331 H.]

POWER IN SINGLE-PHASE CIRCUITS IN WHICH E.M.F. AND CURRENT ARE NON-SINUSOIDAL

Let the E.M.F. and current waves be represented by the equations

$$e = E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + \alpha_5) + \ldots$$

$$i = I_{1m}\sin(\omega t + a_1 - \varphi_1) + I_{3m}\sin(3\omega t + a_3 - \varphi_3) + I_{5m}\sin(5\omega t + a_5 - \varphi_5) + \ldots$$

Then the instantaneous power (p) is given by

$$\begin{aligned} p = ei = {} & E_{1m}I_{1m}\sin(\omega t + a_1)\sin(\omega t + a_1 - \varphi_1) \\ & + E_{3m}I_{3m}\sin(3\omega t + a_3)\sin(3\omega t + a_3 - \varphi_3) + \ldots \\ & + E_{1m}I_{3m}\sin(\omega t + a_1)\sin(3\omega t + a_3 - \varphi_3) + \ldots \\ & + E_{3m}I_{1m}\sin(3\omega t + a_3)\sin(\omega t + a_1 - \varphi_1) + \ldots \end{aligned}$$

and the mean power (P) is given by

$$\begin{aligned} P = \frac{1}{\pi}\int_0^\pi ei\, d\omega t = \frac{1}{\pi}\int_0^\pi \{ & E_{1m}I_{1m}\sin(\omega t + a_1)\sin(\omega t + a_1 - \varphi_1) \\ & + E_{3m}I_{3m}\sin(3\omega t + a_3)\sin(3\omega t + a_3 - \varphi_3) + \ldots \\ & + E_{1m}I_{3m}\sin(\omega t + a_1)\sin(3\omega t + a_3 - \varphi_3) + \ldots \\ & + E_{3m}I_{1m}\sin(3\omega t + a_3)\sin(\omega t + a_1 - \varphi_1) + \ldots \} d\omega t \end{aligned}$$

Now the integral, taken between 0 and π, of all products of different frequencies is zero, and the integral of terms such as

$$E_mI_m\sin(\omega t + a)\sin(\omega t + a - \varphi) = \tfrac{1}{2}\pi E_mI_m\cos\varphi.*$$

Hence the expression for the mean power reduces to

$$\begin{aligned} P &= \frac{1}{\pi}\left(\tfrac{1}{2}\pi E_{1m}I_{1m}\cos\varphi_1 + \tfrac{1}{2}\pi E_{3m}I_{3m}\cos\varphi_3 + \tfrac{1}{2}\pi E_{5m}I_{5m}\cos\varphi_5 + \ldots\right) \\ &= \tfrac{1}{2}(E_{1m}I_{1m}\cos\varphi_1 + E_{3m}I_{3m}\cos\varphi_3 + E_{5m}I_{5m}\cos\varphi_5 + \ldots) \\ &= E_1I_1\cos\varphi_1 + E_3I_3\cos\varphi_3 + E_5I_5\cos\varphi_5 + \ldots \qquad (108) \end{aligned}$$

Thus the mean power due to distorted current and E.M.F. waves is equal to the sum of the mean powers due to the several harmonic components.

Power Factor of Circuits in which Current and E.M.F. are Non-sinusoidal. In Chapter IV power factor was defined in two ways : (1) the ratio of the mean power to the apparent power, or volt-amperes ; (2) the cosine of the angle of phase difference between impressed E.M.F. and current. The power factor determined according to the first definition is a definite quantity for any particular circuit, whether the supply E.M.F. is sinusoidal or non-sinusoidal. The second definition of power factor, however, requires further consideration when the E.M.F. and current waves are non-sinusoidal. Thus, with sinusoidal current and E.M.F.

* Thus, $\int_0^\pi \sin(\omega t + a)\sin(\omega t + a - \varphi)d\omega t = \int_0^\pi [\sin(\omega t + a)\sin(\omega t + a)\cos\varphi - \cos(\omega t + a)\sin\varphi]d\omega t = \int_0^\pi \sin^2(\omega t + a)\cos\varphi\, d\omega t - \int_0^\pi (\sin(\omega t + a)\cos(\omega t + a)\sin\varphi)d\omega t = \tfrac{1}{2}\pi\cos\varphi.$

the phase difference, when the "constants" of the circuit are invariable, is constant at every instant, but with distorted waves the phase difference between the maximum values of E.M.F. and current is not necessarily the same as that between the zero values of these quantities, since the current and E.M.F. waves may be of different shapes. In these circumstances ambiguity may be avoided by employing the term "phase difference" (or angle of lag, or lead) only in connection with the *equivalent* sine waves of current and E.M.F. (i.e. the sine waves having the same frequency and R.M.S. values as the distorted waves*). Hence, if φ_e is the phase difference between these (equivalent sine) waves, then $\cos \varphi_e$ will represent the equivalent power factor of the circuit; i.e.

$$\cos \varphi_e = P/EI,$$

where P is the mean power (as measured by a wattmeter or calculated from equation (108) and E, I are the R.M.S. values of E.M.F. and current, respectively

Distorted E.M.F. and Current Waves in Polyphase Circuits

The method of treatment is in general similar to that given above for single phase circuits, viz., each harmonic is treated separately, and the current or voltage harmonics for the several "phases" of the circuit are compounded geometrically in the same manner as these quantities were compounded in the case of simple sine waves.

Expressions for "Phase" E.M.Fs. Considering the symmetrical three-phase system, let the "phase" E.M.Fs. be represented by the equations

$$\begin{aligned} e_{\text{I}} &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \dots \\ e_{\text{II}} &= E_{1m}\sin(\omega t + a_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + a_3 - 3 \times \tfrac{2}{3}\pi) \\ &\qquad + E_{5m}\sin(5\omega t + a_5 - 5 \times \tfrac{2}{3}\pi) + \dots \\ e_{\text{III}} &= E_{1m}\sin(\omega t + a_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + a_3 - 3 \times \tfrac{4}{3}\pi) \\ &\qquad + E_{5m}\sin(5\omega t + a_5 - 5 \times \tfrac{4}{3}\pi) + \dots \end{aligned}$$

These, upon simplification, become

$$\begin{aligned} e_{\text{I}} &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \dots \qquad (109) \end{aligned}$$

* See p. 16 for method of determining the equivalent sine wave.

$$e_{\text{II}} = E_{1m}\sin(\omega t + a_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5 + \tfrac{2}{3}\pi) + \text{.} \quad (110)$$

$$e_{\text{III}} = E_{1m}\sin(\omega t + a_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5 + \tfrac{4}{3}\pi) + \text{..} \quad (111)$$

$$= E_{1m}\sin(\omega t + a_1 + \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5 - \tfrac{2}{3}\pi) + \text{.} \quad (111a)$$

Thus all harmonics of triple frequency as well as their multiples (i.e. the 9th, 15th, 21st, etc.) are equal in all the phases of the circuit. Further, at any instant these E.M.Fs. have the same direction (i.e. in a star-connected system they are all directed either away from, or towards, the neutral point, and in a Δ-connected system they all act in the *same* direction around the circuit). Moreover, all harmonics which are not a multiple of three are displaced 120° from one another, and can be dealt with in the usual manner. It is to be observed, however, that these harmonics do not have all the same phase rotation as the fundamental; the phase rotation for the 5th, 11th, 17th, 23rd, 29th, etc., harmonics being opposite to that of the fundamental, and that for the 7th, 13th, 19th, 25th, 31st, etc., harmonics being the same as that of the fundamental.

Expressions for the "Line" E.M.Fs. in a Star-connected Three-phase System. Having obtained the equations to the "phase" E.M.Fs. the equations to the "line" E.M.Fs. readily follow, thus

$$\begin{aligned}
v_{\text{I II}} = e_{\text{I}} - e_{\text{II}} &= E_{1m}\{\sin(\omega t + a_1) - \sin(\omega t + a_1 - \tfrac{2}{3}\pi)\} \\
&\quad + E_{3m}\{\sin(3\omega t + a_3) - \sin(3\omega t + a_3)\} \\
&\quad + E_{5m}\{\sin(5\omega t + a_5) - \sin(5\omega t + a_5 - \tfrac{4}{3}\pi)\} + \ldots \\
&= 2E_{1m}\{\cos\tfrac{1}{2}[2(\omega t + a_1) - \tfrac{2}{3}\pi]\sin(\tfrac{1}{2} \times \tfrac{2}{3}\pi)\} \\
&\quad + 2E_{5m}\{\cos\tfrac{1}{2}[2(5\omega t + a_5) - \tfrac{4}{3}\pi]\sin(\tfrac{1}{2} \times \tfrac{4}{3}\pi)\} + \ldots \\
&= 2E_{1m}\{\tfrac{1}{2}\sqrt{3}\cos(\omega t + a_1 - \tfrac{1}{3}\pi)\} \\
&\quad + 2E_{5m}\{\tfrac{1}{2}\sqrt{3}\cos(5\omega t + a_5 - \tfrac{2}{3}\pi)\} + \ldots \\
&= \sqrt{3}\{E_{1m}\sin(\omega t + a_1 - \tfrac{1}{3}\pi + \tfrac{1}{2}\pi) \\
&\quad + E_{5m}\sin(5\omega t + a_5 - \tfrac{2}{3}\pi + \tfrac{1}{2}\pi) + \ldots\} \\
&= \sqrt{3}\{E_{1m}\sin(\omega t + a_1 + \tfrac{1}{6}\pi) + E_{5m}\sin(5\omega t \\
&\quad + a_5 - \tfrac{1}{6}\pi) + \ldots\} \quad \text{. . } (112)
\end{aligned}$$

$$\begin{aligned}
v_{\text{II-III}} = e_{\text{II}} - e_{\text{III}} &= \sqrt{3}\{E_{1m}\sin(\omega t + a_1 - \tfrac{1}{2}\pi) \\
&\quad + E_{5m}\sin(5\omega t + a_5 + \tfrac{1}{2}\pi) + \ldots\} \quad \text{. } (113)
\end{aligned}$$

$$v_{\text{III-I}} = e_{\text{III}} - e_{\text{I}} = \sqrt{3}\{E_{1m}\sin(\omega t + a_1 + \tfrac{5}{6}\pi) + E_{5m}\sin(5\omega t + a_5 - \tfrac{5}{6}\pi) + \ldots\} \qquad (114)$$

Thus in a star-connected symmetrical three-phase system no E.M.F. of triple frequency, or a multiple thereof, appears in the line pressures, notwithstanding that these E.M.Fs. may be present in the phase pressures.

If, in the above equations of line E.M.Fs. we replace $(\omega t + \frac{1}{6}\pi)$ by $\omega t'$, which means that time is now reckoned from an instant 30° in advance of the previous zero, we have

$$v_{\text{I-II}} = \sqrt{3}\{E_{1m}\sin(\omega t' - a_1) - E_{5m}\sin(5\omega t' - a_5) + \ldots\} \qquad (112a)$$

$$v_{\text{II-III}} = \sqrt{3}\{E_{1m}\sin(\omega t' - a_1 - \tfrac{2}{3}\pi) - E_{5m}\sin(5\omega t' - a_5 - \tfrac{4}{3}\pi) + \ldots\} \qquad (113a)$$

$$v_{\text{III-I}} = \sqrt{3}\{E_{1m}\sin(\omega t' - a_1 - \tfrac{4}{3}\pi) - E_{5m}\sin(5\omega t' - a_5 - \tfrac{2}{3}\pi) + \ldots\} \qquad (114a)$$

These equations are now similar to those for the phase E.M.Fs. except that (1) there are no third harmonic terms; (2) the signs of the fifth harmonics are changed*; and (3) the factor $\sqrt{3}$ is introduced.

R.M.S. Values of "Phase" and "Line" E.M.Fs. in a Three-phase System. In a symmetrical three-phase system the amplitude of any particular harmonic component of the "phase" E.M.F. is the same for each phase of the circuit, so that when considering R.M.S. values we may denote the R.M.S. value of the fundamental of the phase E.M.F. for each circuit by E_1, and the R.M.S. values of the several harmonics by E_3, E_5, E_7, etc.

Similarly, the R.M.S. value of the fundamental of the line E.M.F. may be denoted by V_1, and the R.M.S. values of the several harmonics by V_5, V_7, etc.

Then, from equations (112), (113), (114),

$$V_1 = \sqrt{3}\,E_1, \quad V_5 = \sqrt{3}\,E_5, \quad V_7 = \sqrt{3}\,E_7, \quad V_{11} = \sqrt{3}\,E_{11}, \quad V_{13} = \sqrt{3}\,E_{13}$$

The R.M.S. value of the line voltage (V) of the system is therefore given by

$$V = \sqrt{(V_1^2 + V_5^2 + V_7^2 + \ldots)}$$

Similarly the R.M.S. value of the "phase" E.M.F. is given by

$$E = \sqrt{(E_1^2 + E_3^2 + E_5^2 + E_7^2 + \ldots)}$$

* The seventh harmonics also have the minus sign.

Hence

$$\frac{V}{E} = \sqrt{\left(\frac{V_1^2 + V_5^2 + V_7^2 + \ldots}{E_1^2 + E_3^2 + E_7^2 + \ldots}\right)}$$

$$= \sqrt{3}\sqrt{\left(\frac{E_1^2 + E_5^2 + E_7^2 + \ldots}{E_1^2 + E_3^2 + E_5^2 + E_7^2 + \ldots}\right)}$$

$$= \sqrt{3}\sqrt{\left(\frac{1 + (E_5/E_1)^2 + (E_7/E_1)^2 + \ldots}{1 + (E_3/E_1)^2 + (E_5/E_1)^2 + (E_7/E_1)^2 + \ldots}\right)} \quad (155)$$

Thus in the case of distorted E.M.F. wave forms the value of the ratio of the "line" and "phase" E.M.Fs. is not the same as that ($\sqrt{3}$) when the E.M.Fs. are sinusoidal. An exception occurs, however, when the third harmonic and its multiples are not present in the phase E.M.Fs.

Example. If the phase E.M.F. of a star-connected, three-phase alternator contains first, third, fifth, seventh, and ninth harmonics of amplitudes 100, 13, 5, 1·5, 1, respectively, the ratio (line E.M.F./phase E.M.F.) is

$$\sqrt{3}\sqrt{\left(\frac{1 + (5/100)^2 + (1{\cdot}5/100)^2 + \ldots}{1 + (13/100)^2 + (5/100)^2 + (1{\cdot}5/100)^2 + (1/100)^2}\right)} = 1{\cdot}732 \times 0{\cdot}9915$$
$$= 1{\cdot}717.$$

If, however, the third and fifth harmonics are 30 per cent and 10 per cent, respectively, of the fundamental, the ratio (line E.M.F./phase E.M.F.) is

$$\sqrt{3}\sqrt{\left(\frac{1 + (10/100)^2 + (1{\cdot}5/100)^2 + \ldots}{1 + (30/100)^2 + (10/100)^2 + (1{\cdot}5/100)^2 + (1/100)^2}\right)}$$
$$= 1{\cdot}732 \times 0{\cdot}9583$$
$$= 1{\cdot}66.$$

In the former case, which is taken from actual practice, the ratio of the "line" and "phase" E.M.Fs. is 0·87 per cent less than the value for sinusoidal waves, while in the latter case, which represents excessive wave distortion, the ratio is 4·15 per cent less than the value for sinusoidal waves.

Circulating Current in Delta-connected Alternator in which E.M.F. Wave is Non-sinusoidal. Consider the case of an alternator in which the phase E.M.Fs. are symmetrical and represented by the equations

$$e_{\text{I}} = E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5) + \ldots$$

$$e_{\text{II}} = E_{1m}\sin(\omega t + a_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5 + \tfrac{2}{3}\pi) + \ldots$$

$$e_{\text{III}} = E_{1m}\sin(\omega t + a_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5 + \tfrac{4}{3}\pi) + \ldots$$

Then the resultant E.M.F. acting in the delta-connected armature winding is

$$e_{\text{I}} + e_{\text{II}} + e_{\text{III}} = 3E_{3m}\sin(3\omega t + a_3) + 3E_{9m}\sin(9\omega t + a_9) + 3E_{15m}\sin(15\omega t + a_{15}) + \ldots$$

Hence, if R is the resistance per phase and L the inductance per phase of the armature winding, the circulating current (i_c) due to the resultant E.M.F. is

$$i_c = \frac{3E_{3m}\sin(3\omega t + a_3)}{3\sqrt{(R^2 + 9\omega^2L^2)}} + \frac{3E_{9m}\sin(9\omega t + a_9)}{3\sqrt{(R^2 + 81\omega^2L^2)}} + \frac{3E_{15m}\sin(15\omega t + a_{15})}{3\sqrt{(R^2 + 225\omega^2L^2)}} + \ldots$$

$$= \frac{E_{3m}\sin(3\omega t + a_3)}{\sqrt{(R^2 + 9\omega^2L^2)}} + \frac{E_{9m}\sin(9\omega t + a_9)}{\sqrt{(R^2 + 81\omega^2L^2)}} + \frac{E_{15m}\sin(15\omega t + a_{15})}{\sqrt{(R^2 + 225\omega^2L^2)}} + \ldots$$

and its R.M.S. value is

$$I_c = 0{\cdot}707\sqrt{\,[E_{3m}{}^2/(R^2 + 9\omega^2L^2) + E_{9m}{}^2/(R^2 + 81\omega^2L^2) + E_{15m}{}^2/(R^2 + 225\omega^2L^2) + \ldots]}$$

Example. A three-phase 50-cycle alternator has a delta-connected armature winding for which the resistance and inductance per phase are 0·025 Ω. and 0·4 mH. respectively. The no-load E.M.F. wave-form contains third, ninth, and fifteenth harmonics (together with others) which have amplitudes of 4 per cent, 2 per cent, and 1·5 per cent of that of the fundamental. Calculate the circulating current at no-load when the excitation is such that the amplitude of the fundamental of the E.M.F. is 850 volts.

The reactance per phase (ωL) at 50 frequency

$$= 314 \times 0{\cdot}4 \times 10^{-3} = 0{\cdot}126\,\Omega.$$

Hence, for the 3rd, 9th, and 15th harmonics we have

$$9\omega^2L^2 = 0{\cdot}142\,\Omega.;\ 81\omega^2L^2 = 1{\cdot}28\,\Omega.;\ 225\omega^2L^2 = 3{\cdot}55\,\Omega.,$$

whence the values of the corresponding (impedance per phase)^2 are

$$R^2 + 9\omega^2L^2 = 0{\cdot}1426\,\Omega.;\ R^2 + 81\omega^2L^2 = 1{\cdot}2806\,\Omega.;$$
$$R^2 + 225\omega^2L^2 = 3{\cdot}55\,\Omega.$$

The maximum values, or amplitudes, of the E.M.Fs. due to the 3rd, 9th, and 15th harmonics are

$$E_{3m} = 0{\cdot}04 \times 850 = 34\text{ V.};\ E_{9m} = 0{\cdot}02 \times 850 = 17\text{ V.};$$
$$E_{15m} = 0{\cdot}015 \times 850 = 12{\cdot}7\text{ V.}$$

Hence $I_c = 0{\cdot}707\sqrt{\,[(34^2/0{\cdot}1426) + (17^2/1{\cdot}28) + (12{\cdot}7^2/3{\cdot}55)]} = 64{\cdot}8$ A.

Power in Three-phase Circuits in which the E.M.Fs. and Currents are Non-sinusoidal. The total power in a polyphase system in which the E.M.Fs. and currents are non-sinusoidal is equal to the algebraic sum of the power supplied by the several harmonic components of the current and E.M.F. Hence in the case of a three-phase, three-wire system with balanced loads we have

$$P = \sqrt{3}.V_1I_1\cos\varphi_1 + \sqrt{3}.V_5I_5\cos\varphi_5 + \sqrt{3}.V_7I_7\cos\varphi_7 + \ldots\ldots \qquad (156)$$

where V_1, V_5, V_7, ... I_1, I_5, I_7, ... denote the line voltages and currents, respectively, due to the several harmonics.

In the case of a four-wire system, the neutral wire provides a path for the circulation of currents of triple frequency and multiples thereof. The power is therefore given by

$$P = \sqrt{3}.V_1I_1\cos\varphi_1 + \sqrt{3}.V_3I_3\cos\varphi_3 + \sqrt{3}.V_5I_5\cos\varphi_5 + \sqrt{3}.V_7I_7\cos\varphi_7 + \quad \ldots \qquad (157)$$

which is greater than that in the corresponding three-wire system by the amount of the power due to the third harmonic and its multiples.

Harmonic Analysis

General. The process of determining the magnitude, order, and phase of the several harmonics in a complex periodic curve is called *harmonic analysis*. The analysis may be effected by analytical, graphical, or mechanical methods, the mechanical method involving the use of a special instrument called a " harmonic analyser."

All methods of analysis are based upon Fourier's theorem, which states that any single-valued complex periodic function can be resolved into a series of simple harmonic curves, the first of which has a frequency equal to that of the complex function. Thus, if y_θ denotes the magnitude of any ordinate at abscissa θ of the complex curve, then, generally,

$$y_\theta = F_0 + F_1\sin(\theta + a_1) + F_2\sin(2\theta + a_2) + F_3\sin(3\theta + a_3) + \ldots$$

where F_0, F_1, F_2, F_3, ... are constants, and a_1, a_2, a_3 ... are the phase differences of the harmonic components with respect to the complex wave. The constant F_0 is required only in cases where the mean value of the ordinates for a cycle of the complex wave is not zero. Hence if the reference axis is so drawn that the mean value of the ordinates for a cycle is zero, the equation to the curve is given by

$$y_\theta = F_1\sin(\theta + a_1) + F_2\sin(2\theta + a_2) + F_3\sin(3\theta + a_3) + \ldots \qquad (158)$$

and, when only odd harmonics are present, we have

$$y_\theta = F_1\sin(\theta + a_1) + F_3\sin(3\theta + a_3) + F_5\sin(5\theta + a_5) + \ldots \qquad (158a)$$

This equation is representative of the complex E.M.F. and current waves which are met with in alternating-current work.

The number of terms which may be necessary to express the function with absolute exactness depends upon the extent to which the complex curve deviates from a sine curve.

The number of terms to be included in the analysis depends on the degree of accuracy desired. For commercial purposes analysis as far as the fifth harmonic is generally sufficient, as the principal harmonics which occur in alternating-current systems are the third and the fifth. In special cases it may be desirable to carry out the analysis so as to include harmonics of the 17th, or higher, orders. These higher harmonics occur principally in generators and are due to magnetic pulsations; their amplitude is usually very small in comparison with that of the fundamental.

Analytical Methods of Analysis. With these methods the constants F_1, F_3, . . . , and the phase angles α_1, α_3, . . . , are obtained indirectly. Thus, since $\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$, each term of the series

$$y_\theta = F_1 \sin(\theta + \alpha_1) + F_3 \sin(3\theta + \alpha_3) + F_5 \sin(5\theta + \alpha_5) + \ldots$$

can be expressed as the sum of a sine and cosine term.

Effecting this transformation, we have—

$$\begin{aligned} y_\theta = {} & F_1 \sin\theta\cos\alpha_1 + F_3 \sin 3\theta \cos\alpha_3 + F_5 \sin 5\theta \cos\alpha_5 + \ldots \\ & + F_1 \cos\theta \sin\alpha_1 + F_3 \cos 3\theta \sin\alpha_3 + F_5 \cos 5\theta \sin\alpha_5 + \ldots \end{aligned} \quad (158b)$$

Substituting A_1 for $F_1 \cos\alpha_1$, A_3 for $F_3\cos\alpha_3$, etc.; B_1 for $F_1 \sin\alpha_1$, B_3 for $F_3 \sin\alpha_3$, etc., we have

$$\begin{aligned} y = {} & A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots \\ & + B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots \end{aligned} \quad (159)$$

Hence $A_1^2 + B_1^2 = F_1^2 \cos^2\alpha_1 + F_1^2 \sin^2\alpha_1 = F_1^2$,

Whence $F_1 = \sqrt{(A_1^2 + B_1^2)}$

Similarly $F_3 = \sqrt{(A_3^2 + B_3^2)}$; $F_5 = \sqrt{(A_5^2 + B_5^2)}$; etc.

Also $A_1/B_1 = F_1 \cos\alpha_1 / F_1 \sin\alpha_1 = \cot\alpha_1$

Whence $\alpha_1 = \tan^{-1} B_1/A_1$

Similarly $\alpha_3 = \tan^{-1} B_3/A_3$;

$\alpha_5 = \tan^{-1} B_5/A_5$; etc.

The coefficients A_1, A_3, A_5, . . . , B_1, B_3, B_5, . . . , are best evaluated by a summation, or integration, process, and the labour may be shortened considerably by suitably selecting the ordinates

and grouping the terms as shown later. The summation process is based upon the following theorems of the integral calculus—

$$\int_0^{\pi} \sin^2 \quad d\theta = \tfrac{1}{2}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

$$\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \begin{Bmatrix} 0 \text{ when } m \gtrless n \\ 0 \text{ when } m = 0, n = 0 \end{Bmatrix} . \quad . \quad (161)$$

$$\int_0^{\pi} \cos^2 \theta \, d\theta = \tfrac{1}{2}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

$$\int^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{Bmatrix} 0 \text{ when } m \gtrless n \\ \pi \text{ when } m = 0, n = 0 \end{Bmatrix} \quad . \quad (163)$$

$$\int_0^{2\pi} \sin m\theta \cos n\theta \, d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

where m and n are any positive integers.

For instance, to evaluate A_1 multiply equation (159) throughout by $\sin \theta$ and integrate between the limits 0 and π,* thus

$$\int_0^{\pi} (y_\theta \sin \theta) d\theta = \int_0^{\pi} \{A_1 \sin^2 \theta + A_3 \sin 3\theta \sin \theta$$
$$+ A_5 \sin 5\theta \sin \theta + \ldots$$
$$+ B_1 \cos \theta \sin \theta + B_3 \cos 3\theta \sin \theta$$
$$+ B_5 \cos 5\theta \sin \theta + \ldots\} d\theta$$
$$= \tfrac{1}{2}\pi A_1$$

Hence

$$A_1 = 2 \times \frac{1}{\pi} \int_0^{\pi} (y_\theta \sin \theta) \, d\theta$$
$$= 2 \times \text{mean value of } y_\theta \sin \theta \text{ for the half period.}$$

Similarly,

$$A_3 = 2 \times \text{mean value of } y_\theta \sin 3\theta \text{ for the half period.}$$

. . . . . .

$$B_1 = 2 \times \text{mean value of } y_\theta \cos \theta \text{ for the half period.}$$
$$B_3 = 2 \times \text{mean value of } y_\theta \cos 3\theta \text{ for the half period.}$$

. . . . . .

Method of Summation with Ungrouped Terms. The mean value of $y_\theta \sin \theta$ may be obtained simply by dividing the half period into a number of equal parts (the number of parts depending upon the accuracy desired), by means of ordinates erected at equidistant intervals, the first ordinate being erected at abscissa zero;

* With wave-forms in which the two half-waves are symmetrical, only one half of the wave need be considered in analysis.

multiplying each ordinate by the sine of the angle corresponding to the abscissa ; adding the products and finally dividing the sum by the number of intervals.

The mean value of the other products—$y \sin 3\theta$, $y \sin 5\theta$, etc.—is obtained in a similar manner, except that the ordinates are multiplied by $\sin 3\theta$, $\sin 5\theta$, etc., instead of $\sin \theta$.

For example, if the half period is divided into six equal parts by ordinates erected at abscissæ 0°, 30°, 60°, 90°, 120°, 150°, 180°, and the values of these ordinates are denoted by y_0, y_{30}, y_{60}, y_{90}, y_{120}, y_{150}, y_{180}, respectively, then

$$A_1 = 2\left\{\tfrac{1}{6}(y_{30}\sin 30° + y_{60}\sin 60° + y_{90}\sin 90° + y_{120}\sin 120° + y_{150}\sin 150° + y_{180}\sin 180°)\right\}$$

Similarly

$$A_3 = 2\left\{\tfrac{1}{6}(y_{30}\sin 90° + y_{60}\sin 180° + y_{90}\sin 270° + y_{120}\sin 360° + y_{150}\sin 450° + y_{180}\sin 540°)\right\}$$

. . . . . .

$$B_1 = 2\left\{\tfrac{1}{6}(y_{30}\cos 30° + y_{60}\cos 60° + y_{90}\cos 90° + y_{120}\cos 120° + y_{150}\cos 150° + y_{180}\cos 180°)\right\}$$

$$B_3 = 2\left\{\tfrac{1}{6}(y_{30}\cos 90° + y_{60}\cos 180° + y_{90}\cos 270 + y_{120}\cos 360° + y_{150}\cos 450° + y_{180}\cos 180°)\right\}$$

On the assumption that all harmonics present in the wave-form have been included in the analysis, a check upon the accuracy of the calculations may be obtained by the application of the following rules*—

(1) Ordinate at 90° $= y_{90} = A_1 - A_3 + A_5 - A_7 + \dots$

(2) Ordinate at 0° $= y_0 = B_1 + B_3 + B_5 + B_7 + \dots$

Example. Analysis of E.M.F. wave (Fig. 202) for odd harmonics up to, and including, the fifth.

The half-period is divided into twelve equal parts by ordinates erected at abscissæ 0°, 15°, 30°, etc., the values of the ordinates being as follows—

Abscissa	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Ordinate	0	24·5	51	84·2	126	171	191·5	180	152·3	119	82	41	0

The calculations are best carried out in tabular form and are given in Table VII.

* $y_{90} = A_1 \sin 90° + A_3 \sin(3 \times 90)° + A_5 \sin(5 \times 90)° \dots + B_1 \cos 90° + B_3 \cos(3 \times 90)° + B_5 \cos(5 \times 90)° + \dots$
$= A_1 - A_3 + A_5 - \dots$
$y_0 = A_1 \sin 0° + A_3 \sin 0° + A_5 \sin 0° + \dots + B_1 \cos 0° + B_3 \cos 0° + B_5 \cos 0° + \dots$
$= B_1 + B_3 + B_5 + \dots$

TABLE VII

Analysis of E.M.F. Curve Up to Fifth Harmonic

		Sine Terms						Cosine Terms					
θ	y_θ	$\sin\theta$	$y\sin\theta$	$\sin 3\theta$	$y\sin 3\theta$	$\sin 5\theta$	$y\sin 5\theta$	$\cos\theta$	$y\cos\theta$	$\cos 3\theta$	$y\cos 3\theta$	$\cos 5\theta$	$y\cos 5\theta$
15°	24·5	0·259	6·34	0·707	17·33	0·966	23·7	0·966	23·7	0·707	17·33	0·259	6·34
30°	51	0·5	25·5	1·0	51	0·5	25·5	0·866	44·1	0	0	−0·866	−44·1
45°	84·2	0·707	59·5	0·707	59·5	−0·707	−59·5	0·707	59·5	−0·707	−59·5	−0·707	−59·5
60°	126	0·866	109	0	0	−0·866	−109	0·5	63	−1·0	−126	0·5	63
75°	171	0·966	165·2	−0·707	−121	0·259	44·3	0·259	44·3	−0·707	−121	0·966	165·2
90°	191·5	1·0	191·5	−1·0	−191·5	1·0	191·5	0	0	0	0	0	0
105°	180	0·966	173·9	−0·707	−127·2	0·259	46·6	−0·259	−46·6	0·707	127·2	−0·966	−173·9
120°	152·3	0·866	132	0	0	−0·866	−132	−0·5	−76·1	1·0	152·3	−0·5	−76·1
135°	119	0·707	84·1	0·707	84·1	−0·707	−84·1	−0·707	−84·1	0·707	84·1	0·707	84·1
150°	82	0·5	41	1·0	82	0·5	41	−0·866	−71	0	0	0·866	71
165°	41	0·259	10·62	0·707	29	0·966	39·6	−0·966	−39·6	−0·707	−29	−0·259	−10·6
180°	0	0	0	0	0	0	0	−1·0	0	−1·0	0	−1·0	0
Total . .			998·7		−116·8		27·6		−82·8		45·4		25·4
$\frac{1}{6}$ × total .			(A_1) 166·4		(A_3) −19·5		(A_5) 4·6		(B_1) −13·8		(B_3) 7·6		(B_5) 4·2
Check .		$A_1 - A_3 + A_5 = y_{90}$; i.e. $166·4 + 19·5 + 4·6 = 190·5$*						$B_1 + B_3 + B_5 = 0$; i.e. $-13·8 + 7·6 + 4·2 = -2$*					

$F_1 = \sqrt{(166·4^2 + 13·8^2)} = 167$ $F_3 = \sqrt{(19·5^2 + 7·6^2)} = 20·9$ $F_5 = \sqrt{(4·6^2 + 4·2^2)} = 6·23$

$a_1 = \tan^{-1} - 13·8/166·4 = -4·8°$ $a_3 = \tan^{-1} 7·6/-19·5 = 158·7°$ $a_5 = \tan^{-1} 4·2/4·6 = 42·4°$

Equation to curve—$e = 167 \sin(\theta - 4·8°) + 20·9 \sin(3\theta + 158·7°) + 6·2 \sin(5\theta + 42·4°)$

* These discrepancies indicate that other harmonics are present in the wave-form, and that the present analysis only gives an approximation to the equation of the wave.

Summation Methods Employing Grouped Terms. With these methods the A and B coefficients are obtained by so choosing the number of ordinates that certain terms may be grouped together to enable the multiplication operations to be effected upon an assemblage of terms instead of upon individual terms. By these means the labour involved in the summation is shortened considerably, but in certain cases—chiefly those in which the analysis includes only harmonics of low orders, and the number of ordinates is relatively small—the accuracy is not equal to that of the longer methods in which a relatively large number of ordinates are employed. The accuracy of the shorter methods, however, is usually sufficient for commercial purposes.

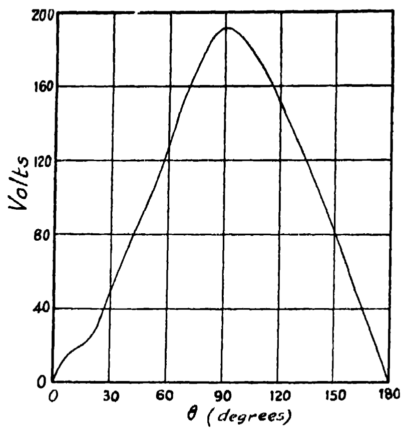


FIG. 202. WAVE-FORM OF E.M.F. ANALYSED IN TABLES VII AND X

In the method due to *Runge and Thompson* the number of intermediate ordinates in the half-period of the wave to be analysed is chosen equal to the order of the highest harmonic required, and these ordinates are treated in supplemental pairs, the sum and difference of each pair being determined. The sums contain only A, or sine, coefficients and the differences contain only B, or cosine coefficients.* The terms are then grouped so as to avoid repetition of the multiplication operations, special groupings being possible

* Thus,

$$\begin{aligned} y + y_{(\pi-\theta)} = {} & [A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots \\ & + B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots] \\ & + [A_1 \sin(\pi-\theta) \\ & + A_3 \sin 3(\pi-\theta) + A_5 \sin 5(\pi-\theta) + \ldots \\ & + B_1 \cos(\pi-\theta) + B_3 \cos 3(\pi-\theta) \\ & + B_5 \cos 5(\pi-\theta) + \ldots] \\ = {} & 2A_1 \sin\theta + 2A_3 \sin 3\theta + 2A_5 \sin 5\theta + \ldots \\ y_\theta - y_{(\pi-\theta)} = {} & [A_1 \sin\theta + A_3 \sin 3\theta + \ldots + B_1 \cos\theta + B_3 \cos 3\theta + \ldots \\ & - [A_1 \sin(\pi-\theta) + A_3 \sin 3(\pi-\theta) + \ldots \\ & + B_1 \cos(\pi-\theta) + B_3 \cos 3(\pi-\theta) + \ldots] \\ = {} & 2B \cos\theta + 2B_3 \cos 3\theta + \ldots \end{aligned}$$

for the third harmonic and its multiples. The manner in which the grouping is effected is best shown by considering a few typical cases.

1. Analysis of a periodic curve for odd harmonics up to, and including, the fifth. Ordinates are erected at abscissæ 30°, 60°, 90°, 120°, 150°. The sums and differences of the ordinates of supplemental angles are determined, the sums being denoted by a_1, a_2, a_3, . . . and the differences by d_1, d_2, d_3, ... This portion of the analysis is best carried out by arranging the quantities in horizontal rows, thus

Ordinates	y_{30} y_{150}	y_{60} y_{120}	y_{90}
Sums	$(y_{30}+y_{150})=a_1$	$(y_{60}+y_{120})=a_2$	$y_{90}=a_3$
Differences	$(y_{30}-y_{150})=d_1$	$(y_{60}-y_{120})=d_2$	—

The method of obtaining the grouping of the terms for the multiplication processes is best shown by carrying out the summation according to the previous method. Thus, from p. 313, we have

$$\begin{aligned} A_1 &= 2\left\{\tfrac{1}{6}(y_{30}\sin 30^\circ + y_{60}\sin 60^\circ + y_{90}\sin 90^\circ + y_{120}\sin 120^\circ \right.\\ &\qquad \left. + y_{150}\sin 150^\circ)\right\} \\ &= \tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 30^\circ + (y_{60}+y_{120})\sin 120^\circ \right.\\ &\qquad \left. + y_{90}\sin 90^\circ\right\} \\ &= \tfrac{1}{3}(a_1\sin 30^\circ + a_2\sin 60^\circ + a_3\sin 90^\circ) \end{aligned}$$

$$\begin{aligned} A_3 &= 2\left\{\tfrac{1}{6}[y_{30}\sin(3\times 30)^\circ + y_{60}\sin(3\times 60)^\circ + y_{90}(3\times 90)^\circ \right.\\ &\quad \left. + y_{120}\sin(3\times 120)^\circ + y_{150}\sin(3\times 150)^\circ]\right\} \\ &= \tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 90^\circ + (y_{60}+y_{120})\sin 180^\circ \right.\\ &\qquad \left. + y_{90}\sin 270^\circ\right\} \\ &= \tfrac{1}{3}(a_1\sin 90^\circ + a_2\sin 180^\circ + a_3\sin 270^\circ) \\ &= \tfrac{1}{3}(a_1 - a_3)\sin 90^\circ \\ &= \tfrac{1}{3}c_1, \end{aligned}$$

where $c_1 = a_1 - a_3$.

$$\begin{aligned} A_5 &= 2\left\{\tfrac{1}{6}[y_{30}\sin(5\times 30)^\circ + y_{60}\sin(5\times 60)^\circ \right.\\ &\quad + y_{90}\sin(5\times 90)^\circ + y_{120}\sin(5\times 120)^\circ \\ &\qquad \left. + y_{150}\sin(5\times 150)^\circ]\right\} \\ &= \tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 150^\circ + (y_{60}+y_{120})\sin 300^\circ \right.\\ &\qquad \left. + y_{90}\sin 450^\circ\right\} \\ &= \tfrac{1}{3}(a_1\sin 30^\circ - a_2\sin 60^\circ + a_3\sin 90^\circ) \end{aligned}$$

$$
\begin{aligned}
B_1 &= 2\left\{\tfrac{1}{6}(y_{30}\cos 30^\circ + y_{60}\cos 60^\circ + y_{90}\cos 90^\circ \right.\\
&\qquad\left. + y_{120}\cos 120^\circ + y_{150}\cos 150^\circ)\right\}\\
&= \tfrac{1}{3}\left\{(y_{30}-y_{150})\cos 30^\circ + (y_{60}-y_{120})\cos 60^\circ \right.\\
&\qquad\left. + y_{90}\cos 90^\circ\right\}\\
&= \tfrac{1}{3}(d_1\cos 30^\circ + d_2\cos 60^\circ)
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= 2\left\{\tfrac{1}{6}(y_{30}\cos(3\times 30)^\circ + y_{60}\cos(3\times 60)^\circ + y_{90}\cos(3\times 90)^\circ \right.\\
&\quad\left. + y_{120}\cos(3\times 120)^\circ + y_{150}\cos(3\times 150)^\circ\right\}\\
&= \tfrac{1}{3}\left\{(y_{30}-y_{150})\cos 90^\circ + (y_{60}-y_{120})\cos 180^\circ\right\}\\
&= \tfrac{1}{3}\,d_2\cos 180^\circ\\
&= -\tfrac{1}{3}\,d_2
\end{aligned}
$$

$$
\begin{aligned}
B_5 &= 2\left\{\tfrac{1}{6}(y_{30}\cos(5\times 30^\circ) + y_{60}\cos(5\times 60^\circ) + y_{90}\cos(5\times 90^\circ) \right.\\
&\quad\left. + y_{120}\cos(5\times 120^\circ) + y_{150}\cos(5\times 150^\circ)\right\}\\
&= \tfrac{1}{3}\left\{(y_{30}-y_{150})\cos 150^\circ + (y_{60}-y_{120})\cos 300^\circ\right\}\\
&= \tfrac{1}{3}(-d_1\cos 30^\circ + d_2\cos 60^\circ)
\end{aligned}
$$

Observe that the same products are involved in the coefficients of both the first and fifth harmonics. Hence, if the products are arranged alternately in two vertical columns (see Table VIII), one-third of the sum of the totals of the columns gives the coefficient for the first harmonic, or fundamental, and one-third of the difference of the totals of the columns gives the coefficient for the fifth harmonic. Observe also that terms a_1, a_3, may be grouped together and treated as a single term (c_1) in obtaining the products for the third harmonics.

II. Analysis of a periodic curve for odd harmonics up to, and including, the eleventh. In this case we can obtain special groupings for both third and ninth harmonics.

Thus, denoting the eleven equidistant intermediate ordinates (erected at abscissæ 15°, 30°, 45°, etc.) by y_{15}, y_{30°, etc., and taking the sums and differences of ordinates at supplemental angles, as above, we have

	y_{15°	y_{30°	y_{45°	y_{60°	y_{75°	y_{90°
	y_{165°	y_{150°	y_{135°	y_{120°	y_{105°	
Sum	a_1	a_2	a_3	a_4	a_5	a_6
Difference	d_1	d_2	d_3	d_4	d_5	—

TABLE VIII

Schedule for Analysis of Periodic Curve up to Fifth Harmonic

Order of Harmonic.			1st.	5th.	3rd.	1st.	5th.	3rd.
θ	$90° - \theta$	$\sin\theta = \cos(90° - \theta)$	Sine Terms.			Cosine Terms.		
30° 60° 90°	60° 30° 0°	0·5 0·866 1·0	$a_1 \sin 30°$ $a_3 \sin 90°$	$a_2 \sin 60°$	$c_1 \sin 90°$	$d_2 \cos 60°$	$d_1 \cos 30°$	$-d_2 \cos 0$
Total 1st column . . . . Total 2nd column								
$\frac{1}{3}$ × Sum of totals . . . . $\frac{1}{3}$ × Difference of totals . . .			A_1 A_5		A_3	B_1 B_5		B_3
Check			$A_1 - A_3 + A_5 = a_3$			$B_1 + B_3 + B_5 = 0$		

$F_1 = \sqrt{(A_1^2 + B_1^2)}$. $F_3 = \sqrt{(A_3^2 + B_3^2)}$. $F_5 = \sqrt{(A_5^2 + B_5^2)}$.

$a_1 = \tan^{-1} B_1/A_1$. $a_3 = \tan^{-1} B_3/A_3$. $a_5 = \tan^{-1} B_5/A_5$

Equation to curve—$y = F_1 \sin(\theta + a_1) + F_3 \sin(3\theta + a_3) + F_5 \sin(5\theta + a_5)$.

Now the coefficients for the third harmonics are given by

$$A_3 = \tfrac{1}{6}(a_1 \sin 45° + a_2 \sin 90° + a_3 \sin 135° + a_4 \sin 180° + a_5 \sin 225° + a_6 \sin 270°)$$

$$= \tfrac{1}{6}(a_1 \sin 45° + a_2 \sin 90° + a_3 \sin 45° + 0 - a_5 \sin 45° - a_6 \sin 90°)$$

$$= \tfrac{1}{6}\{(a_1 + a_3 - a_5) \sin 45° + (a_2 - a_6) \sin 90°\}$$

$$= \tfrac{1}{6}(c_1 \sin 45° + c_2 \sin 90°)$$

where $c_1 = (a_1 + a_3 - a_5)$, and $c_2 = (a_2 - a_6)$

$$B_3 = \tfrac{1}{6}(d_1 \cos 45° + d_2 \cos 90° + d_3 \cos 135° + d_4 \cos 180° + d_5 \cos 225° + d_6 \cos 270°)$$

$$= \tfrac{1}{6}(d_1 \cos 45° + 0 - d_3 \cos 45° - d_4 - d_5 \cos 45° + 0)$$

$$= \tfrac{1}{6}\{d_1 - d_3 - d_5) \cos 45° - d_4\}$$

$$= \tfrac{1}{6}(g_1 \cos 45° - d_4)$$

where $g_1 = (d_1 - d_3 - d_5)$

The coefficients for the ninth harmonic reduce to

$$A_9 = \tfrac{1}{6}\{(a_1 + a_3 - a_5) \sin 45° - (a_2 - a_6) \sin 90°\}$$

$$= \tfrac{1}{6}(c_1 \sin 45° - c_2 \sin 90°)$$

$$B_9 = \tfrac{1}{6}\{(-d_1 + d_3 + d_5) \cos 45° - d_4\}$$

$$= \tfrac{1}{3}(-g_1 \cos 45° - d_4)$$

Hence the coefficients for the third and ninth harmonics can be evaluated together.

Similarly, the coefficients for the first and eleventh harmonics can be evaluated together, as well as those for the fifth and seventh harmonics. Thus

$$A_1 = \tfrac{1}{6}(a_1 \sin 15° + a_2 \sin 30° + a_3 \sin 45° + a_4 \sin 60° + a_5 \sin 75° + a_6 \sin 90°)$$

$$A_{11} = \tfrac{1}{6}(a_1 \sin 15° - a_2 \sin 30° + a_3 \sin 45° - a_4 \sin 60° + a_5 \sin 75° - a_6 \sin 90°)$$

$$B_1 = \tfrac{1}{6}(d_1 \cos 15° + d_2 \cos 30° + d_3 \cos 45° + d_4 \cos 60° + d_5 \cos 75°)$$

$$B_{11} = \tfrac{1}{6}(-d_1 \cos 15° + d_2 \cos 30° - d_3 \cos 45° + d_4 \cos 60° - d_5 \cos 75°)$$

$$A_5 = \tfrac{1}{6}(a_1 \sin 75^\circ + a_2 \sin 30^\circ - a_3 \sin 45^\circ - a_4 \sin 60^\circ + a_5 \sin 15^\circ + a_6 \sin 90^\circ)$$

$$A_7 = \tfrac{1}{6}(a_1 \sin 75^\circ - a_2 \sin 30^\circ - a_3 \sin 45^\circ + a_4 \sin 60^\circ + a_5 \sin 15^\circ - a_6 \sin 90^\circ)$$

$$B_5 = \tfrac{1}{6}(d_1 \cos 75^\circ - d_2 \cos 30^\circ - d_3 \cos 45^\circ + d_4 \cos 60^\circ - d_5 \cos 15^\circ)$$

$$B_7 = \tfrac{1}{6}(-d_1 \cos 75^\circ - d_2 \cos 30^\circ + d_3 \cos 45^\circ + d_4 \cos 60^\circ + d_5 \cos 15^\circ)$$

The schedule is therefore arranged according to Table IX.

Example. Analysis of the E.M.F. curve given in Fig. 202 for odd harmonics up to, and including, the eleventh.

The values of the ordinates at abscissæ 15°, 30°, . . are as follow—

Abscissa	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
Ordinate	24·5	51	84·2	126	171	191·5	180	152·3	119	82	41

Taking sums and differences of ordinates of supplemental angles, we have

Ordinates	24·5	51	84·2	126	171	191·5
	41	82	119	152·3	180	
Sums	65·5($=a_1$)	133 ($=a_2$)	203·2($=a_3$)	278·3($=a_4$)	351($=a_5$)	191·5($=a_6$)
Differences	$-16·5$($=d_1$)	-31($=d_2$)	$-34·8$($=d_3$)	$-26·3$($=d_4$)	-9($=d_5$)	

Grouping for third harmonic—

$$c_1 = a_1 + a_3 - a_5 = 65{\cdot}5 + 203{\cdot}2 - 351 = -82{\cdot}3$$

$$c_2 = a_2 - a_6 = 133 - 191{\cdot}5 = -58{\cdot}5$$

$$g_1 = d_1 - d_3 - d_5 = -16{\cdot}5 + 34{\cdot}8 + 9 = 17{\cdot}3$$

The remaining calculations are given in Table X.

III. Analysis of a periodic curve for odd harmonics up to, and including, the seventeenth. In this case special groupings are obtained for the third, ninth, and fifteenth harmonics.

Seventeen equidistant intermediate ordinates are erected at abscissæ 10°, 20°, etc.

The sums and differences of ordinates at supplemental angles are obtained and tabulated thus—

	$y_{10°}$	$y_{20°}$	$y_{30°}$	$y_{40°}$	$y_{50°}$	$y_{60°}$	$y_{70°}$	$y_{80°}$	$y_{90°}$
	$y_{170°}$	$y_{160°}$	$y_{150°}$	$y_{140°}$	$y_{130°}$	$y_{120°}$	$y_{110°}$	$y_{100°}$	$y_{90°}$
Sums	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
Differences	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	

TABLE IX

Schedule for Analysis of Periodic Curve up to Eleventh Harmonic

Order of Harmonic.			Sine 1st.	Sine 11th.	Sine 3rd.	Sine 9th.	Sine 5th.	Sine 7th.	Cosine 1st.	Cosine 11th.	Cosine 3rd.	Cosine 9th.	Cosine 5th.	Cosine 7th.
θ	$90°-\theta$	$\sin\theta = \cos(90°-\theta)$	Sine Terms.						Cosine Terms.					
15°	75°	0·259	$a_1 \sin 15°$				$a_5 \sin 15°$			$d_5 \cos 75°$				$d_1 \cos 75°$
30°	60°	0·5		$a_2 \sin 30°$				$a_2 \sin 30°$	$d_4 \cos 60°$				$d_4 \cos 60°$	
45°	45°	0·707	$a_3 \sin 45°$		$c_1 \sin 45°$		$-a_3 \sin 45°$			$d_3 \cos 45°$		$g_1 \cos 45°$		$-d_3 \cos 45°$
60°	30°	0·866		$a_4 \sin 60°$				$-a_4 \sin 60°$	$d_2 \cos 30°$				$-d_2 \cos 30°$	
75°	15°	0·966	$a_5 \sin 75°$				$a_1 \sin 75°$			$d_1 \cos 15°$				$+d_5 \cos 15°$
90°	0	1·0		$a_6 \sin 90°$		$c_2 \sin 90°$		$a_6 \sin 90°$			$-d_4 \cos 0°$			
Total 1st column .														
Total 2nd column .														
Sum														
Difference . .														
½ × Sum . . .			A_1		A_3		A_5		B_1		B_3		B_5	
½ × Difference . .			A_{11}		A_9		A_7		B_{11}		B_9		B_7	
Check . . .			$A_1 - A_3 + A_5 - A_7 + A_9 - A_{11} = a_6$						$B_1 + B_3 + B_5 + B_7 + B_9 + B_{11} = 0$					

$F_1 = \sqrt{(A_1^2 + B_1^2)}$. $F_3 = \sqrt{(A_3^2 + B_3^2)}$. $F_5 = \sqrt{(A_5^2 + B_5^2)}$. $F_7 = \sqrt{(A_7^2 + B_7^2)}$. $F_9 = \sqrt{(A_9^2 + B_9^2)}$. $F_{11} = \sqrt{(A_{11}^2 + B_{11}^2)}$.

$\alpha_1 = \tan^{-1} B_1/A_1$. $\alpha_3 = \tan^{-1} B_3/A_3$. $\alpha_5 = \tan^{-1} B_5/A_5$. $\alpha_7 = \tan^{-1} B_7/A_7$. $\alpha_9 = \tan^{-1} B_9/A_9$. $\alpha_{11} = \tan^{-1} B_{11}/A_{11}$.

Equation to curve—$y = F_1 \sin(\theta + \alpha_1) + F_3 \sin(3\theta + \alpha_3) + F_5 \sin(5\theta + \alpha_5) + F_7 \sin(7\theta + \alpha_7) + F_9 \sin(9\theta + \alpha_9) + F_{11} \sin(11\theta + \alpha_{11})$.

TABLE X

Analysis of E.M.F. Curve up to Eleventh Harmonic

Order of Harmonic.			1st.	11th.	3rd.	9th.	5th.	7th.	1st.	11th.	3rd.	9th.	5th.	7th.
θ	$90^\circ-\theta$	$\sin\theta=\cos(90^\circ-\theta)$	Sine Terms.						Cosine Terms.					
15°	75°	0·259	17				91			−2·33				−4·27
30°	60°	0·5		66·5				66·5	−13·15				−13·15	
45°	45°	0·707	143·6		−58·2		−143·6			−24·6		12·23		24·6
60°	30°	0·866		241				−241	−26·9				26·9	
75°	15°	0·966	339				63·3			−15·94				8·7
90°	0°	1·0		191·5		−58·5		191·5			26·3			
Total 1st column			499·6		−58·2		10·7		−40		26·3		13·75	
Total 2nd column			499		−58·5		17		−42·9		12·23		29·03	
Sum			998·6		−116·7		27·7		−82·9		38·5		42·78	
Difference			0·6		0·3		−6·3		2·9		14·1		−15·28	
½ × Sum			166·5 ($=A_1$)		−19·45 ($=A_3$)		4·6 ($=A_5$)		−13·82 ($=B_1$)		6·4 ($=B_3$)		7·13 ($=B_5$)	
½ × Difference			0·1 ($=A_{11}$)		0·05 ($=A_9$)		−1·05 ($=A_7$)		0·48 ($=B_{11}$)		2·35 ($=B_9$)		−2.55 ($=B_7$)	
Check			$A_1-A_3+A_5-A_7+A_9-A_{11}=a_6$ i.e. $166·5+19·45+4·6+1·05+0·05-0·1=191·55$						$B_1+B_3+B_5+B_7+B_9+B_{11}=0$ i.e. $-13·82+6·4+7·13-2·55+2·35+0·48=0·01$					

$F_1=\sqrt{[166·5^2+(-13·82)^2]}=167.$ $\quad F_3=\sqrt{[(-19·45)^2+6·4^2]}=20·5.$ $\quad F_5=\sqrt{(4·6^2+7·13)}=8·9.$

$F_7=\sqrt{[(-1·05)^2+(-2·55)^2]}=2·76.$ $\quad F_9=\sqrt{(0·05^2+2·35)^2}=2·35.$ $\quad F_{11}=\sqrt{(0·1^2+0·49)}=0·49.$

$a_1=\tan^{-1}-13·82/166·5=-4·8^\circ.$ $\quad a_3=\tan^{-1}6·4/-19·45=161·8^\circ.$ $\quad a_5=\tan^{-1}7·13/4·6=57·2^\circ.$

$a_7=\tan^{-1}-2·55/-1·05=247·6^\circ.$ $\quad a_9=\tan^{-1}2·35/0·05=88·8^\circ.$ $\quad a_{11}=\tan^{-1}0·48/0·1=78·2^\circ.$

Equation to curve—$e=167\sin(\theta-4·8^\circ)+20·5\sin(3\theta+161·8^\circ)+8·9\sin(5\theta+57·2^\circ)-2·76\sin(7\theta+67·6^\circ)$
$+2·35\sin(9\theta+88·8^\circ)+0·49\sin(11\theta+78·2^\circ).$

The coefficients for the third, ninth, and fifteenth harmonics reduce to the following expressions—

	15th.	5th.	13th.	7th.	11th.	9th.
	Cosine Terms.					
1(		$-d_3 \cos 80°$		$d_4 \cos 80°$		
2(			$-d_5 \cos 70°$		$d_1 \cos 70°$	
3(0°		$d_6 \cos 60°$		$d_6 \cos 60°$		
4(			$d_1 \cos 50°$		$-d_7 \cos 50°$	
5(		$d_8 \cos 40°$		$-d_2 \cos 40°$		
6(	$g_1 \cos 30°$		$-d_8 \cos 30°$		$-d_2 \cos 30°$	
7(		$-d_4 \cos 20°$		$-d_8 \cos 20°$		
8(			$d_7 \cos 10°$		$d_5 \cos 10°$	
9(0°						$g_3 \cos 0°$
	B_3 B_{15}	B_5 B_{13}		B_7 B_{11}		B_9

$+ B_9 + B_{11} + B_{13} + B_{15} + B_{17} = 0$

$= \sqrt{(A_{13}^2 + B_{13}^2)}$. $F_{15} = \sqrt{(A_{15}^2 + B_{15}^2)}$. $F_{17} = \sqrt{(A_{17}^2 + B_{17}^2)}$

$= \tan.^{1} B_{13}/A_{13}$. $a_{15} = \tan^{-1} B_{15}/A_{15}$. $a_{17} = \tan^{-1} B_{17}/A_{17}$

$3\theta + a_{13}) + F_{15} \sin(15\theta + a_{15}) + F_{17} \sin(17\theta + a_{17})$

(T

preparation of schedules for these cases are effected by methods similar to those already given.

The coefficients for the third, ninth, and fifteenth harmonics reduce to the following expressions—

$$A_3 = \tfrac{1}{9}\{(a_1 + a_5 - a_7)\sin 30^\circ + (a_2 + a_4 - a_8)\sin 60^\circ + (a_3 - a_9)\sin 90^\circ\}$$

$$= \tfrac{1}{9}(c_1 \sin 30^\circ + c_2 \sin 60^\circ + c_3 \sin 90^\circ)$$

where $c_1 = (a_1 + a_5 - a_7)$, $c_2 = (a_2 + a_4 - a_8)$ and $c_3 = (a_3 - a_9)$

$$A_9 = \tfrac{1}{9}\{(a_1 - a_3 + a_5 - a_7 + a_9)\sin 90^\circ\} = \tfrac{1}{9}c_4 \sin 90^\circ$$

where $c_4 = (a_1 - a_3 + a_5 - a_7 + a_9)$

$$A_{15} = \tfrac{1}{9}\{(a_1 + a_5 - a_7)\sin 30^\circ - (a_2 + a_4 - a_8)\sin 60^\circ + (a_3 - a_9)\sin 90^\circ\}$$

$$= \tfrac{1}{9}(c_1 \sin 30^\circ - c_2 \sin 60^\circ + c_3 \sin 90^\circ)$$

$$B_3 = \tfrac{1}{9}\{(d_1 - d_5 - d_7)\cos 30^\circ + (d_2 - d_4 - d_8)\cos 60^\circ - d_6\}$$

$$= \tfrac{1}{9}(g_1 \cos 30^\circ + g_2 \cos 60^\circ - d_6)$$

where $g_1 = (d_1 - d_5 - d_7)$, and $g_2 = (d_2 - d_4 - d_8)$

$$B_9 = \tfrac{1}{9}(-d_2 + d_4 - d_6 + d_8) = \tfrac{1}{9}g_3$$

$$B_{15} = \tfrac{1}{9}\{(-d_1 + d_5 + d_7)\cos 30^\circ + (d_2 - d_4 - d_8)\cos 60^\circ - d_6\}$$

$$= \tfrac{1}{9}(g_2 \cos 60^\circ - d_6 - g_1 \cos 30^\circ)$$

Hence the groupings for these harmonics are

	For Sine terms.	For Cosine terms.
Third and fifteenth harmonics	$a_1 + a_5 - a_7 = c_1$ $a_2 + a_4 - a_8 = c_2$ $a_3 - a_9$	$d_1 - d_5 - d_7 = g_1$ $d_2 - d_4 - d_8 = g_2$
Ninth harmonic	$a_1 - a_3 + a_5 - a_7 + a_9 = c_4$	$-d_2 + d_4 - d_6 + d_8 = g_3$

The schedule is arranged according to Table XI.

IV. Analyses for harmonics higher than the seventeenth. If an analysis is to include the 23rd harmonic, special groupings can be obtained for the 3rd, 9th, 15th, and 21st harmonics; while, if the 29th harmonic is to be included, special groupings can be obtained for the 3rd, 5th, 9th, 15th, 21st, 25th, and 27th harmonics.

In general, as the analysis is extended to include still higher harmonics, the number of harmonics for which special groupings of the terms may be obtained, increases.

The determination of the "grouping coefficients" (c, g) and the preparation of schedules for these cases are effected by methods similar to those already given.

CHAPTER XV

MAGNETIZATION OF IRON BY ALTERNATING CURRENTS

Effects Due to Alternating Magnetization. The magnetization of iron by alternating current produces phenomena which are absent when the iron is magnetized by direct current. In the latter case, assuming a closed magnetic circuit, a given steady current in the magnetizing coil results in a definite flux in the iron, and hysteresis manifests itself only when changes occur in the magnetizing current, the resulting flux due to a given current being then dependent upon whether this current is greater or less than the previous current.

With a steady magnetizing current (i), the impressed E.M.F. (v) at the terminals of the magnetizing coil is given by $v = iR$, where R is the resistance of the coil. But, with a varying current, the impressed E.M.F. contains a component which balances the E.M.F. induced in the coil by the variations of the flux. Moreover, the changes in the flux also induce E.M.Fs. in the iron core and set up *eddy currents* therein. The eddy currents cause a loss of energy—which is expended in heating the core—and produce a magnetic reaction tending to damp out the variations of the flux. An additional loss of energy, due to hysteresis, occurs in the iron core during each cycle of magnetization, and is expended in heating the core. Further, both hysteresis and eddy currents cause a phase displacement between the magnetizing current and the flux. Hence with alternating magnetization the current (maximum value) required to produce a given flux (maximum value) is larger than that required when the flux is non-alternating.

Therefore when iron is magnetized by alternating current, both the current in the coil and the potential difference at its terminals will have higher maximum values, for a given maximum value of the flux, than those when the magnetization is produced by steady current. Also, due to the varying permeability of the iron, and hysteresis the wave-form of the current is distorted and is no longer of similar shape to that of the impressed E.M.F.

[NOTE. The distortion becomes greater as the magnetic saturation increases ; it is affected similarly by the hysteresis loss.]

The wave-form of the current may be easily obtained when the hysteresis loop of the iron is available and the wave-form of the

flux is known; the method is a graphical one and is described later (p. 326).

Flux Wave-form. The wave-form of the flux can be obtained when that of the impressed E.M.F. is known, provided that the voltage drop due to the resistance of the magnetizing coil is negligible. Thus, if v denotes the impressed E.M.F., i the current, R the resistance of the magnetizing coil, and e the E.M.F. induced in the latter due to the variations of the flux, we have

$$v = -(e + iR)$$

and, when iR is negligible, $v = -e$.

Now $e = -(N \times 10^{-8})d\Phi/dt$, where N denotes the number of turns in the magnetizing coil, and Φ denotes the value of the flux at the instant t. Hence

$$\Phi = -\frac{10^8}{N}\int e \, . \, dt = \frac{10^8}{N}\int v \, . \, dt \quad . \quad . \quad . \quad . \quad (165)$$

Thus the curve obtained by the integration of the wave-form of the impressed E.M.F. represents the wave-form of the flux.

If the impressed E.M.F. is sinusoidal, i.e. $v = V_m \sin \omega t$,

$$\Phi = \frac{V_m \times 10^8}{N}\int \sin \omega t \, . \, dt = -\frac{V_m \times 10^8}{N\omega}\cos \omega t$$

$$= \frac{V_m \times 10^8}{N\omega}\sin(\omega t - \tfrac{1}{2}\pi) \quad (166)$$

Thus the flux is sinusoidal and lags 90° with respect to the impressed E.M.F.

The maximum value (Φ_m) of the flux is given by

$$\Phi_m = V_m \times 10^8/N\omega = \sqrt{2} \, . \, V \times 10^8/2\pi f N$$
$$= V \times 10^8/4{\cdot}44 f N \quad . \quad . \quad . \quad . \quad . \quad (167)$$

where V denotes the R.M.S. value of the impressed E.M.F., and f the frequency.

If the impressed E.M.F. is non-sinusoidal, and only odd harmonics are present, let V_{av} denote the mean value of the E.M.F. during the half-period ($= \frac{1}{2}T$ seconds) between two successive zero values. Then

$$V_{av} = \int_0^{\frac{1}{2}T} \frac{v \, . \, dt}{\frac{1}{2}T}$$

and

$$\int_0^{\frac{1}{2}T} v \, . \, dt = \tfrac{1}{2}T V_{av}$$

Now $\int_0^{\frac{1}{2}T} v\,.\,dt$ represents, under the conditions stated, the maximum value of the indefinite integral $\int v\,.\,dt$, and therefore if the value $(\frac{1}{2}TV_{av})$ of the definite integral is substituted in equation (165), the resulting expression $[(\frac{1}{2}TV_{av})10^8/N]$ represents the maximum change $(\Delta\Phi)_m$ in the flux during a half-period, i.e. $(\Delta\Phi)_m = (\frac{1}{2}TV_{av})10^8/N$. Since the E.M.F. wave-form is assumed to contain only odd harmonics, then only odd harmonics are present in the flux wave-form, and the positive and negative half-waves are identical in shape. Hence, if Φ_m denotes the maximum value, or amplitude, of the flux wave, then the maximum change in the flux during a half-period is equal to twice the amplitude and is given by $(\Delta\Phi)_m = 2\Phi_m$.

Substituting the above value for $(\Delta\Phi)_m$, we have

$$2\Phi_m = (\tfrac{1}{2}TV_{av})10^8/N$$

Whence

$$\Phi_m = TV_{av} \times 10^8/4N$$

$$= 10^8 V_{av}/4fN \qquad \text{. . . . . (167}a\text{)}$$

since $T = 1/f$.

Hence for a given frequency (f) and number of turns (N), the maximum value of the flux depends only upon the *mean value* of the impressed E.M.F. Conversely, the mean E.M.F. induced in a coil by an alternating flux depends only upon the maximum value of the flux—the frequency and number of turns being constant—and is unaffected by the shape of the wave-form of the flux.

Denoting the form factor—i.e. the ratio (R.M.S. value/mean value)—of the E.M.F. wave by k_f, we have

$$V_{av} = V/k_f$$

where V is the R.M.S. value of the impressed E.M.F

Hence

$$\Phi_m = V \times 10^8/4k_f fN \qquad \text{. . . . . (167}b\text{)}$$

If the E.M.F. in sinusoidal, $k_f = 1{\cdot}11$, and the expression (167b) reduces to (167), i.e.

$$\Phi_m = V \times 10^8/4{\cdot}44fN$$

Wave-form of the Magnetizing Current. The wave-form of the magnetizing current is deduced as follows—

Assuming the hysteresis loop and the wave-form of the flux, or flux density, to have been plotted in rectangular co-ordinates, select a convenient number of points, say twelve, on the flux curve and determine the magnetizing current for each point from the corresponding points on the hysteresis loop (Fig. 203). Plot

the currents so obtained to the same abscissæ as the flux curve, and draw a curve through the points. Thus, in Fig. 203, the flux-density curve, B, has been plotted to the same ordinate scale as the hysteresis loop (which has been plotted in terms of flux density, B, and magnetizing current, I). By arranging these curves side by side, the selected points, 1–12, on the flux curve are projected across to the hysteresis loop, and the corresponding values of currents are marked off as points 1′–12′ on the respective ordinates of the flux curve.

A similar method can be used to determine the wave-form of

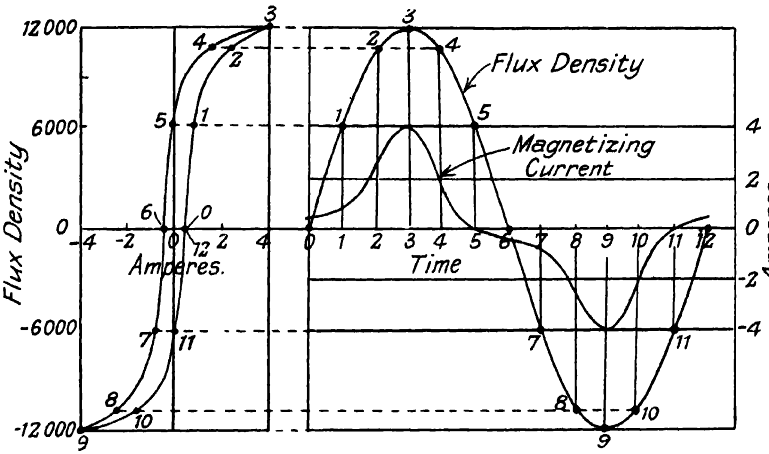


FIG. 203. METHOD OF DETERMINING WAVE-FORM OF MAGNETIZING CURRENT

the flux corresponding to a given wave-form of current. Fig. 204 gives the wave-form of the flux when the magnetizing current follows a sine law and the hysteresis loop is the same as that given in Fig. 203.

Wave-form of Induced E.M.F. When the shape of the flux curve is known, the wave-form of the induced E.M.F. may be readily obtained since each point on the E.M.F. wave is proportional to the differential coefficient of the corresponding point on the flux wave.

[NOTE. The differential coefficient of any point on the flux wave may be determined either graphically, by drawing the tangent at that point and calculating its value, or analytically, by calculating the quantity $\Delta\Phi/\Delta t$.]

Referring to Fig. 203 we observe that when the flux is sinusoidal

the magnetizing current wave-form is distorted. Moreover, although the maximum ordinate of the current wave coincides with the maximum value of the flux, the zero values of current occur earlier than those of the flux. This non-coincidence of the zero values of current and flux is due to hysteresis.*

We also observe that the half-waves of current are similar and symmetrical. Therefore only odd harmonics are present in the current wave.

Analysis of Current Wave-form. If the current wave is analysed

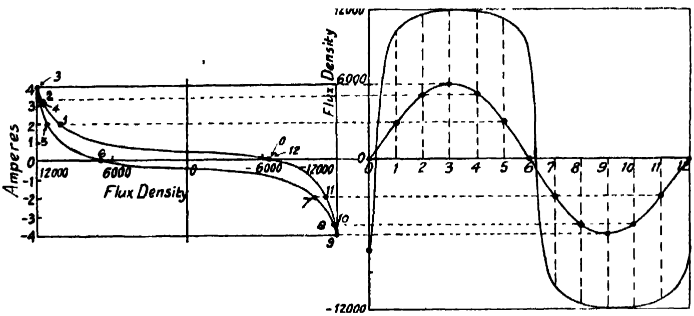


FIG. 204. METHOD OF DETERMINING WAVE-FORM OF FLUX

into its harmonics the results will show the presence of a fundamental, and third, fifth, and higher harmonics; the harmonics higher than the fifth being small in comparison with the third and fifth harmonics and the fundamental. For example, the analysis of the current wave in Fig. 203 gives the following result—

$$i = 2{\cdot}707 \sin(\theta + 9{\cdot}3^\circ) - 1{\cdot}11 \sin(3\theta + 10{\cdot}3^\circ) + 0{\cdot}255 \sin (5\theta - 48{\cdot}2) - 0{\cdot}1 \sin(7\theta - 59{\cdot}3^\circ) + 0{\cdot}085 \sin(9\theta - 83{\cdot}2^\circ) - 0{\cdot}054 \sin(11\theta + 82{\cdot}6)^\circ$$

In Fig. 205 the fundamental of the current wave of Fig. 203 is denoted by i_1 and the higher harmonics are grouped together in the curve marked i_2.

Equivalent Exciting Current. The power necessary to carry the iron through a cycle of magnetization is given by the product of the impressed E.M.F. and the fundamental of the current curve (since, with sinusoidal E.M.F. and non-sinusoidal current, only the fundamental of the current wave has a frequency equal to that of the E.M.F.), and this power is equal to the hysteresis loss. The

* If hysteresis were absent, or negligible, the current wave-form would be in phase with the flux wave-form, but the former would still show distortion due to the variation of permeability of the iron with variation of magnetizing current.

fundamental of the current wave may therefore be resolved into power and wattless components with respect to the impressed E.M.F. wave. These components are shown in Fig. 205 by the curves i_{1p} and i_{1w}.

The wattless component, i_{1w}, of the fundamental of the current wave, when compounded with the curve, i_2, representing the third, fifth, and higher harmonics (which are wattless with respect to a sinusoidal E.M.F.) gives the curve marked i_w, which represents the total wattless component of the current wave and is the true

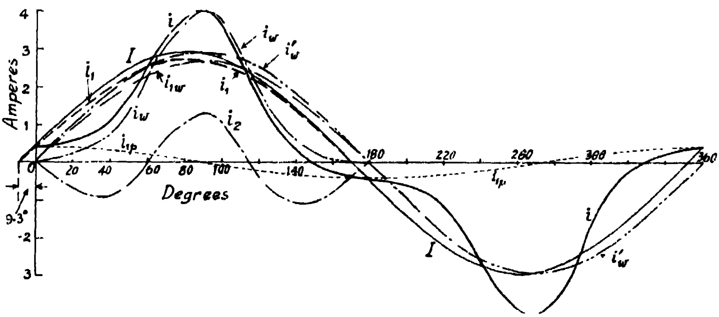


FIG. 205. COMPONENTS OF MAGNETIZING AND EXCITING CURRENTS

Explanation and data of curves—
i, non-sinusoidal magnetizing current, Fig. 187
i_1, fundamental of (i)—equation $i_1 = 2·7 \sin(\theta + 9·3°)$
i_2, higher harmonics grouped together
i_1p, power component of i_1—equation $i_1p = 0·137 \cos\theta$
i_1w, wattless component of i_1—equation $i_1w = 2·672 \sin\theta$
w, total wattless component of current wave
w', equivalent sine curve for w—equation $w' = 2·9 \sin\theta$
I, equivalent exciting current—equation $I_{inst} = 2·935 \sin(\theta + 9·3°)$

magnetizing current. This curve, i_w, may be replaced by an equivalent sine wave, i_w', having an R.M.S. value equal to I_w. Compounding the sine curve i_w' with the power component i_{1p} of the fundamental of the current wave, we obtain the sine curve I, which represents the equivalent sinusoidal current necessary to carry the iron through a magnetic cycle. This current, since it is compounded from the true magnetizing current and the power component of the current supplying the hysteresis loss, is called the *exciting current*, and the angle by which it leads the flux is called the *hysteretic angle of advance*, a.

The vector diagram representing these conditions is given in Fig. 206, in which the vector $O\Phi_m$ represents the flux, OI_o the exciting current, OI_w the magnetizing current, and OI_p the power component of the exciting current supplying the losses. Hence

$$I_o = \sqrt{(I_p^2 + I_w^2)}.$$

Now the R.M.S. value of the exciting current may be obtained from the reading of an ammeter connected in the circuit of the magnetizing winding, and if the hysteresis loss is denoted by P_h watts, the R.M.S. value of the power component of the fundamental of the current wave is equal to P_h/E, where E denotes the R.M.S. value of the E.M.F. induced in the magnetizing winding

Whence $$I_w = \sqrt{[I_o^2 - (P_h/E)^2]}$$

Thus the R.M.S. value of the magnetizing current may be obtained when the exciting current, the hysteresis loss, and the E.M.F. induced in the magnetizing winding are known. If the pressure drop in the winding is small in comparison with the impressed E.M.F., then the induced E.M.F. may be taken as equal to the impressed E.M.F.

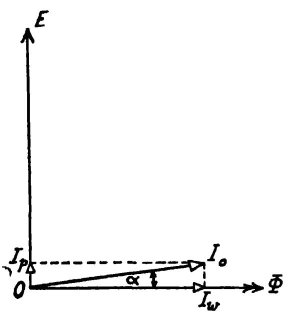


FIG. 206. VECTOR DIAGRAM SHOWING EFFECT OF HYSTERESIS

Effect of Hysteresis and Magnetic Saturation on the "Constants" of an Iron-cored Choking Coil. The general effect of hysteresis and magnetic saturation is to cause the power-factor of any iron-cored coil to be higher than that calculated from the resistance and inductance of the magnetizing winding. Thus, with sinusoidal impressed E.M.F. and non-sinusoidal exciting current, the power component of the fundamental of the exciting current supplies the losses, which, in the present case, include the I^2R loss in the magnetizing winding and the hysteresis loss (with which should be included the eddy-current loss, if any) in the iron core. If P_i denotes the iron losses, the total loss, P, is given by

$$P = P_i + I^2R,$$

where I is the exciting current and R the resistance of the magnetizing winding. Dividing throughout by I^2, we have

$$P/I^2 = P_i/I^2 + R$$

The quantity P/I^2 represents the effective resistance R_{eff} of the choking coil, and includes the true resistance (R) of the winding and the apparent resistance (P_i/I^2) due to the losses in the iron core. The effective resistance also includes the apparent resistance due to any eddy currents in the conductors.

If E denotes the impressed E.M.F., the effective impedance (Z_{eff}) of the choking coil is given by

$$Z_{eff} = E/I$$

Hence the effective reactance is given by

$$X_{eff} = \sqrt{[(E/I)^2 - (P/I^2)^2]}$$

Now the non-sinusoidal exciting current (I) is larger than the sinusoidal magnetizing current which would be required to produce the same flux, and therefore both the effective reactance and the effective impedance of the choking coil are smaller than the calculated values obtained from the resistance and inductance of the magnetizing winding.

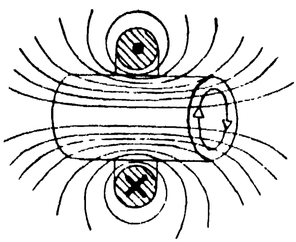


FIG. 207. PRODUCTION OF EDDY CURRENTS IN SOLID CORE

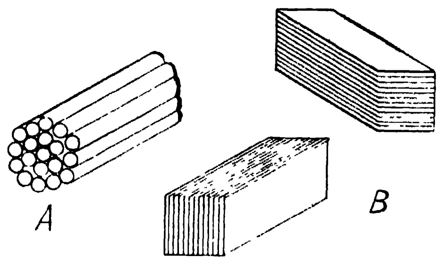


FIG. 208. LAMINATION OF CORES SUBJECTED TO ALTERNATING FLUX

Hence, since the power factor is given by (effective resistance/ effective impedance), it follows that, due to iron losses and magnetic saturation the actual power-factor of the choking coil will be higher than that calculated from the resistance and inductance of the magnetizing winding.

Eddy Currents. When a solid iron or metal core is placed longitudinally in an alternating magnetic field, the lines of force cut the core radially, the motion of the flux being towards the centre of the core when the magnetizing force is decreasing, and *vice versâ* when the magnetizing force is increasing. Hence, E.M.Fs. are induced in the core, and these E.M.Fs. give rise to currents which circulate in the core in a direction parallel and opposite to that of the current in the magnetizing solenoid (*see* Fig 207). These circulating currents are called *eddy currents*, and cause heating of the core.

The energy, which is dissipated in the form of heat, is obtained by electromagnetic induction from the circuit supplying current to

the magnetizing solenoid. Moreover, the circulating currents produce a magneto-motive force which acts in opposition to that producing magnetization, and therefore the flux due to a given current in the solenoid is smaller than that which would be obtained if the core were removed or replaced by one of non-magnetic material.

Hence when large masses of iron—such as the cores of transformers and alternating-current machines, or heavy conductors—are subjected to alternating magnetic fields, means must be adopted

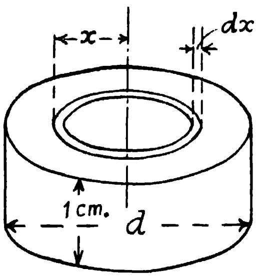


FIG. 209

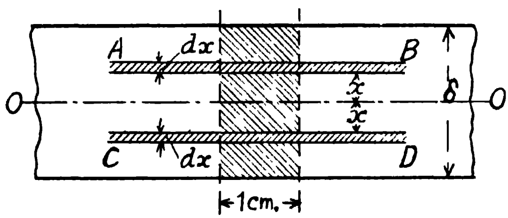


FIG. 210

PERTAINING TO THE CALCULATION OF EDDY CURRENTS IN WIRES AND PLATES

to reduce the eddy currents to such values that the resulting heating and loss of energy is not detrimental to the efficiency and operation.

The most effective means of reducing the eddy current loss is to *laminate* the material subjected to the alternating flux, the laminations being parallel to the direction of the magnetic lines of force. For example, the core of Fig. 207 would be constructed either of iron wires of small diameter, or of thin iron sheets, as indicated at A and B, Fig. 208, respectively. In each case the individual wires or laminations must be lightly insulated from one another by a thin coating of insulating varnish, as, if the laminations are in electrical contact, the eddy current loss will be practically the same as with a solid core.

Calculation of Eddy-current Loss. The loss due to eddy-currents in wires and laminations may be calculated easily when the flux is uniformly distributed over the cross section of the material.

I. Eddy-current loss in round wires. Consider an elementary coaxial annulus of radius x cm. and width dx cm., in the cross section of the wire (Fig. 209). Then if d cm. denotes the external diameter of the wire, B_m, the maximum value of the flux density (assumed to be uniform throughout the cross section of the wire),

the flux linked with the annulus is $B_m \pi x^2$, and the R.M.S. value (E_x) of the induced E.M.F. is given by

$$E_x = 4k_f f B_m \pi x^2 \times 10^{-8} \text{ volts},$$

where k_f is the form-factor or the flux wave and f is the frequency.

The direction of the induced E.M.F. is along the mean perimeter of the annulus, and for unit length (1 cm.) of wire the resistance in the path of current is

$$r = 2\pi x \rho / dx,$$

where ρ denotes the specific resistance of the material in ohm-cm. units.

Hence the loss, in watts, in this elementary ring due to the current circulating in it is

$$\begin{aligned} P_e &= E_x^2/r = E_x^2\, dx/2\pi x \rho \\ &= (4k_f f B_m \pi x^2 \times 10^{-8})^2\, dx/2\pi x \rho \\ &= (8\pi k_f^2 f^2 B_m^2 x^3 \times 10^{-16}) dx/\rho \end{aligned}$$

Whence the loss in watts per cm. length of wire is

$$\begin{aligned} P_{e1} &= \int_0^{\frac{1}{2}d} P_e \,.\, dx = \left(\frac{8\pi k_f^2 f^2 B_m^2}{\rho \times 10^{16}}\right) \int^{\frac{1}{2}d} x^3 \,.\, dx \\ &= \frac{\pi k_f^2 f^2 B_m^2 d^4}{\rho \times 8 \times 10^{16}} \end{aligned}$$

Therefore the loss in watts per cubic cm of the wire is

$$\begin{aligned} P &= P_{e1}/(\tfrac{1}{4}\pi d^2) \\ &= \tfrac{1}{2} k_f^2 f^2 B_m^2 d^2 \times 10^{-16}/\rho \qquad \qquad (168) \\ &= \tfrac{1}{2} \cdot \frac{1}{\rho \times 10^6}\left(d\, \frac{f}{100}\, \frac{k_f B_m}{1000}\right)^2 \end{aligned}$$

and the loss per kilogram of core is

$$\begin{aligned} P_{kg} &= P/\sigma \times 10^{-3} \\ &= \tfrac{1}{2}\left(\frac{1}{\sigma\rho \times 10^3}\right)\left(d\, \frac{f}{100}\, \frac{k_f B_m}{1000}\right)^2 \qquad \qquad (169) \end{aligned}$$

where σ is the density of the wire in grammes per cubic cm.

II. Eddy-current loss in thin plates In this case, the thickness, δ, of the plate is assumed to be very small in comparison with the width, i.e. the dimension parallel to the direction in which the eddy currents circulate. The eddy currents may therefore be considered to flow parallel to the surfaces of the plate and will have opposite directions on either side of the centre line as indicated in Fig. 210.

Consider two parallel elementary layers, AB, CD, in a plate at distances x cm. on either side of the centre line OO, Fig. 210. Let the thickness of each element be dx cm. and its length perpendicular to the paper be 1 cm. Then the flux enclosed per cm. length of the central portion of the plate (shown shaded in Fig. 210) bounded by the elements is

$$B_m(2x \times 1) = 2x \,.\, B_m$$

Hence the E.M.F. (E_x) induced in the circuit formed by the elements (which is equivalent to an electric circuit of one turn) is

$$E_x = 4k_f f B_m \,.\, 2x \times 10^{-8} \text{ volts.}$$

Now if the thickness of the plate is very small in comparison with the width, the length of the elementary circuit $ABCD$ may be considered as equal to twice the distance between the transverse boundary planes, and the resistance of this circuit per cm. length and depth of the plate is

$$r = 2\rho/(dx \times 1) \text{ ohms.}$$

Hence the loss, in watts, due to the circulating current in this circuit is

$$\begin{aligned} P_e &= E_x^2/r = E_x^2 dx/2\rho \\ &= (4k_f f B_m \,.\, 2x \times 10^{-8})^2 dx/2\rho \end{aligned}$$

and the total loss, in watts, per cm. length and depth of plate (i.e. for a volume δ cubic cm.) is

$$P_{e1} = \frac{1}{2\rho}\int_0^{\frac{1}{2}\delta} E_x^2\, dx = \frac{4}{3\rho}\, k_f^2 f^2 B_m^2 \delta^3 \times 10^{16}$$

Therefore the eddy current loss, in watts, per cubic cm. of the plate is

$$P = \frac{P_{e1}}{\delta} = \frac{4}{3\rho}\, k_f^2 f^2 B_m^2 \delta^2 \times 10^{-16} \quad . \quad . \quad . \quad (170)$$

and the loss per kilogram is

$$P_{kg} = \frac{4}{3}\left(\frac{1}{\sigma\rho \times 10^3}\right)\left(\delta\,\frac{f}{100}\,\frac{k_f B_m}{1000}\right)^2 \quad . \quad . \quad . \quad (171)$$

Example. Calculate the eddy-current loss in watts per kg. of alloyed iron plate for which the thickness is 0·5 mm., the specific resistance is 50×10^{6} ohms per cm. cube, the density is 7·5 gm. per cubic cm., and the maximum value of the sinusoidal flux density is 13,000 lines per square cm. at a frequency of 50 cycles per second.

From the data we have

$\sigma = 7{\cdot}5$	$f = 50$
$\rho = 50 \times 10^{-6}$	$k_f = 1{\cdot}11$
$\delta = 0{\cdot}05$	$B_m = 13{,}000$

Hence, substituting in equation (171),

$$P_{k\theta} = \frac{4 \times 10^6}{3 \times 7{\cdot}5 \times 50 \times 10^3}\left(0{\cdot}05 \times \frac{50}{100} \times \frac{1{\cdot}11 \times 13000}{1000}\right)^2$$
$$= 0{\cdot}462 \text{ W.}$$

NOTE. In practice the eddy-current loss is always in excess of that calculated by the above method, the discrepancy being due to a number of causes, such as imperfect insulation between the plates, non-uniform distribution of the flux throughout the cross section of the core due to differences in the length of the magnetic path for different portions of the core, non-uniform distribution of the flux throughout the cross section of individual plates due to the effect of eddy currents (see p. 339).

Effect of Eddy Currents on Flux Distribution in Iron Cores. The E.M.F. (E_x) induced in an element of the cores of Figs. 209, 210, by the alternations of the (sinusoidal) flux may be represented by a vector, OE_x, Fig. 211, lagging 90° with respect to the vector, OI_w, which represents the equivalent (sinusoidal) magnetizing current. The circulating, or eddy, current due to the E.M.F., E_x, may be represented by a vector, OI_x, lagging $\varphi°$ with respect to OE_x owing to the inductance of the path of the eddy current.

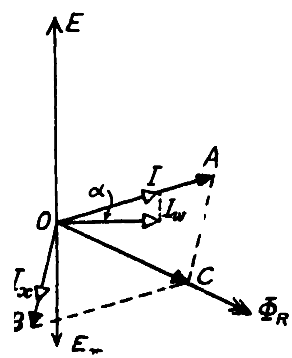


FIG. 211. VECTOR DIAGRAM SHOWING EFFECT OF EDDY CURRENTS

Now when the core is concentric with the magnetizing winding, the path of this eddy current is concentric with that of the current in the magnetizing winding. Therefore the resultant magneto-motive force, to which the flux in the core is due, is equal to the vector sum of the M.M.Fs. due to the current in the magnetizing winding and the eddy current in the core. The component M.M.Fs are represented by OA, OB in Fig. 211, the resultant M.M.F. is represented by OC, and the flux in the core (which is in phase with the resultant M.M.F.) is represented by $O\Phi_R$. In general, the resultant M.M.F. will be smaller than that due to the magnetizing current, and therefore the resultant flux will not be in phase with the magnetizing current, and will be smaller than the flux which would be obtained if there were no eddy currents, other conditions remaining unaltered.

When the effects of all the eddy currents in the core are considered—which requires analytical treatment and is given later—the results show that, except in the cases of very thin wires and plates, the flux distribution over the cross section of the core is not

uniform; the flux tending to concentrate towards the surface, or outer layers, of the plates, or wires, forming the core. The eddy currents, therefore, produce a magnetic screening effect on the central portions of the plate. Hence, with cores constructed of thick plates, a large percentage of the cross-section would be rendered magnetically ineffective by eddy currents, and therefore the average value of the flux density in the core may be very different from the computed value on the assumption of uniform flux distribution.

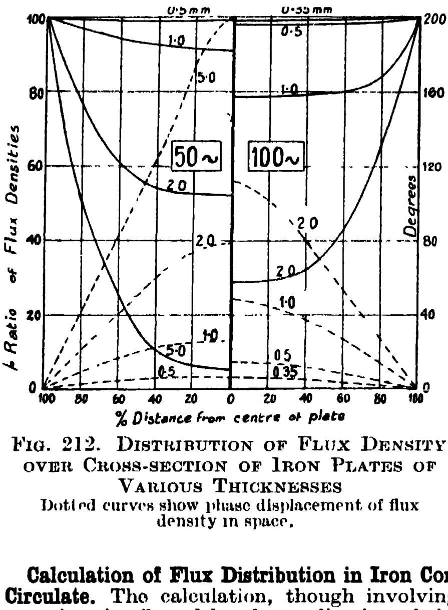


FIG. 212. DISTRIBUTION OF FLUX DENSITY OVER CROSS-SECTION OF IRON PLATES OF VARIOUS THICKNESSES
Dotted curves show phase displacement of flux density in space.

The curves of Fig. 212 show, for a particular brand of iron, that to obtain approximate uniform distribution of flux throughout the cross section of the plate, the thickness should not exceed about 0·5 mm. (0·014 in.) when the frequency is 50 cycles per second, and about 0·35 mm. (0·01 in.) when the frequency is 100 cycles per second.

Calculation of Flux Distribution in Iron Core-plates in which Eddy Currents Circulate. The calculation, though involving the solution of a differential equation, is effected by the application of elementary principles. Thus, the change in flux density over the cross-section of a core-plate is related to the change, with respect to the cross-section, of M.M.F. due to eddy currents, and the latter is proportional to the change in the E.M.F. induced in the plate. In the calculations we shall assume that the flux in the core varies sinusoidally with respect to time; that the individual core-plates are insulated from one another, and that the thickness of each plate is small in comparison with the width of the plate.

Then, considering a single plate of thickness δ cm., let two elementary parallel layers, each of thickness dx, be chosen in the cross-section at distances x cm. from the centre line, as represented in Fig. 210, p. 332. Let B_{mx} denote the maximum value of the flux density (assumed to be uniform*) over these layers, and Φ_{mx} the maximum value of the flux enclosed in unit width of plate by the inner surfaces, AB, CD, of the layers. Hence the maximum value of the E.M.F. induced in unit width of plate at a distance x from the centre line (i.e. along the inner surfaces) is given by

$$E_{mx} = \omega\Phi_{mx} \times 10^{-8} = 2\pi f\Phi_{mx} \times 10^{-8}.$$

* This assumption is justifiable if the thickness dx is very small in comparison with δ.

Similarly, the induced E.M.F. at a distance $(x + dx)$ from the centre line (i.e. along the outer surfaces) is given by

$$E_{m(x+dx)} = 2\pi f(\Phi_{mx} + 2B_{mx}dx) \times 10^{-8}.$$

Whence

$$E_{mx} - E_{m(x+dx)} = dE_{mx} = 2\pi f(2B_{mx}\,dx) \times 10^{-8} = 4\pi f B_{mx}dx \times 10^{-8}.$$

If ρ denotes the specific resistivity of the plate, the resistance of the circuit formed by the elementary layers, per unit width and depth of plate, is equal to $2\rho/dx$. Hence the eddy current circulating in this circuit is given by

$$I_{mx} = E_{mx}/(2\rho/dx) = E_{mx}dx/2\rho.$$

The magneto-motive force (maximum value) due to this current is equal to $0{\cdot}4\pi I_{mx}$, and this represents the increment in magneto-motive force due to eddy currents for the elementary layers under consideration. Therefore the increment in flux density for the elementary layers is given by

$$dB_{mx} = \mu(0{\cdot}4\pi I_{mx}) = (0{\cdot}2\pi\, E_{mx}\mu/\rho)dx$$

where μ is the permeability.

Expressing dB_{mx} and dE_{mx} in symbolic notation, and taking the flux density as the quantity of reference, we have

$$\frac{d\dot{B}_{mx}}{dx} = \frac{0{\cdot}2\pi\mu}{\rho}\dot{E}_{mx} \qquad (\alpha)$$

$$\frac{d\dot{E}_{mx}}{dx} = -j\,(4\pi f\dot{B}_{mx} \times 10^{-8}) \qquad (\beta)$$

Whence $$\frac{d^2\dot{B}_{mx}}{dx^2} = \frac{0{\cdot}2\pi\mu}{\rho}\cdot\frac{d\dot{E}_{mx}}{dx}$$

Substituting for $d\dot{E}_{mx}/dx$ from equation (β), we have

$$\frac{d^2\dot{B}_{mx}}{dx^2} = -j\left[\dot{B}_{mx}\left(0{\cdot}8\pi^2\frac{\mu}{\rho}f \times 10^{-8}\right)\right]$$

$$= -j\,2c^2\dot{B}_{mx} \qquad (\gamma)$$

where $c^2 = 0{\cdot}4\pi^2 f\mu/10^8\rho$,

or $$c = \frac{2\pi}{10^4}\sqrt{\frac{f\mu}{10\rho}} \qquad (172)$$

The solution of the differential equation (γ) is

$$\dot{B}_{mx} = \dot{C}_1\varepsilon^{x\sqrt{(-j2c^2)}} + \dot{C}_2\varepsilon^{-x\sqrt{(-j2c^2)}}$$

where $\dot{C}_1$ and $\dot{C}_2$ are complex constants. In the present case these constants have equal values, since the flux density has the same value, but is of opposite sign, at points on opposite sides of, and equidistant from, the centre line of the plate. Whence

$$\dot{B}_{mx} = \dot{C}(\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}})$$

Denoting the flux density at the surface layers of the plate (for which x has the value $\frac{1}{2}\delta$) by $B_{m\delta}$, we have

$$\dot{B}_{m\delta} = \dot{C}\,(\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}})$$

Whence $$\frac{\dot{B}_{mx}}{\dot{B}_{m\delta}} = \frac{\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}}}{\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}}}$$

or $$\dot{B}_{mx} = \dot{B}_{m\delta}\,\frac{\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}}}{\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}}}$$

which ultimately reduces to*

$$\underset{\cdot}{B}_{mx} = \underset{\cdot}{B}_{m\delta} \frac{(\varepsilon^{cx} + \varepsilon^{-cx}) \cos cx - j(\varepsilon^{cx} - \varepsilon^{-cx}) \sin cx}{(\varepsilon^{\frac{1}{2}\delta c} + \varepsilon^{-\frac{1}{2}\delta c}) \cos \frac{1}{2}\delta c - j(\varepsilon^{\frac{1}{2}\delta c} - \varepsilon^{-\frac{1}{2}\delta c}) \sin \frac{1}{2}\delta c} \quad . \quad (173)$$

$$= \underset{\cdot}{B}_{m\delta} \frac{\cosh cx \cos cx - j \sinh cx \sin cx}{\cosh \frac{1}{2}\delta c \cos \frac{1}{2}\delta c - j \sinh \frac{1}{2}\delta c \sin \frac{1}{2}\delta c} \quad . \quad . \quad . \quad (173a)$$

The denominator of this expression represents a complex number having a constant value for given conditions. The numerator is equal to unity when $x = 0$, and becomes equal to the denominator when $x = \frac{1}{2}\delta$. For values of x intermediate between 0 and $\frac{1}{2}\delta$ the numerator represents a complex number having a value intermediate between unity and the value of the denominator. Hence the equation (173) to the flux density at any point in the cross-section of the plate is of the form

$$\underset{\cdot}{B}_{mx} = \underset{\cdot}{B}_{m\delta}(\pm \boldsymbol{a} \pm j\boldsymbol{b})$$

where $\pm \boldsymbol{a} + j\boldsymbol{b}$ represents the value of the quotient in equation (173).

Therefore at any instant the flux densities at different portions of the cross section of the plate not only differ in magnitude but also have a space displacement with respect to one another, i.e. these flux densities have different directions (in space) with respect to the direction of the flux density at the surface layers (which coincides with that of the magnetizing ampere-turns). A reversal in direction (i.e. a space displacement of 180°) occurs when $cx = \pi$, or when $x = \frac{10^4}{2}\sqrt{\frac{10\rho}{\mu}}$. For example, if $\rho = 10^{-5}$, $\mu = 1000$, $f = 50$, then $x = 0{\cdot}2235$ cm. Hence, for these conditions, the flux density at the centre of a plate 4·47 mm. thick has a direction opposite to that of the flux density at the surface layers. The manner in which the magnitude and space displacement of the flux density varies over the cross-section of a similar plate 5 mm. thick is shown in the worked example which follows.

Example. Calculation of flux distribution in an iron plate 5 mm. thick when subjected to an alternating magnetization of 50 frequency. Specific resistance $= 10^{-5}$ ohm per cm. cube; permeability $= 1000$ (assumed to be constant).

From equation (172) the space displacement between the flux densities at centre and surface layers is

$$\theta = \tfrac{1}{2} \times 0{\cdot}5 \times \frac{2\pi}{10^4}\sqrt{\frac{50 \times 1000 \times 10^5}{10}} = 3{\cdot}51 \text{ radians} = 201°$$

Hence it will be convenient to calculate the flux densities at space intervals corresponding to displacements of $\frac{1}{6}\pi$ radians, or 30°, i.e. values of cx in equation (173a) will be taken at intervals of $\frac{1}{6}\pi$ radains.

The evaluation of equation (173a) is effected with the aid of tables of hyperbolic functions and presents no difficulties. The values of $\cosh cx$, $\sinh cx$, $\cos cx$, $\sin cx$ required for the calculations are given in Table XII.

Since $\cosh 3{\cdot}51 = 16{\cdot}82$; $\cos 201° = -0{\cdot}933$
$\sinh 3{\cdot}51 = 16{\cdot}78$; $\sin 201° = -0{\cdot}359$

the denominator of equation (173a) reduces to

$$16{\cdot}82 \times (-0{\cdot}933) + j16{\cdot}78 \times 0{\cdot}359 = -15{\cdot}7 + j6{\cdot}02$$

* The reduction depends upon the theorems—

$(\alpha)\ .\ .\ .\ \varepsilon^{x\sqrt{-j2}} = \varepsilon^{x(1-j1)} = \varepsilon^{x} \cdot \varepsilon^{-jx} = \varepsilon^{x}(\cos x - j \sin x)$

$(\beta)\ .\ .\ .\ \varepsilon^{-x\sqrt{-j2}} = \varepsilon^{-x(1-j1)} = \varepsilon^{-x} \cdot \varepsilon^{jx} = \varepsilon^{-x}(\cos x + j \sin x)$

The steps in the reduction of the exponent $x\sqrt{-j2}$ are—

$$\begin{aligned} x\sqrt{(0 - j2)} = \sqrt{(0 - j2x^2)} &= \sqrt{(2x^2\, \varepsilon^{-j\frac{1}{2}\pi})} \\ &= (x\sqrt{2})\varepsilon^{-j\frac{1}{4}\pi} \\ &= x\sqrt{2}(\cos \tfrac{1}{4}\pi - j \sin \tfrac{1}{4}\pi) \\ &= x(1 - j1) \end{aligned}$$

The values of the numerators and quotients are calculated in the usual manner and the results are given in tabular form in Table XII, and are plotted in Fig. 213*a*, from which it will be observed that the central portion of the plate for a distance of about 1·5 mm. on each side of the centre line, is almost entirely ineffective.

0·0622 mm. from the surface), and the remaining radii-vectores are drawn at angles of 10° from one another (corresponding to successive points 0·1244 mm. apart in the cross-section of the plate along the direction perpendicular to the surface of the plate). The values of the projections, taken in order and expressed as a percentage of $B_{m\delta}$, are 92, 74, 52·6, 43·5, 32, 22·1, 13·7, 7·2, 2, − 1·5, − 3·6, − 5, − 5·8, − 6·2, − 6·4, − 6·6, − 6·6, − 6·4, − 6·2, − 5·6 ; and their algebraic sum is 279·2.

Whence the average flux density over the cross-section of the plate is (13,000 × 279·2/2000 =) 1815 lines per square cm., and the flux transmitted per cm. of width is (1815 × 0·5 × 1 =) 907 lines.

Hence the thickness of plate required to transmit this flux at a uniform flux density of 13,000 lines per square cm. (i.e. on the assumption of no eddy-currents) is (907/13000 =) 0·0698 cm., or about two-thirds of a millimetre, which is only 14 per cent of the thickness of the 5 mm. plate.

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sinh cx, cos cx, sin cx required for the ...

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$$(\alpha) \ldots \quad \varepsilon^{x\sqrt{-j2}} = \varepsilon^{x(1-j1)} = \varepsilon^{x} . \varepsilon^{-jx} = \varepsilon^{x}(\cos x - j \sin x)$$

$$(\beta) \ldots \quad \varepsilon^{-x\sqrt{-j2}} = \varepsilon^{-x(1-j1)} = \varepsilon^{-x} . \varepsilon^{jx} = \varepsilon^{-x}(\cos x + j \sin x)$$

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The values of the numerators and quotients are calculated in the usual manner and the results are given in tabular form in Table XII, and are plotted in Fig. 213*a*, from which it will be observed that the central portion of the plate for a distance of about 1·5 mm. on each side of the centre line, is almost entirely ineffective.

It is of interest to calculate the flux which is transmitted per cm. width of plate when the flux density at the surface (i.e. $B_{m\delta}$) is 13,000 lines per square cm., and thence deduce the thickness of plate required to transmit this flux at a uniform flux density of 13,000 lines per square cm. (i.e. on the assumption of no eddy currents).

The calculation may be effected quite easily by dividing the cross-section of the plate into narrow strips and determining for each strip the component

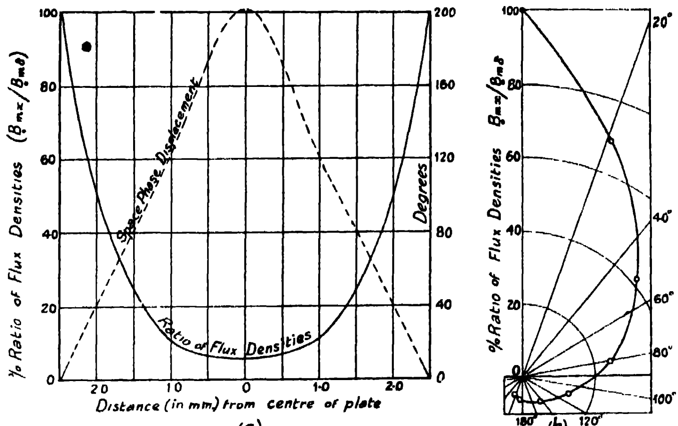

FIG. 213. DISTRIBUTION OF FLUX DENSITY OVER CROSS-SECTION OF IRON PLATE 5 MM. THICK

of the mean flux density in the direction parallel to the axis of magnetization. Probably the simplest method is to plot the ratio $B_{mx}/B_{m\delta}$ and θ in polar co-ordinates, as in Fig. 213*b*, divide this into twenty parts by radii-vectores spaced 10° apart, draw the mid-point radii-vectores, and measure their projections on the axis of reference (which is vertical in Fig. 213*b*). Thus the first mid-point radius-vector is drawn at an angle of 5° from the vertical (corresponding to a point in the cross-section of the plate distant [2·5(5/201) =] 0·0622 mm. from the surface), and the remaining radii-vectores are drawn at angles of 10° from one another (corresponding to successive points 0·1244 mm. apart in the cross-section of the plate along the direction perpendicular to the surface of the plate). The values of the projections, taken in order and expressed as a percentage of $B_{m\delta}$, are 92, 74, 52·6, 43·5, 32, 22·1, 13·7, 7·2, 2, − 1·5, − 3·6, − 5, − 5·8, − 6·2, − 6·4, − 6·6, − 6·6, − 6·4, − 6·2, − 5·6; and their algebraic sum is 279·2.

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Hence the thickness of plate required to transmit this flux at a uniform flux density of 13,000 lines per square cm. (i.e. on the assumption of no eddy-currents) is (907/13000 =) 0·0698 cm., or about two-thirds of a millimetre, which is only 14 per cent of the thickness of the 5 mm. plate.

In consequence of the non-uniform distribution of the flux in thick laminations, the maximum value of the flux density may be considerably higher than the value calculated on the assumption of uniform flux distribution. Therefore the eddy-current loss in such plates will be higher than that calculated from equation (171).

Effect of Frequency and Wave-form on Hysteresis and Eddy-current Losses in Iron Cores. *Effect of Frequency on Hysteresis Loss.* Experimental results of hysteresis tests, by the "ballistic" method, on magnetic materials show that the hysteresis loss in a given material is proportional to nearly the 1·6th power of the flux density. Thus the energy expended per cubic cm. of material per magnetic cycle is expressed by

$$W_h = \eta B_m^{1.6}$$

where η is a coefficient the value of which depends upon the quality of the iron and the units in which W_h and B_m are expressed. For iron laminations such as are used in electrical machinery, the value of η varies from 0·001 to 0·003 when W_h and B_m are expressed in C.G.S. units.

Assuming the hysteresis loop to be the same with alternating magnetization as when determined ballistically, the hysteresis loss, with alternating magnetization, will be proportional to the first power of the frequency and to the 1·6th power of the flux density. The proportionality of hysteresis loss and frequency at constant flux density holds only for low frequencies, as, for frequencies of the order of 100 cycles per second and above, the width of the hysteresis loop, corresponding to a given maximum flux density, increases with increasing frequency, and, therefore, the hysteresis loss per magnetic cycle becomes larger as the frequency increases.

Effect of wave-form of impressed E.M.F. on hysteresis loss. For low frequencies the hysteresis loss per cubic cm. is given by $P_h = \eta f B_m^{1.6}$.

Now the flux density B_m is given by Φ_m/A, where A is the magnetic cross-section of the core. Also $\Phi_m = E \times 10^8/(4k_f f N)$, where E is the R.M.S. value of the E.M.F. induced in the magnetizing winding, k_f the form-factor of this E.M.F., f the frequency, and N the number of turns in the magnetizing winding. Whence

$$P_h = \eta f[E \times 10^8/(4k_f fNA)]^{1.6}$$

$$= \eta\left(\frac{(E \times 10^8)}{4NA}\right)^{1.6} \frac{\eta}{k_f^{1.6} f^{0.6}} \quad . \quad . \quad . \quad (174)$$

Hence, if the resistance of the magnetizing winding is negligible, and the impressed E.M.F. and frequency are constant, the hysteresis loss is inversely proportional to the 1·6th power of the form-factor

of the impressed E.M.F. Therefore the hysteresis loss with peaked E.M.F. waves will be lower than that with flat-topped E.M.F. waves of the same R.M.S. value.

The following values indicate the extent to which the form-factor of the impressed E.M.F. affects the hysteresis loss—

Form-factor.	1·0	1·11	1·2	1·3	1·4
Relative hysteresis loss (constant E.M.F. (R.M.S. value) and frequency) .	1·18	1·0	0·88	0·78	0·69

The *eddy-current loss* in a given plate or wire has been shown to be proportional to the squares of the flux density, frequency, and form factor. Thus

$$P_e = \xi(k_f \,.\, f \,.\, B_m)^2$$

Substituting for B_m in terms of the induced E.M.F., frequency, etc., we have

$$P_e = \xi[k_f \,.\, f \,.\, E \times 10^8/(4k_f f NA)]^2$$
$$= \xi[E \times 10^8/(4\,NA)]^2 \quad . \quad . \quad . \quad . \quad (175)$$

Hence, in a given magnetic core magnetized by alternating current the eddy-current loss is proportional to the square of the impressed E.M.F. (R.M.S. value) and is independent of the frequency and wave-form of the latter. This statement, however, holds only for cases where the flux density is uniformly distributed over the cross-section of the core (e.g. when the laminations are very thin and the frequency is low).

The *total loss* in a magnetic core magnetized by alternating current is, therefore, dependent upon the form factor of the impressed E.M.F., when the frequency and R.M.S. value of impressed E.M.F. are constant.

When the impressed E.M.F. is constant and the frequency is varied, the total loss decreases as the frequency increases, since the eddy-current loss is constant and the hysteresis loss decreases with increasing frequency. But when the flux, or flux density, is constant and the frequency is varied, the total iron loss increases as the frequency increases, since the hysteresis loss varies directly as the frequency and the eddy-current loss varies as the square of the frequency, the form-factor of the impressed E.M.F. being assumed to be constant.

In practice the total iron loss is usually expressed in the form of

curves which show the loss (in watts per lb., or kg.) at various flux densities and constant frequency. Typical curves are given in Fig. 214.

Separation of Iron Loss into Hysteresis and Eddy-current Components. If the specific iron loss follows the equation

$$P_{i(kg)} = \eta f B_m^{\ x} + \xi f^2 B_m^{\ y},$$

where the first term represents the hysteresis loss and the second term the eddy-current loss, these components may be separated by determining

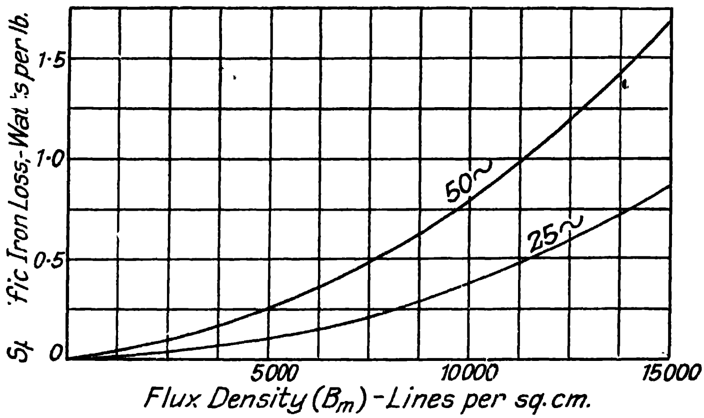


FIG. 214. SPECIFIC IRON LOSS FOR 0·018 IN. (0·45 MM.) ALLOYED IRON LAMINATIONS

$P_{i(kg)}$ for two different frequencies at a given flux density. The exponents x, y, may be evaluated by determining $P_{i(kg)}$ for two different flux densities at a given frequency.

Thus, if observations are made at frequencies f_1, f_2, and flux density B_m, we have

$$\frac{P_{i(kg)1}}{f_1} = \eta B_m^{\ x} + \xi f_1 B_m^{\ y} = a + bf_1$$

$$\frac{P_{i(kg)2}}{f_2} = \eta B_m^{\ x} + \xi f_2 B_m^{\ y} = a + bf_2$$

where a, b, denote the hysteresis and eddy-current losses, respectively, per kg. per cycle at the flux density B_m.

Whence
$$a = \frac{(P_{i(kg)2}/f_2)f_1 - (P_{i(kg)1}/f_1)f_2}{f_1 - f_2}$$

$$b = \frac{P_{i(kg)1}/f_1 - P_{i(kg)2}/f_2}{f_1 - f_2}$$

If $f_1 = 2f_2$ the computation is greatly simplified, for then

$$a = 2P_{i(kg)2} - P_{i(kg)1}$$

$$b = \frac{1}{f_2}\left(\frac{P_{i(kg)1}}{f_1} - \frac{P_{i(kg)2}}{f_2}\right)$$

Calculation of Magnetizing and Exciting Currents for an Iron-cored Magnetic Circuit with Alternating Flux. *Case I, in which the distorting effect of hysteresis and magnetic saturation on the wave-form of the magnetizing current is ignored.* If both flux and magnetizing current are sinusoidal, the calculation of the ampere turns, or magnetizing current, required to produce a given flux is very simple. The procedure is the same as that when the magnetization is produced by direct current except that, with alternating magnetization, the *maximum* value of the flux, or flux density, is usually given (or is obtained by calculation), and the R.M.S. value of the magnetizing current to produce this flux has to be determined.

For example, if the magnetization, or B-H, curve determined by the ballistic method for a magnetic material, shows that F_{cm} ampere turns per cm. of magnetic length are necessary to obtain a flux density equal to B_m, then, with alternating (sinusoidal) magnetization and the same flux density (*maximum* value in this case), the *maximum* value of the ampere-turns will be equal to F_{cm}, and the R.M.S. value of the ampere-turns will be equal to $F_{cm}/\sqrt{2}$.

Hence, if l is the length (in cm.) of a magnetic circuit formed of this material, then, if the flux density B_m is the same at all parts of the circuit, the magnetizing current will be given by

$$I_m = lF_{cm}/(\sqrt{2} \,.\, N.)$$

If the magnetic circuit consists of a number of parts having different cross-sections and magnetization curves, the problem is treated by determining the ampere-turns required for each part of the circuit and adding these to obtain the total ampere turns. Thus, if l_1, l_2, l_3, . . . denote the several magnetic lengths, F_{1cm}, F_{2cm}, F_{3cm}, . . . the ampere turns per cm. required to obtain the requisite flux densities, corresponding to a given flux, in the several parts of the circuit, then the magnetizing current is given by

$$I_w = \frac{1}{N\sqrt{2}}\,(l_1F_{1cm} + l_2F_{2cm} + l_3F_{3cm} + \;\ldots)$$

The exciting current may be determined when the losses in the iron core and magnetizing winding are known. Thus, if P denotes the total losses and E the impressed E.M.F., the power component of the exciting current is given by $I_p = P/E$, and, therefore, the exciting current is given by

$$I_o = \sqrt{(I_p^2 + I_w^2)}.$$

Example. A closed magnetic circuit is built up of 0·35 mm. laminations and consists of two cores which are magnetically connected by yokes. The magnetic cross-section of each core is 100 square cm., and that of each yoke is 108 square cm. The magnetic length of each core is 20 cm., and that of

each yoke is 11 cm. Each core is wound with a magnetizing coil having 75 turns and a resistance of $0{\cdot}18\ \Omega$. The two coils are connected in series and excited from a 50 cycle circuit to give a maximum flux of $1{\cdot}2 \times 10^6$ magnetic lines in the cores and yokes. Calculate the exciting current.

Data of the magnetic properties of the laminations are—

Maximum flux density (B_m) (lines per square cm.)	11,100	12,000
Magnetizing ampere turns per cm. of length (F_{cm})	3·9	6
Core loss at 50 frequency (watts per lb.). .	0·76	0·88

The flux density (maximum value) in each core is ($1{\cdot}2 \times 10^6/100 =$) 12,000 lines per square cm., and that in each yoke is ($1{\cdot}2 \times 10^6/108 =$) 11,100 lines per square cm.

Hence the magnetizing ampere turns (maximum value) required are equal to

$$2 \times 11 \times 3{\cdot}9 + 2 \times 20 \times 6 = 326,$$

and the R.M.S. value of the magnetizing current is

$$326/(150 \times \sqrt{2}) = 1{\cdot}535 \text{ A}.$$

The weight of laminations in the two cores is

$$2 \times 20 \times 100 \times 0{\cdot}28/2{\cdot}54^3 = 68{\cdot}3 \text{ lb.},$$

and that in the two yokes is

$$2 \times 11 \times 108 \times 0{\cdot}28/2{\cdot}54^3 = 40{\cdot}5 \text{ lb.}$$

Whence the total iron loss is

$$0{\cdot}88 \times 68{\cdot}3 + 0{\cdot}76 \times 40{\cdot}5 = 90{\cdot}8 \text{ watts}.$$

The E.M.F. induced in the magnetizing coils is

$$4{\cdot}44 \times 1{\cdot}2 \times 10^6 \times 50 \times 2 \times 75 = 400 \text{ V}.$$

Hence a first approximation to the exciting current is given by

$$\sqrt{[1{\cdot}535^2 + (90{\cdot}8/400)^2]} = \sqrt{(1{\cdot}535^2 + 0{\cdot}227^2)} = 1{\cdot}552 \text{ A},$$

and the angle by which this current leads the flux is equal to $\tan^{-1} 0{\cdot}227/1{\cdot}535$ or 8°.

Since the pressure drop in the magnetizing coils due to the exciting current is $1{\cdot}552 \times 2 \times 0{\cdot}18 = 0{\cdot}56$ V., and has a phase difference of 81° with respect to the induced E.M.F. in those coils, the impressed E.M.F. is practically equal to the induced E.M.F. and the first approximation to the exciting current is sufficiently accurate for practical purposes.

Case II, in which the distortion of the wave-form of the magnetizing current is considered. If the magnetic circuit is a simple one (i.e. the magnetic cross-section and material is the same throughout the circuit) the magnetizing current corresponding to a given maximum sinusoidal flux may be determined without difficulty when the hysteresis loop for the given magnetic conditions is available. The procedure is similar to that given on p. 326 for the determination of the wave-form of magnetizing current.

Having determined the wave-form of the current necessary to carry the magnetization through a cycle, the R.M.S. value of this current is deduced, together with the R.M.S. values of the magnetizing and power components. To the power component is added the component of the supply current which supplies the eddy-current loss. The exciting current is then obtained from the resultant power component and the magnetizing component.

This method of procedure becomes too involved when complex magnetic circuits have to be calculated, as an equivalent hysteresis loop for the complete magnetic circuit would have to be deduced before the wave-form of the current could be determined. Moreover, even for simple magnetic circuits, the above method is too tedious for practical purposes. A shorter method, however, is available which, although not possessing the same accuracy as the preceding method, nevertheless possesses sufficient accuracy for practical purposes and, moreover, takes into account wave-form distortion due to magnetic saturation.

For this method, magnetization curves corresponding to alternating magnetization are required, and when such curves are available the method of procedure in calculating the magnetizing current is similar to that for a magnetic circuit excited with direct current.

The magnetization curves are obtained experimentally on a sample of the material by measuring the exciting current and the power supplied to the magnetizing winding at various impressed voltages and constant frequency. The flux and flux density in the specimen are calculated from the induced E.M.F., the number of turns in the magnetizing winding, and the magnetic cross-section of the specimen. The power supplied, when corrected for the I^2R loss in the magnetizing winding, represents the hysteresis and eddy-current losses in the specimen. If the exciting current is split up into power and wattless components, the latter represents the equivalent magnetizing current and takes into account the effects of hysteresis and eddy currents.

The magnetizing ampere turns per cm. of magnetic length ($F_{w \cdot cm}$) are calculated from the wattless component (I_w) of the exciting current, the number of turns (N) in the magnetizing winding and the mean length of magnetic path (l). Thus

$$F_{w \cdot cm} = NI_w/l.$$

If the exciting current is denoted by I_0, the hysteresis and eddy-current losses by P_i, and the impressed E.M.F. by E, then the power component of the exciting current which supplies the hysteresis

and eddy-current losses is given by $I_p = P_i/E$, and the wattless component is given by

$$I_w = \sqrt{(I_o^2 - I_p^2)} = \sqrt{[I_o^2 - (P_i/E)^2]}.$$

Whence the magnetizing ampere turns per cm. of magnetic length are

$$F_{w \cdot cm} = NI_w/l = \sqrt{[(NI_o/l)^2 - (NI_p/l)^2]}$$
$$= \sqrt{(F_{cm}^2 - F_{p \cdot cm}^2)}$$

where F_{cm} denotes the exciting ampere turns per cm. of magnetic path, and $F_{p \cdot cm}$ the ampere turns per cm. of magnetic length for supplying the hysteresis and eddy-currents; these ampere turns having a phase difference of 90° with respect to the magnetizing ampere turns.

If curves are plotted, showing the relationship between the maximum flux density, B_m, and $F_{w \cdot cm}$ and $F_{p \cdot cm}$, we have available a simple means for obtaining the two components of the exciting ampere turns.

CHAPTER XVI

COMMERCIAL MEASURING INSTRUMENTS

Classification and Principles of Operation. Commercial measuring instruments are classified according to both the quantity measured by the instrument and the principle of operation. Four general principles of operation are available: (1) electromagnetic, which utilizes the magnetic effects of electric currents; (2) electro-thermic, which utilizes the heating effect; (3) electrostatic, which utilizes the forces between electrically-charged conductors; (4) rectification.

Electromagnetic instruments may be subdivided according to the nature of the movable system and the method by which the deflecting, or operating, torque is produced. The sub-classes are: (*a*) moving-iron instruments, (*b*) electro-dynamic, or dynamometer, instruments, (*c*) induction instruments.

In *moving-iron instruments* the movable system consists of one or more pieces of specially-shaped soft iron, which are so pivoted as to be acted upon by the magnetic field produced by the current, or currents, in one or more fixed coils.

In *electro-dynamic instruments* the operating torque is due to the interaction of the magnetic fields produced by currents in a system of fixed and movable coils. The movable system of these instruments consists of one or more coils, which are so pivoted as to move in the magnetic field produced by the fixed coils. The currents in both fixed and movable coils are obtained from a common source.

In *induction instruments* the movable system consists of a pivoted non-magnetic conducting disc or drum, a portion of which moves in the magnetic field produced by an electromagnet, or a system of electromagnets, excited with alternating current. The magnetic field induces currents in the movable disc, or drum, and the magnetic reaction of the induced currents and the alternating magnetic field produces the operating torque.

Electro-thermic instruments may be subdivided into two distinct types: (*a*) the "expansion" type (commonly called "hot-wire," or "hot-strip" instruments) in which the operation depends upon the linear expansion of a wire, or strip, heated by a current; (*b*) the "thermo-couple" type (commonly called "thermo" instruments), in which one or more thermo-couples are heated, either directly or indirectly, by the current to be measured, and the

thermo-E.M.F. is caused to operate a direct-current instrument of the permanent-magnet moving-coil type.

Rectifier instruments consist of a sensitive direct-current milli-ammeter of the permanent-magnet moving-coil type connected to a full-wave metal rectifier. A small fraction of the alternating voltage or current to be measured is applied to the rectifier, and the rectified current is measured by the milli-ammeter.

Application of Operating Principles: Relative Advantages and Disadvantages. The *electromagnetic principle of operation* has an extensive application to all kinds of commercial instruments, and is the only operating principle which can be employed for energy meters, power-factor meters, frequency meters, and synchroscopes.

The *electro-dynamic form of construction* is applicable to all classes of measuring instruments, as well as to synchroscopes, galvanometers, and oscillographs. Well-designed electro-dynamic measuring instruments possess the advantage that the instrument readings are practically independent of the wave-form and frequency* of the current passing through the instrument. The instruments can therefore be used on either alternating- or direct-current circuits without re-calibration.

The *moving-iron form of construction* is applicable to ammeters, voltmeters, power-factor meters, frequency meters, synchroscopes, and galvanometers. With measuring instruments, however, the effects of hysteresis and eddy-currents in the iron elements will cause the instrument readings to be affected by the wave-form and frequency of the operating current, but by suitable design the errors due to these causes may be made almost negligible for commercial frequencies and wave-forms. Moving-iron instruments are, in general, inferior to electro-dynamic instruments with respect to accuracy and power consumption, but their first cost is lower than that of either electro-dynamic or induction instruments of similar range and size.

It is important to note that, unless the electromagnetic portion of the mechanism of electro-dynamic and moving-iron instruments is magnetically shielded, the indications are liable to be seriously affected by external magnetic fields, due, for example, to heavy currents in neighbouring conductors.

The *induction form of construction* is applicable to ammeters, voltmeters, wattmeters, energy meters, and frequency meters. Induction measuring instruments are more susceptible to errors due to frequency and wave-form than other forms of measuring

* This statement applies only to frequencies within the range adopted in electric lighting and power supply systems.

instruments, and, in consequence, their application is limited to circuits of constant frequency. These instruments are used principally on switchboards, and are almost immune from the effects of external magnetic fields.

Induction watt-hour meters have entirely superseded the electrodynamic form for the measurement of energy on commercial electric lighting and power circuits.

The *electro-thermic principle* is applicable to current-measuring instruments; it has been applied to oscillographs of the hot-wire type and to galvanometers of the thermo-electric type. Electrothermic ammeters and voltmeters possess the advantage that the instrument readings are independent of the wave-form and frequency of the current passing through the instrument; they can, therefore, be used on either alternating- or direct-current circuits. The instruments, however, possess the disadvantages of sluggishness, relatively high power consumption (particularly in the "expansion" type of instrument), and liability to damage with small overloads.

The *electrostatic principle* is usually confined to instruments for measuring potential difference, although the principle has recently been applied to oscillographs. Electrostatic voltmeters possess two important advantages over other types of alternating-current voltmeters, viz. (1) the power-consumption is negligible; (2) the instruments can be constructed for direct connection to high-voltage circuits. Electrostatic voltmeters also possess the advantage that the instrument readings are independent of frequency and wave-form.

Ammeters and Voltmeters

General Requirements. The indications of alternating-current ammeters and voltmeters must represent the R.M.S. values of the current, or potential difference, respectively, applied to the instruments. The scale divisions, therefore, must be proportional to the mean squared values of the corresponding currents or potential differences. Hence, in the case of an ammeter or a voltmeter, the mean deflecting torque corresponding to a given deflection of the pointer must be proportional to the mean squared value of the current or potential difference applied to the instrument.

Moreover, if the readings are to be correct when the instruments are used on both direct- and alternating-current circuits, the torque must be uninfluenced by both the frequency and the wave-form of the applied current or potential difference.

Hot-wire Ammeters and Voltmeters. In these instruments the deflection of the pointer is produced by the linear expansion of a

wire, or strip, heated by the current (or a definite fraction thereof) to be measured. Under ideal conditions the expansion is proportional to the square of the current, and if the angular deflections of the pointer are proportional to the expansion, the scale divisions will follow a parabolic, or square, law. Moreover, in such a case the instrument will read correctly with direct and alternating currents. But in practice a number of variable factors enter into the relationship between expansion and current, and the scale divisions have to be determined by calibration. If, however, an instrument is calibrated with direct current it will read correctly with alternating current provided that during a period, or half-period, the temperature of the wire remains sensibly constant.

Construction of Hot-wire Instrument (Expansion Type). The mechanism* of a typical hot-wire ammeter or voltmeter of the expansion type, in which the double sag principle is utilized as a multiplying device, is shown diagrammatically in Fig. 215(*a*). The hot-wire, *A*, is usually of platinum-iridium (formerly platinum-silver was employed) and is stretched between supports which are fixed to a metallic base-plate, *B*, of bimetal.

The magnifying device consists of (1) a fine phosphor-bronze wire, *C*, stretched between the hot-wire, *A*, and an insulated pillar fixed to the base-plate ; (2) a silk fibre, *D*, which is attached to *C* near its mid-point, lapped round a small grooved pulley, *E*, fixed to the spindle carrying the pointer, and is maintained in tension by a spring, *F*. The silk fibre is clamped to the pulley to prevent slipping. Hence any change in length of the hot-wire relatively to the base-plate causes an angular movement of the spindle and a deflection of the pointer.

The spindle also carries a light aluminium damping disc, *G*, which moves in the narrow air gap of a horse-shoe magnet, *H*, fixed to the base of the instrument.

Ammeter ranges up to 5 A. are obtained either by varying the diameter of the hot-wire or by dividing the wire into a number of sections connected in parallel. For example, for a range of 0·5 A. the wire has a diameter of 0·005 in., and the whole current traverses the wire ; but for a range of 5 A. the wire has a diameter of 0·01 in. and it is divided electrically into four sections, which are connected in parallel as shown at (*b*) in Fig. 215.

For currents exceeding 5 A. the wire has a diameter of about

* The term *mechanism* applied to instruments refers to the arrangement for producing and controlling the motion of the pointer. It includes all the essential parts necessary to produce this result, but does not include the base, cover, scale, or any other accessories, such as series resistances or shunts, the function of which is to make the readings agree with the scale markings.

0·01 in. and is connected in parallel with a non-inductive shunt contained in the case of the instrument. At radio frequencies, however, the shunt would introduce errors, and for these frequencies a special form of *hot-strip* instrument is employed. In this instrument the current to be measured passes through a number of narrow and thin strips of a material having a high specific resistance. The strips are arranged symmetrically with respect to one another

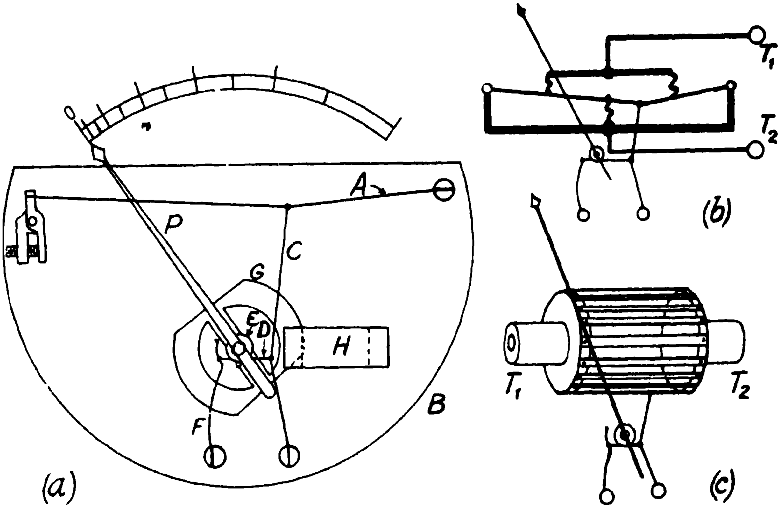


FIG. 215. HOT-WIRE AND HOT-STRIP INSTRUMENTS

and the terminals (see Fig. 215c), and are adjusted to have equal effective inductances and resistances. The distribution of current in the strips will, therefore, be independent of frequency, and the linear expansion of one strip will be a measure of the current in the circuit. This expansion is indicated by a pointer and scale in a manner similar to that adopted in the hot-wire instrument.

Voltmeter ranges up to about 400 V. are obtained by the use of a fine wire (having a diameter of about 0·0015 in.) and a non-inductive series resistance, the current required for full-scale deflection being about 0·25 A.

Thermo-couple Ammeters and Galvanometers. In these instruments the heating effect of the current is utilized indirectly to deflect the pointer. The mechanism of the ammeter consists essentially of: (1) a moving coil of fine copper wire which carries a pointer and is pivoted in the magnetic field of a permanent magnet, the ends of the coil being connected to a thermo-couple; (2) a heater, of the resistance type, which is located close to the thermo-couple and through which passes the current to be measured, or a definite fraction thereof. In the case of the galvanometer (which is

of the reflecting type) the coil consists of a single turn and is suspended by a quartz fibre. The principle of the construction is shown in Fig. 216.

In the pivoted instrument the pivots are arranged inside the moving coil to allow the thermo-couple and heater to be mounted at the end of the coil remote from the pointer. The control consists of a flat spiral spring and the damping is electromagnetic, as in the ordinary type of direct current moving-coil instrument.

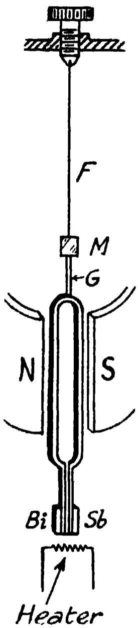


FIG. 216

PRINCIPLE OF THERMO-COUPLE GALVANOMETER

The heater consists either of a short filament of wire or a grid (having an area of about 0·2 sq. cm.) of platinized mica, according to the range of the instrument.

Ammeter ranges from 10 mA. to 100 mA. are obtained by heaters of different resistances, but higher ranges are obtained by shunting the heater with non-inductive shunts. In the galvanometer the sensitivity, with a given heater, may be varied by altering the distance between the thermo-couple and the heater.

The power taken at full-scale deflection by an unshunted instrument is very small (about 0·015 watt), and the instrument will withstand safely an overload of about three times the normal current.

Due to the small dimensions of the heaters, the unshunted instrument and the galvanometer possess extremely small self-inductance and capacity. They are, therefore, particularly suitable for high-frequency measurements. Moreover, the deflections are practically proportional to the mean squared value of the current passing through the heater, and are independent of wave-form and frequency.

The instruments are standardized and calibrated on a direct-current circuit with the aid of standard direct-current instruments.

Form of Scale. In an ideal instrument the whole of the heat produced by the current passing through the heater is radiated to the thermo-junction, and the temperature of the latter is proportional to the square of this current. Now the E.M.F. of a thermo-junction is proportional to its temperature, and therefore the current in the moving coil of the indicator will be proportional to the square of the current in the heater. Hence, since for a permanent magnet moving-coil instrument the deflection is proportional to the current in the moving coil, the deflection in the case of the thermo-ammeter will be proportional to the square of the current in the heater. Conversely, the current (I) in the heater is proportional to the square-root of the deflection (θ) of the indicator, i.e. $I = k\sqrt{\theta}$, where k is a constant. Thus the scale must be divided according to a square law. For example, if the full-scale deflection is 70° and corresponds to a current of 120 mA., the intermediate scale divisions and the corresponding deflections are as follow—

Current (mA.)	120	100	80	60	40	20	0
$\theta°(= 70 \times \text{current}^2/120^2)$	70	48·6	31·1	17·5	7·78	1·945	0

Thermo-electric Ammeters and Galvanometers with Independent Thermo Junctions. Instead of the special construction, described above, in which the thermo-couple and heater form an integral part of the instrument, these parts may be constructed as a separate unit and may then be used in conjunction with an ordinary direct-current moving-coil galvanometer, millivoltmeter, or micro-ammeter. In this case a greater thermo-electric E.M.F. has usually to be provided by the thermo-junction, and either a number of

thermo-couples are connected in series and heated by a common heater, or a single thermo-couple is employed and the temperature of the heater is raised to about 200° C., both heater and thermo-couple being enclosed in a highly-exhausted glass bulb. With these independent thermo-junctions currents exceeding 1 A. may be measured without the use of shunts.

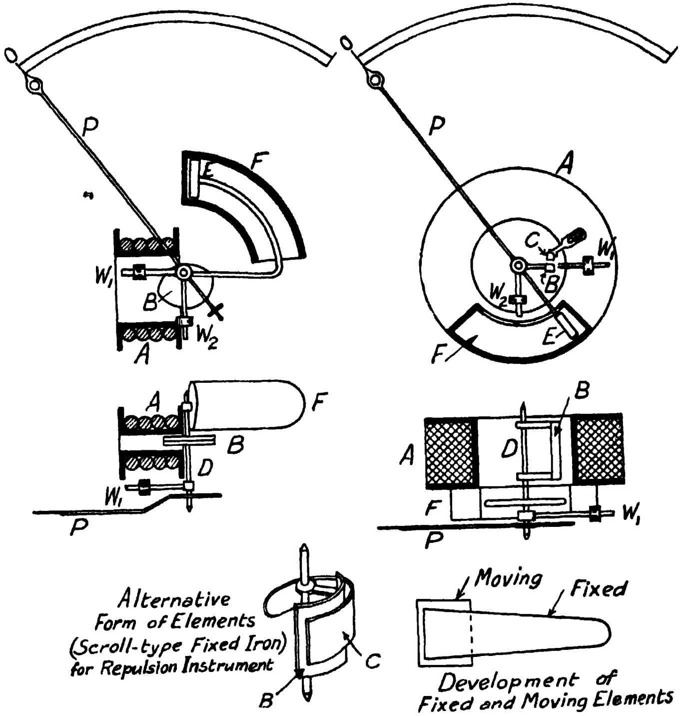


FIG. 217 FIG. 218

ATTRACTION AND REPULSION FORMS OF MOVING IRON AMMETERS AND VOLTMETERS

Construction of Moving-iron Ammeters and Voltmeters. Figs. 217, 218 show the principles of two forms of construction.

In the instrument operating on the attraction principle (Fig. 217), the moving-iron element consists of a few discs, *B*, of soft iron, which are fixed to a spindle, *D*, pivoted in jewelled bearings. The spindle also carries a pointer, *P*, a balance weight, W_1, a controlling weight, W_2, and a damping piston, *E*, which moves in a curved fixed cylinder, *F*. The special shape of the moving-iron discs is for the purpose of obtaining a scale of suitable form.

In the repulsion form of instrument (Fig. 218) the coil A, is cylindrical and is fixed with its magnetic axis perpendicular to the base of the instrument. The moving-iron element, B, is a small bar or rod of soft iron fixed parallel to, and at a small distance from, the spindle, which also carries the pointer, P, and damping vane, E, and is pivoted in jewelled bearings. The fixed iron element, C, is supported by a non-magnetic framework which carries the damping chamber and bearings and forms a clamp for the coil.

Considerable variations in the shape and arrangement of the moving and fixed iron elements are possible and are to be found in commercial instruments. For example, the fixed iron element may be a tongue-shaped piece of thin soft sheet iron bent into cylindrical form and mounted concentric with the spindle. The moving iron may consist of a small piece of similar sheet iron bent to form a cylindric segment and fixed to the spindle so as to move concentrically with respect to the fixed iron tongue. The fixed and moving irons are so arranged that when the pointer is on zero the moving iron is parallel to the broadest part of the surrounding iron tongue, and that as the movement is deflected it moves towards the narrowest part of the tongue. The form of scale depends upon the shape of the fixed iron tongue, and by varying the latter a variety of scale forms are possible.

The instruments may be effectively *shielded* from the influence of external magnetic fields by enclosing the working parts, except the pointer, in a laminated iron cylinder with laminated iron end covers. More generally, however, the complete instrument is enclosed in a cast-iron case which usually gives sufficient shielding for practical purposes.

An interesting form of construction is shown in Fig. 219, a unique feature being that the operating forces are produced by the current in a conductor external to the instrument. The instrument, therefore, has no internal electrical parts or connections, and no terminals.

The essential parts of the instrument comprise: (1) a laminated-iron magnetic circuit A, B, which is provided with an air gap, D, shaped to accommodate the moving-iron elements, E_1, E_2; (2) a pivoted moving system consisting of two specially shaped elements, E_1, E_2, of soft iron, a pointer P, control spring, and damping vane.

The magnetization of the magnetic circuit is produced by current in a single conductor located in the space C, and the outer portion, B, of the magnetic circuit is removable in order that this conductor may be conveniently placed in position. The magnetic circuit is so proportioned that a minimum range of 50 amp. can

be obtained with a single conductor in the space *C*. Higher ranges are obtained by changing the control spring.

The action of the instrument is very simple. Thus when the magnetic circuit, *A*, *B*, is excited, the iron elements, E_1, E_2, tend to move to positions bridging the air gaps, *D*, and so diminishing

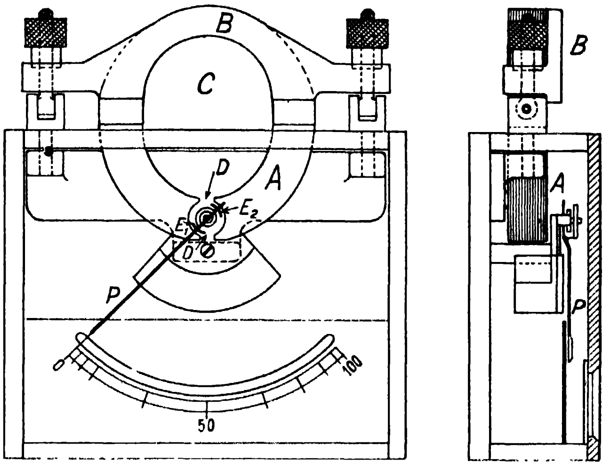


FIG. 219. CROMPTON'S FORM OF MOVING-IRON AMMETER, IN WHICH THE OPERATING TORQUE IS PRODUCED BY CURRENT IN AN EXTERNAL CONDUCTOR

the reluctance of the magnetic circuit. The form of scale depends upon the shape of these elements relative to the air gaps *D*.

Ranges of Ammeters and Voltmeters. For a given moving-iron instrument the ampere-turns necessary to produce full-scale deflection are constant. Hence the ranges of *ammeters* are altered by changing the number of turns and size of conductor in the magnetizing coil. Obviously the maximum range is reached when the coil is wound with one turn. This range is of the order of 300 A.

With *voltmeters* the range may be altered by changing the number of turns, but with a given instrument the range may be increased by connecting a resistance in series with it. Hence the same coil winding specification may be employed for a number of ranges.

Data of Moving-iron Instruments. (1) A commercial moving-iron voltmeter of the type shown in Fig. 217 has a range of 120 volts. The operating coil is wound with 3000 turns of No. 35 S.W.G. copper wire (diameter = 0·0084 in.), the resistance being 140 ohms at 20° C., and the series resistance—of Eureka (constantan) wire—has a resistance of 1060 ohms, giving a total resistance for the instrument of 1200 ohms at 20° C.

The inductance of the operating coil is 0·141 H. with the moving system in its zero position, and 0·1533 H. when the moving system is deflected to the full scale position. The inductances corresponding to intermediate positions of the moving system are given in the following table, together with data from which the form of scale can be determined.

Scale reading (volts)	0	40	60	80	100	120
Deflection (degrees)	0	11·7	29	47·1	61	75
Inductance (henries)	0·141	0·142	0·1455	0·1495	0·1517	0·1533

The following calculated data refer to the operation of the instrument when used on a 50-cycle, 100-volt, circuit of sinusoidal wave-form.

Power loss in instrument $= 100^2/1200 = 8\cdot33$ W.

Reactance of instrument $= 314 \times 0\cdot1517 = 47\cdot6\,\Omega$.

Ratio : reactance/resistance $= 47\cdot6/1200 = 0\cdot0397$

Impedance of instrument at 20° C. $= \sqrt{(1200^2 + 47\cdot6^2)} = 1200\cdot9\,\Omega$.

Power factor $= 1200/1200\cdot9 = 0\cdot99885$

Operating current at 20° C. $= 100/1200\cdot9 = 0\cdot083237$ A.

$$\text{Ratio : } \frac{\text{operating current at 100 V., 50 frequency}}{\text{operating current at 100 V., zero frequency}} = 0\cdot99885$$

Power expended in operating coil at 20° C. $= 140 \times 0\cdot083^2 = 0\cdot97$ W.

Resistance of instrument at 50° C $= 1060 + 140[1 + (50-20)\times 0\cdot0039]$
$= 1216\cdot5\,\Omega$.

Thus the frequency and temperature errors are negligible for practical purposes.

(2) A commercial moving-iron *ammeter*, for a range of 10 amp. and of the type shown in Fig. 217, has the operating coil wound with 29 turns of No. 14 S.W.G. copper wire (diameter = 0·08 in.), the resistance of which at a temperature of 20° C is 0·015 Ω.

The inductances of the instrument with the moving system in a number of positions are given in the following table, together with the corresponding scale markings and angular deflections—

Scale reading (amp.)	0	2	4	6	8	10
Deflection (degrees)	0	6	16	36	56	73
Inductance (μH.) .	16·4	—	16·6	17·1	18·1	19·7

The following calculated data refer to the operation of the instrument, at full-scale reading (10 A.), on an alternating-current circuit of 50 frequency and sinusoidal wave-form—

Power loss = 1·5 W.	Power factor = 0·93
Reactance = 0·006 Ω.	Pressure drop = 0·161V.
Impedance = 0·0161 Ω.	Volt-amperes = 1·61

Compensation for Frequency Error in Moving-iron Voltmeters. The effect of changes in frequency upon the readings of moving-iron voltmeters—as ordinarily arranged with a non-inductive resistance in series with the fixed coil—results in an error which increases as the frequency increases, the error being due to the change of impedance of the instrument with change of frequency.

If, however, an instrument is required to give accurate readings on an A.C. circuit of a particular frequency as well as on a D.C. circuit, the frequency error may be compensated by connecting a condenser in parallel with the series resistance of the instrument as shown in Fig. 220. In this case the condition to be satisfied is that the *numerical* value of the impedance at the particular frequency shall be equal to the total resistance of the instrument.

Thus, if L denotes the inductance of the operating coil corresponding to the scale reading at which compensation is desired; R, the resistance of this coil; R_1, the non-inductive series resistance and C the capacitance of the condenser, then for compensation at a frequency $\omega/2\pi$ we must have

$$R + R_1 = \sqrt{\left[\left(R + \frac{R_1}{1+\omega^2C^2R_1^2}\right)^2 + \omega^2\left(L - \frac{CR_1^2}{1+\omega^2C^2R_1^2}\right)^2\right]}$$

Difficulties arise in the solution of this equation for C, as the fourth power of this quantity is involved.

If, however, the resistance (R) of the operating coil is small in comparison with that of the series resistance (R_1), a close approximation to complete compensation is obtained when

$$R_1 = \sqrt{\left[\left(\frac{R_1}{1+\omega^2C^2R_1^2}\right)^2 + \omega^2\left(L - \frac{CR_1^2}{1+\omega^2C^2R_1^2}\right)^2\right]}$$

the solution of which gives

$$C = \frac{L(\sqrt{[2-(\omega L/R_1)^2]} - 1)}{R_1^2 - \omega^2L^2}$$

Example. A moving-iron voltmeter with a maximum scale reading of 120 V. has a total resistance of 2000Ω of which 200Ω is in the operating coil. The inductance of the latter when the scale reading is 100 is 0·45 H. Calculate the capacitance of the condenser with which the series resistance must be shunted in order that the 100-V. scale reading shall be correct on both D.C. and 50-cycle A.C. circuits.

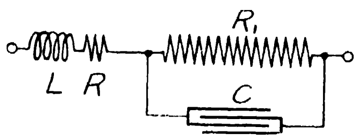


FIG. 220. CONNECTIONS FOR METHOD OF COMPENSATING FREQUENCY ERROR IN MOVING-IRON VOLTMETER

To fulfil the specified conditions the impedance of the instrument at 50 frequency must equal 2000 ohms. Substituting appropriate values for ω, L, R_1, in the above equation we obtain

$$C = 10^6 \times \frac{0{\cdot}45(\sqrt{[2-(314 \times 0{\cdot}45/1800)^2]} - 1)}{1800^2 - (314 \times 0{\cdot}45)^2} = 0{\cdot}0576\mu F$$

It will be of interest to calculate symbolically the impedances of the compensated and uncompensated instrument for a frequency of 50 cycles per second.

The impedance of the compensated instrument is

$$Z = R + \frac{R_1}{1+\omega^2C^2R_1^2} + j\omega\left(L - \frac{CR_1^2}{1+\omega^2C^2R_1^2}\right)$$

Whence $\quad Z = \mathbf{1998{\cdot}1 + j82{\cdot}8}$

and $\quad Z = 2000\ \Omega.$

The impedance of the uncompensated instrument is

$$Z = R + R_1 + j\omega L$$
$$= \mathbf{2000 + j141{\cdot}5}$$

Whence $\quad Z = 2005\ \Omega.$

Electro-dynamic Ammeters and Voltmeters. All electro-dynamic instruments depend for their action upon the dynamic force between adjacent conductors, or coils, carrying electric currents. The application of this principle to measuring instruments is due to

Kelvin and Siemens, and their instruments—the current-balance and the electro-dynamometer—are of the non-deflectional type.

Construction. In deflectional electro-dynamic ammeters and voltmeters the moving coil is wound with fine wire and is mounted on a spindle which carries the pointer, control springs, and damping

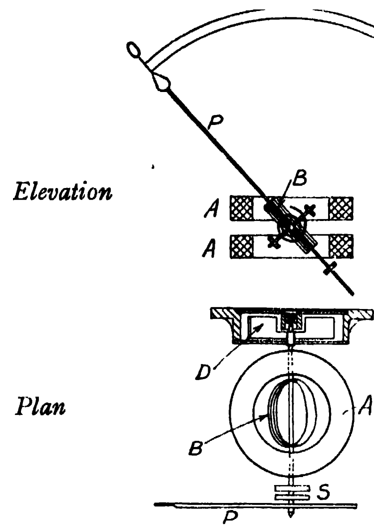


FIG. 221. ARRANGEMENT OF ELECTRO-DYNAMIC INSTRUMENT

A, fixed coils; *B*, moving coil; *D*, damping chamber and vanes; *P*, pointer; *S*, control springs.]

NOTE. The mechanism of an actual instrument is shown in Fig. 234.

vane or piston. This coil is usually pivoted within a pair of coaxial fixed coils (Fig. 221), which, in an ammeter, are wound with thick wire, and are connected in parallel with the moving coil. In a voltmeter the fixed coils are wound with thin wire and are connected in series with the moving coil and a non-inductive resistance.

In general, no iron is employed in the magnetic circuits of the coils, but instruments have been constructed in which a laminated-iron magnetic circuit for the operating coils forms an essential part of the instrument.

The controlling force is supplied by a pair of flat spiral springs, which also act as the leading-in connections for the moving coil. The damping is pneumatic.

Shielding. Electro-dynamic instruments in which iron does not form an essential part of the operating mechanism may be effectively shielded from the effects of external magnetic fields by enclosing

the mechanism (except the tip of the pointer) in a laminated iron hollow cylinder with closed ends. When using such instruments on direct-current circuits, the current through the instrument should be reversed and the mean of the two readings taken. In this manner the effect of the magnetic condition of the shield on the instrument readings is eliminated.

Ranges of Ammeters and Voltmeters. A given size of instruments requires a definite number of ampere-turns to be supplied by the fixed and moving coils to obtain a full-scale deflection. Hence, with *milli-ammeters,* in which the fixed and moving coils are connected in series, the ranges are altered by changing the number of turns and size of conductor in the fixed and moving coils.

With *ammeters* in which only a fraction of the rated current is carried by the moving coil, the range is altered by changing the fixed coils, and in instruments in which two fixed coils are employed (Fig. 221), a double-range instrument may be obtained by connecting these coils either in series or in parallel, the internal connections (which are so arranged that the changes may be effected by either plugs or links) being shown in Fig. 222. The maximum range for which ammeters are usually constructed is 200 A.

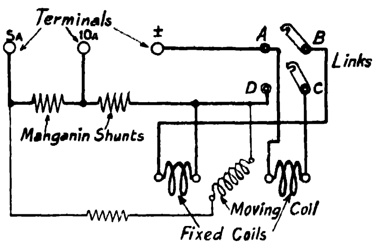


FIG. 222. CONNECTIONS OF DOUBLE-RANGE ELECTRO-DYNAMIC AMMETER

[For higher range, connect A-B, C-D by links; for lower range, connect B-C by links.]

With *voltmeters* the range is altered by changing the number of turns in the coils and the value of the series resistance, but the range of a given instrument may be increased by connecting additional resistance in series with it. For example, the range of a given voltmeter may be doubled by connecting in series with it a non-inductive resistance equal in value to the original resistance of the instrument. The loss in the instrument and series resistance is thereby doubled for a given scale reading.

Theory of Electro-dynamic Ammeters and Voltmeters. With instruments in which the moving coil is circular and is pivoted centrally in a short fixed solenoid or coil, as in Fig. 221, the torque acting upon the moving coil can be easily calculated if the flux density is assumed to be constant throughout the space occupied by the moving coil. Denoting this flux density by B, the current in the moving coil by I_2, the force acting upon an element, of length $r d\alpha$, subtending an angle $d\alpha$ with respect to the centre of the coil, Fig 223, is given by $dF = k_1 B I_2 \, r d\alpha \sin \alpha$, where k_1 is a constant involving the system of units, and α denotes the angular position of the element with

respect to the pivotal axis of the coil. Hence, the torque due to the element is given by

$$d\mathfrak{T}_d = dF\, r \sin a \sin(\beta + \theta) = k_1 B I_2 r^2\, da \sin^2 a \sin(\beta + \theta),$$

where θ is the angular deflection of the coil from its zero position, and β is the angle between the plane containing the zero position of the moving coil and the plane which contains the pivotal axis and is perpendicular to the flux. Therefore the torque for the whole coil of n_2 turns is

$$\mathfrak{T}_d = 4n_2 k_1 B I_2 r^2 \sin(\beta + \theta) \times \int_0^{\frac{1}{2}\pi} \sin^2 a \, . \, da = k_1 n_2 \pi r^2 B I_2 \sin(\beta + \theta)$$

since $\int_0^{\frac{1}{2}\pi} \sin^2 a\, da = \pi/4.$

Now $\pi r^2 B$ is equal to the flux (Φ) linked with the moving coil when its magnetic axis coincides with that of the fixed coil (i.e. when $(\beta + \theta) = 0$ and $(\beta + \theta) = \pi$) and if M_M denotes the mutual inductance under these conditions, $\Phi = M_M I_1 \times 10^8/n_2$.

Therefore the torque is given by

$$\mathfrak{T}_d = k_2 M_M I_1 I_2 \sin(\beta + \theta)$$
$$= k_2 M_M I^2 \sin(\beta + \theta) \quad . \quad . \quad (176)$$

where $k_2 = k_1 \times 10^{-8}$, and $I = I_1 = I_2$, i.e. the coils are connected in series.

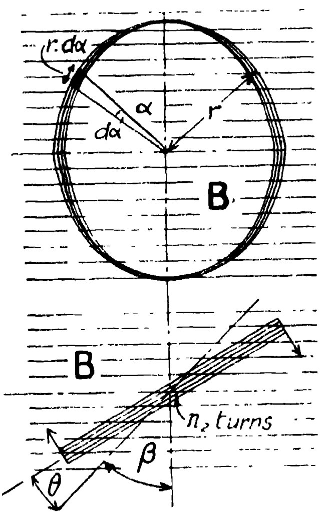


FIG. 223. PERTAINING TO THEORY OF ELECTRO-DYNAMIC INSTRUMENT

Now, the mutual inductance of the coils corresponding to a deflection θ is

$$M_{(\beta+\theta)} = - M_M \cos(\beta + \theta),$$

since, when $\beta + \theta = 0$, the mutual inductance $= - M_M$. Whence the rate of change of mutual inductance with respect to the deflection is

$$\frac{dM_{(\beta+\theta)}}{d(\beta+\theta)} = - M_M \frac{d \cos(\beta+\theta)}{d(\beta+\theta)} = M_M \sin(\beta + \theta)$$

Therefore equation (176) may be written in the form

$$\mathfrak{T}_d = k_2 I_1 I_2 \frac{dM_{(\beta+\theta)}}{d(\beta+\theta)} \qquad (176a)$$

Form of Scale. If the control is due to flat spiral springs, the controlling torque is proportional to the angle of deflection, θ, and for equilibrium,

or $\quad k_2 M_M I^2 \sin(\beta + \theta) = k_3 \theta.$

Whence $\quad I^2 = k \dfrac{\theta}{M_M \sin(\beta+\theta)} = k \dfrac{\theta}{dM_{(\beta+\theta)}/d(\beta+\theta)}$

e. $\quad I = k' \sqrt{\dfrac{\theta}{\sin(\beta+\theta)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$

Typical scales calculated from this equation are shown in Fig. 224.

Conditions for Instrument Readings to be Unaffected by Frequency and Wave-form. The principal conditions are: (1) the currents in the fixed and moving coils must either be equal or have a constant common ratio; (2) eddy currents in the coil supports and conductors must be reduced to a minimum; (3) the reactance of the instrument, when used as a voltmeter, must be very small relatively to its resistance, and the latter must be constant at all temperatures.

With *voltmeters* these conditions are satisfied by (i) connecting the fixed and moving coils in series; (ii) designing these coils for a pressure drop which is only a small fraction of the range of the instrument; and (iii) connecting in

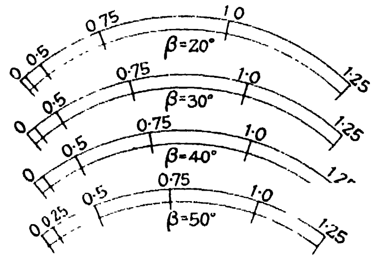


FIG. 224. THEORETICAL SCALE-FORMS FOR ELECTRO-DYNAMIC AMMETER

series with them a non-inductive resistance having a zero temperature-resistivity coefficient.

With *ammeters* for ranges above about 250 mA. the moving coil cannot be connected in series with the fixed coil (on account of the control springs being unsuitable for currents above about 250 mA.). Therefore the moving coil

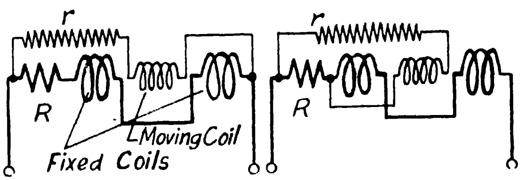


FIG. 225. ALTERNATIVE METHODS OF CONNECTING FIXED AND MOVING COILS IN ELECTRO-DYNAMIC AMMETER

must be connected either in parallel with the fixed coils, Fig. 225(*a*), or across a shunt which is connected in series with the fixed coils, Fig. 225(*b*).

In the instrument with a shunted moving coil, Fig. 225(*b*), the shunt is designed for a relatively large pressure drop (about 0·5 V. in a 5 A. instrument) and only a fraction of this pressure is utilized for operating the moving coil, a non-inductive resistance being connected in series with the latter. Both shunt and non-inductive series resistance are constructed of materials having zero temperature-resistivity coefficients. For extreme accuracy the shunt should be adjusted to have the same ratio of resistance to inductance as the moving-coil circuit.

With the instrument in which the moving coil is connected in parallel with the fixed coils, Fig. 225(*a*), the conditions which must be fulfilled are: (1) the ratio (resistance/reactance) must have the same value for each branch; (2) the percentage change of resistance with temperature must be the same for the two branches.

To satisfy the first condition we must have

$$\omega L_1/R_1 = \omega L_2/R_2, \text{ or } L_1/L_2 = R_1/R_2,$$

where L_1, L_2, are the effective inductances of the fixed- and moving-coil circuits, respectively, and R_1, R_2, the resistances of these circuits.

Now

$$L_1 = L_f \pm M, \quad L_2 = L_m \pm M,$$

where L_f, L_m, denote the true self-inductances of the fixed- and moving-coil circuits, and M denotes their mutual inductance.

Hence for the ratio L_1/L_2 to be constant when $L_f \neq L_m$, M must be either zero or constant. Alternatively, if M is variable, L_f and L_m must be equal.

With all deflectional instruments, however, the mutual inductance varies with the relative positions of the moving and fixed coils. When the axes of these coils are perpendicular to each other the mutual inductance is zero; for other positions, to the right or left of this position, the mutual inductance increases as the angular displacement between the coils increases, and may have either positive or negative values.

In order that the ratio of currents in the fixed and moving coils shall be unaffected by temperature variations, the percentage change of resistance with temperature must be the same for both circuits. This result is best obtained by connecting in series with the coils (which are wound with copper conductors), resistances having zero resistivity-temperature coefficients.

Thus, both forms of electro-dynamic ammeters must have relatively large losses (5 to 7 W. at full scale) if the readings are to be unaffected by variations of frequency and temperature.

Data of Electro-dynamic Instruments. A *voltmeter* has a range of 120 volts and a resistance of 1550 ohms at 20° C., of which 77 ohms is due to the resistances of the fixed and moving coils (which are wound with copper wire) and the remainder—1473 ohms—is a non-inductive resistance of Eureka.

The inductances of the instrument (measured at 50 frequency) for a number of positions of the moving system are given in the following table, together with the corresponding scale markings and angular deflections.

Scale reading (volts)	0	40	60	80	100	120
Angular deflection (degrees)	0	7	13·8	24	37·1	54
Inductance (mH.)	70·1	72·5	74·8	78·3	82·8	88·6

The inductance of the fixed coil is 74·5 mH., and that of the moving coil is 2·2 mH.

The mutual inductance of the fixed and moving coil circuits may be calculated from the above data. Thus, if the mutual inductance is denoted by M, the self-inductances of the fixed and moving coils by L_f, L_m, respectively, and the self-inductance of the instrument by L, we have

$$L = L_f + L_m \pm 2M$$

Whence $M = \frac{1}{2}[L - (L_f + L_m)]$

The calculated values of M for different positions of the moving system are

Scale reading (volts)	0	40	60	80	100	120
Mutual inductance (mH)	− 3·3	− 2·1	− 0·95	+ 0·8	+ 3·05	+ 5·95

These values, when plotted against angular deflection, give a straight line (i.e. $dM/d\theta$ is constant), and theoretically the instrument should have a "square-law" scale. The actual scale closely follows the square law.

The following calculated data refer to the operation of the instrument when used on a 100-volt D.C. circuit and a 50-cycle, 100-volt A.C. circuit.

Power loss in instrument $= 100^2/1550 = 6{\cdot}45$ W.

Reactance of instrument $= 314 \times 0{\cdot}0828 = 26\,\Omega$.

Impedance of instrument at 20° C. $= \sqrt{(1550^2 + 26^2)} = 1550{\cdot}22\ \Omega$.

Operating current at 20° C., 50 frequency = 0·064491 A.

Operating current at 20° C., zero frequency = 0·0645 A.

$$\text{Ratio}: \frac{\text{operating current at 100 V., 50 frequency}}{\text{operating current at 100 V., zero frequency}} = 0{\cdot}99986$$

Hence when the instrument is used on the alternating-current circuit the error is only − 0·014 per cent.

A 5 A. precision *ammeter* of the type illustrated in Fig. 221, with the internal connections arranged as in Fig. 225 (*b*), has a resistance between terminals of 0·188 Ω. at 20° C., of which 0·073 Ω. is due to the fixed coils, and the remainder is the joint resistance of the parallel circuit formed by the moving coil and the manganin shunt. The moving coil itself has a resistance of 0·7 Ω., and the manganin resistance connected in series with it has a resistance of 4·1 Ω. The resistance of the shunt is 0·118 Ω.

The self-inductance of the fixed coils is 0·16 mH., and that of the moving coil is about 1 μH. The mutual inductance of the fixed and moving coils is of the order of 1 μH. Data from which the form of scale may be determined are as follow—

Scale reading (amp.)	0	1	2	3	3·5	4	4·5	5
Deflection (degrees) .	0	2·5	12·7	32·8	45·5	59·5	74	86

The following calculated data refer to the operation of the instrument at full-scale reading (5 A.) on an alternating-current circuit of 50 frequency and sinusoidal wave-form—

Power loss	4·7 W.
Reactance (assuming L = 0·16 mH.) . .	0·05 Ω.
Impedance (,, ,,) . .	0·1945 Ω.
Pressure drop	0·94 V.

Induction Ammeters and Voltmeters. In these instruments the deflecting torque is due to eddy currents induced in a pivoted disc, or drum, by a shifting or travelling magnetic field produced by an alternating-current electromagnet. With both forms of instrument, spring control and electromagnetic (eddy-current) damping are employed, the magnetic field for damping purposes being supplied by a permanent magnet.

*Construction of Disc Instrument.** The arrangement of the essential parts of a disc instrument is shown in Fig. 226. The aluminium disc, C, is fixed to the spindle which carries the pointer, and is pivoted in jewelled bearings. The operating electromagnets, A, B, are mounted adjacent to each other with their pole faces parallel to, and on either side of, the disc. The magnets have series windings for an ammeter and shunt windings for a voltmeter. Both windings are connected in series, but one winding is shunted with a non-inductive resistance so as to produce a phase difference of about

* This instrument (due to Ockenden and made by Messrs. Everett-Edgcumbe) is of modern design, with special compensation for frequency and temperature.

60° between the fluxes of the magnets. A travelling field is, therefore, produced in the air gaps, and this field cuts the disc. The resulting eddy currents react with the field and produce torque, thereby causing the disc to move against the restraining force of the control spring, the movement being damped by the permanent magnet D. The disc is shaped so as to make the torque almost proportional to the deflection over a large portion of the 300° scale.

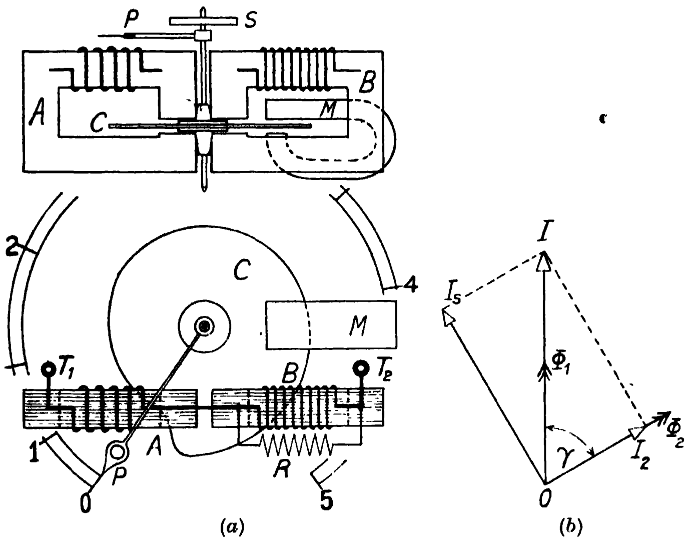


FIG. 226. ARRANGEMENT OF ELECTRIC AND MAGNETIC CIRCUITS (*a*) AND VECTOR DIAGRAM (*b*) OF DISC-TYPE INDUCTION AMMETER

(*Everett, Edgcumbe & Co.*)

This result is obtained by arranging that as the flux in the magnets increases (due to an increase in current), the surface of the disc between the pole faces decreases approximately in proportion to the increase in the flux.

The vector diagram is shown in Fig. 226 (*b*). OI represents the current in the unshunted winding; $O\Phi_1$, the corresponding flux; OI_2, the current in the shunted winding; $O\Phi_2$, the corresponding flux; and OI_s, the current in the non-inductive shunt.

Errors due to variation of frequency and temperature are compensated by the design of the shunted magnet, which is so arranged that—(1) an increase in frequency produces a corresponding decrease in the flux cutting the disc, without affecting appreciably the phase difference, thereby neutralizing the tendency for the E.M.F. induced

in the disc to increase due to the increased frequency; (2) an increase in temperature causes the same percentage variation of resistance in both disc and shunt, this result being obtained by making the shunt of the same material as the disc.

Construction of Drum Instrument. In the Lipman form of instrument, Fig. 227, a thin cylinder or drum, A, of aluminium, is pivoted in the air-gap between the poles of external and internal electromagnets, B and C, respectively. A permanent magnet, D, provides damping.

The external electromagnet is wound with both primary and secondary windings, E, F, respectively. The primary winding, E, is designed for series connection in an ammeter and for shunt connection (together with a suitable series resistance) in a voltmeter. The secondary winding, F, is of low resistance and is connected directly to the magnetizing winding, G, on the inner electromagnet.

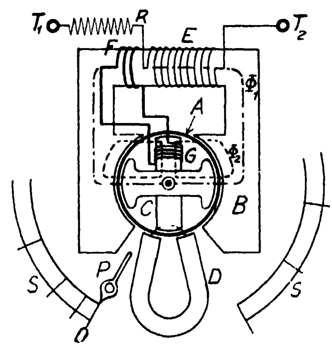


FIG. 227. ARRANGEMENT OF ELECTRIC AND MAGNETIC CIRCUITS OF DRUM-TYPE INDUCTION VOLTMETER

(*Nalder Bros. & Thompson*)

The flux produced by the external magnet, B, is proportional to, and is practically in phase with, the current in the primary winding. This flux takes the path (shown by the chain-dotted line) through the unwound poles of the inner magnet.

The flux produced by the inner magnet is proportional to, and in phase with, the current in the magnetizing winding G. Since this current is obtained by transformer action, it will have a phase difference of nearly 90° from the current in the primary winding. Hence a travelling field is produced in the air-gap between the poles of the internal and external magnets, and a torque is produced on the drum.

Ranges of Ammeters and Voltmeters. Since the deflecting torque is due to an electromagnet, a definite number of ampere-turns are required, with a given instrument, to obtain a full-scale deflection. Hence, with ammeters, the number of turns and size of conductor in the exciting coils must be chosen with reference to the range required. The maximum range for a direct-connected instrument of the above types is about 25 A., above which low range (5 ampere) instruments must be employed in conjunction with current transformers.

With voltmeters, the exciting coil is wound for a lower voltage than that corresponding to the maximum scale reading, and a non-inductive resistance is connected in series with the winding for the purpose of reducing errors due to variations of frequency and temperature.

Data of Induction Instruments. (1) The following data refer to a 50-cycle ammeter, range 0-5 A., of the type shown in Fig. 226.

Scale reading (amp.)	0	1	2	3	4	5
Deflection (degrees)	0	20	80	158	227	294

At full-scale reading—

Voltage at instrument terminals = 1·5 V. Power loss = 5·5 W.
Impedance = 0·3 Ω. Volt-amperes = 7·5.

(2) The following data refer to a 50-cycle voltmeter, range 0-120 V., of the type shown in Fig. 227.

Scale reading (volts)	0	40	60	80	100	110	120
Deflection (degrees)	0	32	78	138	219	255	280

Resistance of primary winding (20° C.) = 79 Ω; total resistance of instrument = 1150 Ω.

At full-scale reading—

Operating current = 100 mA. Power loss = 11·5 W.
Impedance = 1200 Ω. Volt-amperes = 12.

Theory of Disc Type Induction Ammeters and Voltmeters. The theory of these instruments may be developed fairly simply by assuming the fluxes to vary sinusoidally; by neglecting the effects of magnetic saturation, hysteresis, and core loss in the operating electromagnet; and by assuming that the paths of the current in the disc are unrestricted by its dimensions.

Torque. To obtain an expression for the torque, it is first necessary to determine the resultant force acting upon the disc. This force is the difference between the forces due to the interaction of (1) the flux in the shunted pole and the induced current due to the flux in the unshunted pole; and (2) the flux in the unshunted pole and the induced current due to the flux in the shunted pole. Obviously, only the currents in the portion of the disc under the pole faces need be considered.

The E.M.F. induced in the disc by the flux of the unshunted pole is

$$e_1 = -10^{-8}\, d\Phi_1/dt = \omega\Phi_{1m} \times 10^{-8} \sin(\omega t - \tfrac{1}{2}\pi),$$

and that due to the flux in the shunted pole is

$$e_2 = -10^{-8} \times d\Phi_2/dt = \omega\Phi_{2m} \times 10^{-8} \sin(\omega t - \gamma - \tfrac{1}{2}\pi).$$

Each of these E.M.Fs. may be considered to act independently in producing currents in the disc. Assuming each current to circulate in a circular path concentric with the pole face at which the inducing flux is produced, and neglecting the inductance of these paths, the current in an element of the disc (see Fig. 228) under the shunted pole is

$$di_1 = e_1/(2\pi x\rho/\delta\, dx),$$

where dx is the width of the element, x its radius, δ the thickness of the disc (assumed to be uniform), and ρ the specific resistance. Hence the current in the portion of disc under the shunted pole is given by

$$i_1 = \int_{a - \frac{1}{2}b_2}^{a + \frac{1}{2}b_2} di_1 = \frac{e_1\delta}{2\pi\rho}\int_{a-\frac{1}{2}b}^{a+\frac{1}{2}b} \frac{dx}{x}$$

$$= \frac{e_1\delta}{2\pi\rho}\log_e \frac{a + \frac{1}{2}b}{a - \frac{1}{2}b} = k_1 e_1/\rho,$$

where a is the distance between the centres of the poles and b is the breadth of each pole-face.

Similarly, the current in the portion of the disc under the unshunted pole is

$$i_2 = \frac{e_2\delta}{2\pi}\log_e\frac{a+\frac{1}{2}b}{a-\frac{1}{2}b} = k_1e_2/\rho,$$

Substituting for e_1 and e_2 in terms of Φ_{1m} and Φ_{2m}, we have

$$i_1 = \frac{kf\Phi_{1m}}{\rho}\sin(\omega t - \tfrac{1}{2}\pi),$$

$$i_2 = \frac{kf\,\Phi_{2m}}{\rho}\sin(\omega t - \gamma - \tfrac{1}{2}\pi),$$

where f is the frequency ($= \omega/2\pi$) and $k = 2\pi k_1 \times 10^{-8}$.

The instantaneous values of the forces due to the interaction of these currents and the fluxes are

$$F_1 = \tfrac{1}{10}\, i_2\, \Phi_1/bd$$

for the portion of the disc under the unshunted pole, and

$$F_2 = \tfrac{1}{10}\, i_1\, \Phi_2/bd$$

for the portion of the disc under the shunted pole, d being the transverse width of each pole face.

$$\begin{aligned}
F_r = F_2 - F_1 &= k_2(f/\rho)\Phi_{1m}\,\Phi_{2m}[\sin(\omega t-\gamma)\sin(\omega t - \tfrac{1}{2}\pi) - \sin\omega t\sin(\omega t-\gamma-\tfrac{1}{2}\pi)]\\
&= k_2(f/\rho)\Phi_{1m}\,\Phi_{2m}[-\sin(\omega t-\gamma)\cos\omega t + \sin\omega t\cos(\omega t-\gamma)]\\
&= k_2(f/\rho)\Phi_{1m}\,\Phi_{2m}[-\cos\omega t\,(\sin\omega t\cos\gamma - \cos\omega t\sin\gamma)\\
&\qquad + \sin\omega t\,(\cos\omega t\cos\gamma + \sin\omega t\sin\gamma)]\\
&= k_2(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin\gamma
\end{aligned}$$

Therefore the torque acting upon the disc is

$$\mathfrak{T}_d = k'(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin\gamma \quad . \quad . \quad . \quad . \quad . \quad . \qquad (178)$$

where the constant k' involves k_1 and the radius of the resultant force with respect to the pivotal axis of the disc.

Observe that the time angle ωt does not appear in this expression, and therefore the resultant force does not alternate or pulsate but has a constant value for given conditions.

In an uncompensated instrument, Φ_1 and Φ_2 are independent of f and ρ,

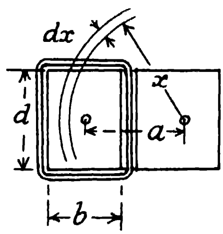


FIG. 228. PERTAINING TO CALCULATION OF CURRENTS IN DISC OF INDUCTION INSTRUMENT

so that the readings of such an instrument would be affected by variations of frequency and temperature.

The effect upon the torque of the inductance of the current paths in the disc may be allowed for in the following manner: Let L denote the inductance corresponding to the mean current path in each of the zones already considered. Then the impedance of each current path is equal to

$$\sqrt{(R^2 + \omega^2L^2)} = R\sqrt{(1 + \omega^2L^2/R^2)} = R\sqrt{(1+\omega^2\tau^2)},$$

where $\tau = L/R$ and R_3 is the resistance of each current path ($= e_1/i_1 = e_2/i_2$). Hence, if φ is the phase difference between the currents and the E.M.Fs., the torque is now given by

$$\frac{k'f}{\rho(1+\omega^2\tau^2)}\,\Phi_{1m}\,\Phi_{2m}\sin\gamma \qquad (178a)$$

The value of τ, however, is usually such as to make the term $\omega^2\tau^2$ small in comparison with unity for the range of commercial supply frequencies, and under these conditions the inductance of the current paths in the disc has little effect upon the torque.

Form of Scale. Considering the case of constant frequency and temperature, the deflecting torque, as given by equation (178), may be written

$$\mathfrak{T}_d = k_d \, \Phi_{1m} \, \Phi_{2m} \sin \gamma$$

Since the controlling torque is due to a spring, we have $\mathfrak{T}_c = k_c \, \theta$, where θ is the angle of deflection. Hence, for equilibrium, $\mathfrak{T}_d = \mathfrak{T}_c$, or

$$k_d \, \Phi_{1m} \, \Phi_{2m} \sin \gamma = k_c \theta.$$

Now when the effects of magnetic saturation, hysteresis, and eddy currents in the operating magnet are ignored, we may write $\Phi_{1m} \Phi_{2m} \sin \gamma = kI^2$, where I is the current. Whence, from the preceding equation,

$$\theta = kI^2$$

or

$$I = k\sqrt{\theta}.$$

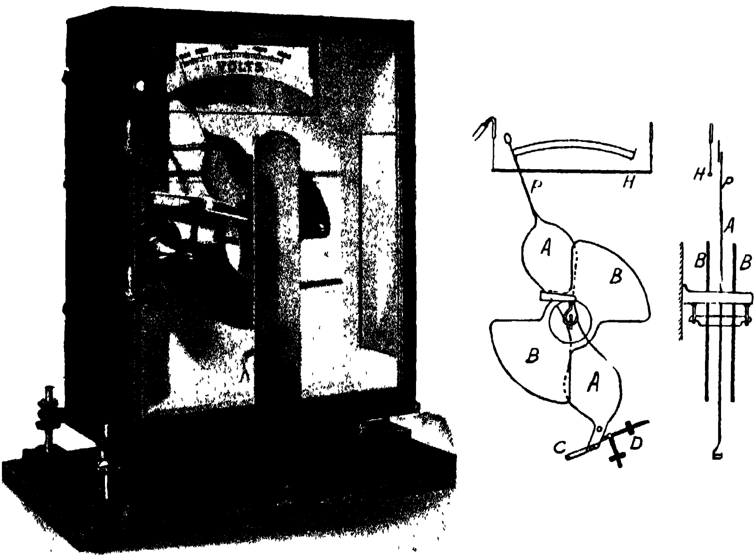
Thus the scale must be divided according to a parabolic law, and, in consequence, the divisions at the upper limit of the scale are considerably more extended than those at the lower limit. In order to obtain a more uniform scale, the disc is cam-shaped, so that the surface of the portion acted upon by the operating magnet decreases as the deflection increases.

Electrostatic Voltmeters. In these instruments the deflecting torque is due to the electric force (attraction or repulsion) between two charged conductors. The instruments, therefore, are free from errors due to magnetic and heating effects, frequency, and wave-form. Moreover, by suitable design, electrostatic instruments may be employed for the direct measurement of high voltages with a high degree of accuracy and with an almost negligible expenditure of energy. Electrostatic instruments, particularly those for low-voltage circuits, are characterized by the smallness of the operating torque in comparison with that of electromagnetic instruments, and, in consequence, special features are necessary to avoid errors due to pivot friction ; for example, the moving system may be supported by a unifilar or bifilar suspension instead of being pivoted, or, alternatively, knife-edge supports may be employed instead of pivots.

Construction. The majority of electrostatic voltmeters are virtually modified forms of the Kelvin electrometers (quadrant and attracted-disc, or absolute, types). The simplest form of Kelvin instrument (Fig. 229), which is suitable for pressures from about 800 to 10,000 volts, is a modified form of quadrant electrometer. It consists essentially of a single flat paddle-shaped aluminium "needle," A, which is connected to one terminal and is supported on knife edges, so as to swing between two vertical quadrant plates, B, which are both connected to the other terminal of the instrument. A pointer, P, is attached to the upper extremity of the

needle, and the lower extremity is extended into a curved arm, C, to which the lever, D, carrying the control and balance weights is attached. Carefully adjusted weights (supplied with the instrument) may also be hung from the extremity of the arm so as to increase the controlling force and thereby extend the range of the instrument (which is designed for test-room work).

The damping device is operated by the observer and consists of a stiff horizontal wire, H, which can be brought lightly against



(a) (b)

FIG. 229. KELVIN ELECTROSTATIC VOLTMETER

(a) View of instrument showing tinfoil screen; (b) Diagram showing essential parts

(*Kelvin, Bottomley & Baird*)

the back of the pointer and thereby damp the movements by mechanical friction. This wire is suspended so as to hang normally clear of the pointer and is operated from the outside of the case by an insulated lever.

Low-voltage instruments, in which flat needles are employed, are of the multi-cellular type with a suspended moving system Twelve or more needles, and a corresponding number of pairs of fixed quadrants, may be necessary, according to the range. The suspension usually consists of a fine phosphor-bronze wire or strip which supplies the control. Liquid damping is employed.

An alternative construction (due to Ayrton and Mather) employs a cylindrical "needle" and fixed cylindrical segments concentric with the needle. The needle is mounted vertically in pivots which enable a very small clearance to be adopted between the moving and fixed parts. Spring control and pneumatic damping are employed.

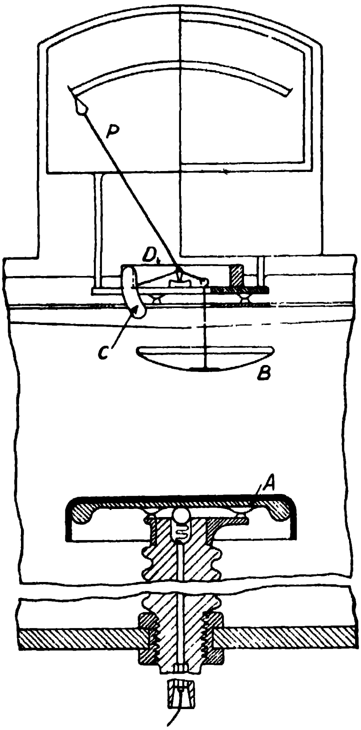


FIG. 230. EXTRA HIGH-VOLTAGE ELECTROSTATIC VOLTMETER
(*Kelvin, Bottomley & Baird*)

Extra-high voltage instruments (for measuring voltages above 20,000 V.) are usually modifications of the absolute, or attracted disc, electrometer. Special precautions must be taken in the design to shield the working parts from the influence of neighbouring conductors, to avoid the formation of brush discharges, and to secure adequate insulation.

The essential parts of one form of instrument are shown diagrammatically in Fig. 230. The fixed disc, A, is of aluminium, the edges being turned-up and rounded. It is mounted horizontally upon an insulating pillar fixed to a substantial slate base The moving disc, B, is dished to a spherical shape and is suspended from a lever fixed to a horizontal spindle which carries the pointer, P, and an aluminium sector, C, which moves between the poles of a permanent magnet, D, and provides the damping. Gravity control is employed, and the whole of the working parts are enclosed in a metal case, which is provided with a small window for observing the scale.

Theory of Electrostatic Voltmeters. Consider an instrument of the quadrant type having a single needle and a pair of double quadrants arranged symmetrically with respect to the needle. The instrument is then equivalent to a condenser, the capacitance of which varies with the deflection of the needle. Let C be the capacitance when the deflection is θ and the voltage between needle and quadrants is E. Then the energy of the system is given by $W = \frac{1}{2}CE^2$. Let the deflection be now increased by $\delta\theta$ due to an increase of

voltage δE, and let δC be the change in capacitance. Then the increment in energy is

$$\delta W = \tfrac{1}{2}\delta C(E + \delta E)^2.$$

Hence if $\mathfrak{T}_d$ is the torque corresponding to the deflection θ and $\delta\mathfrak{T}_d$ the increment in torque, the work done in increasing the deflection of the system is

$$\delta W' = (\mathfrak{T}_d + \delta\mathfrak{T}_d)\delta\theta.$$

Whence $$(\mathfrak{T}_d + \delta\mathfrak{T}_d)\delta\theta = \tfrac{1}{2}\delta C(E + \delta E)^2,$$

or $$\mathfrak{T}_d + \delta\mathfrak{T}_d = \tfrac{1}{2}(E + \delta E)^2\, \delta C/\delta\theta,$$

and, in the limit,

$$\mathfrak{T}_d = \tfrac{1}{2}E^2 dC/d\theta$$

Multipliers for Electrostatic Voltmeters The range of a low-voltage instrument may be extended by means of a *resistance*

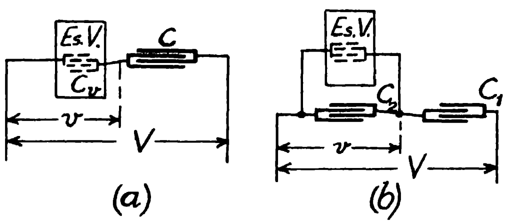


FIG. 231. ALTERNATIVE FORMS OF CONDENSER-MULTIPLIER FOR ELECTROSTATIC VOLTMETER

multiplier (or potential divider), in which the voltmeter is connected across a definite fraction of a high non-inductive resistance (of from 10,000 to 100,000 Ω), which is connected across the supply circuit.

For high-voltage circuits *condenser-multipliers* are employed. Two methods of connection are possible; in one case, Fig. 231 (*a*), a condenser of suitable capacitance is connected in series with the voltmeter; in the other case, Fig. 231 (*b*), two or more condensers are connected in series across the supply circuit and the voltmeter is connected across one of them.

With the first method, the additional capacitance required is extremely small and may only be a fraction of the capacitance of the voltmeter itself. In consequence, the multiplying factor will vary with the deflection of the voltmeter owing to the variation of the capacitance of the latter.

The capacitance of the condenser-multiplier is easily determined. Thus, if the reading, v, of a voltmeter is to represent the voltage, V,

and the capacitance of the voltmeter at this reading is C_v, the capacitance, C, of the condenser-multiplier is given by

$$\omega C_v v = \omega\left(\frac{1}{1/C + 1/C_v}\right) V$$

or

$$\frac{v}{V} = \frac{C}{C + C_v}$$

whence

$$C = C_v/(V/v - 1)$$

For example, if the "100" reading of a 120 V. voltmeter is to represent 10,000 volts, and the capacitance of the voltmeter at this reading is 70 $\mu\mu$F., the capacitance required for the condenser-multiplier is equal to [70/(100 - 1) =] 0·707 $\mu\mu$F.

With the alternative method shown in Fig. 231 (*b*), the effect of the variation of the capacitance of the voltmeter may be made extremely small by arranging that the capacitance of the condenser across which the voltmeter is connected is large in comparison with that of the voltmeter itself. Under these conditions,

$$C_1 = C_2/(V/v - 1).$$

For example, if $C_2 = 7\ \mu$F. (i.e. 100 times the capacitance of the voltmeter in the preceding case), and $V/v = 100$,

$$C_1 = 7/(100 - 1) = 0{\cdot}707\ \mu\text{F}$$

With all condenser-multipliers it is highly important that the condensers have low dielectric losses and high insulation resistance.

Rectifier Voltmeters. Electromagnetic voltmeters, on account of their relatively large operating currents are entirely unsuitable for voltage measurements in high-impedance circuits on account of the disturbing effect of the instrument on the circuit. Although electrostatic instruments are free from this objection, they cannot be employed for measuring low voltages. Moreover, the form of scale is such that accurate readings cannot be obtained when the voltage to be measured is less than about one-third of that corresponding to full scale.

These difficulties are surmounted in the rectifier-type of voltmeter. This instrument consists of a sensitive direct-current milliammeter or micro-ammeter, of the permanent-magnet moving-coil type, combined with a full-wave metal rectifier, of the copper-oxide type, and suitable series resistances. The connections are shown in Fig. 232.

The rectifier is a specially small size of the ordinary copper-oxide rectifier, and must be suitable for the maximum operating current of the milli-ammeter. This current should not exceed 10 mA., and

smaller values (e.g. 1 mA. or lower) are desirable. The resistance of the milli-ammeter should be as low as is consistent with the permissible errors due to temperature variation.

The R.M.S. value of the alternating voltage at the terminals of the rectifier to obtain full-scale deflection on a 1 mA., 100-ohm milli-ammeter is about 1 volt. Such an instrument is made commercially for a minimum voltage range of 10 V. If a lower voltage range is required, this instrument should be used in conjunction with a suitably designed potential transformer, rather than with a lower value of series resistance, as in the latter case the "forward" resistance of the rectifier will become an appreciable fraction of the whole resistance of the circuit between the instrument terminals, and errors will arise due to the non-linear

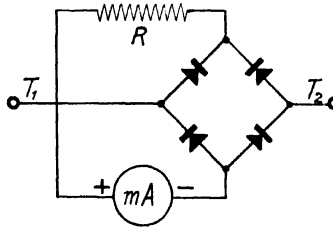


FIG. 232. CIRCUIT DIAGRAM OF RECTIFIER VOLTMETER

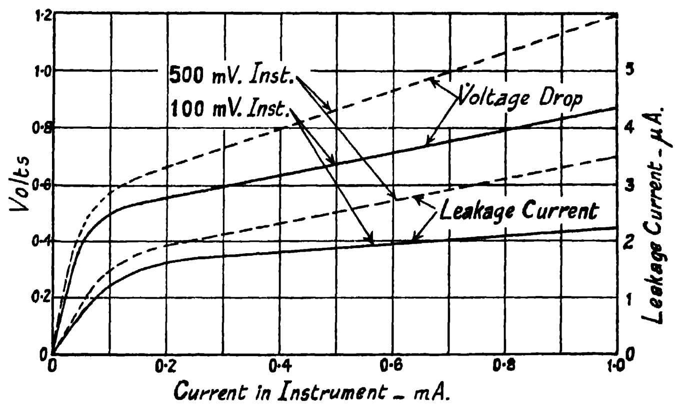


FIG. 233. CHARACTERISTIC CURVES FOR TYPICAL RECTIFIER INSTRUMENTS
(*Westinghouse Brake and Signal Co.*)

characteristic of the rectifier, a typical example being shown in Fig. 233.

Rectifier Ammeters. The rectifier principle may be applied to the measurement of small currents (e.g. below about 2 A.) by supplying the rectifier from a suitable current transformer. For such purposes a rectifier instrument possesses the great advantage that the power consumption is only about $\frac{1}{1000}$th of that required by an electro-magnetic A.C. ammeter of the same range. Owing to the small

output (about 1 mVA.) required from the secondary of the current transformer, this can be built with a mu-metal core of small size, thereby giving a very compact and portable combination.

Errors of Rectifier Instruments. The principal sources of error are due to wave-form and temperature.

Errors Due to Wave-form. Due to the fact that the D.C. milli-ammeter measures the mean value of the rectified current, the scale readings will only be absolutely correct when the wave-form of the voltage or current supplied to the rectifier is the same as that of the voltage employed for calibration. The errors resulting from the use of an instrument (which has been calibrated with a sine-wave voltage) on a circuit having distorted wave-forms containing second and third harmonics are shown in the accompanying table—

Order of harmonic . . .	2nd	2nd	3rd	3rd	3rd	3rd
Amplitude of harmonic (per cent of amplitude of fundamental).	10	20	10	20	30	20
Phase difference of harmonic with respect to fundamental . .	0°	0°	0°	0°	0°	180°
Percentage error of instrument .	− 0·5	− 2·0	+ 2·7	+ 4·4	+ 5·0	− 9·2

Temperature Errors. The temperature error due to self-heating is negligible owing to the very small amount of power dissipated in the rectifier and instrument. Errors, however, will occur if the ambient temperature has a value different from that at which the instrument was calibrated. The general effect of a rise in temperature is to lower the voltage drop across the A.C. terminals of the rectifier and to increase the leakage or "reverse" current. These two effects, however, are inter-related because the leakage current is related to the voltage drop in the manner shown in Fig. 233. Hence the total error due to ambient temperature is less than the separate errors due to the temperature coefficients of the "forward" resistance of the rectifier and the resistance of its leakage paths.

In the case of an instrument calibrated at an ambient temperature of 20° C., the errors are well within those allowable for a sub-standard instrument.

Wattmeters

Electro-dynamic Single-phase Wattmeters. The mechanism of a *commercial electro-dynamic wattmeter* closely resembles that of an electro-dynamic ammeter, but the moving coil of the wattmeter

has a high non-inductive resistance connected in series with it and is provided with separate terminals. The fixed coil is connected either directly in series with the circuit, or in the secondary circuit of a "series" or "current" transformer, the primary of which is connected in series with the main circuit. The moving coil, together

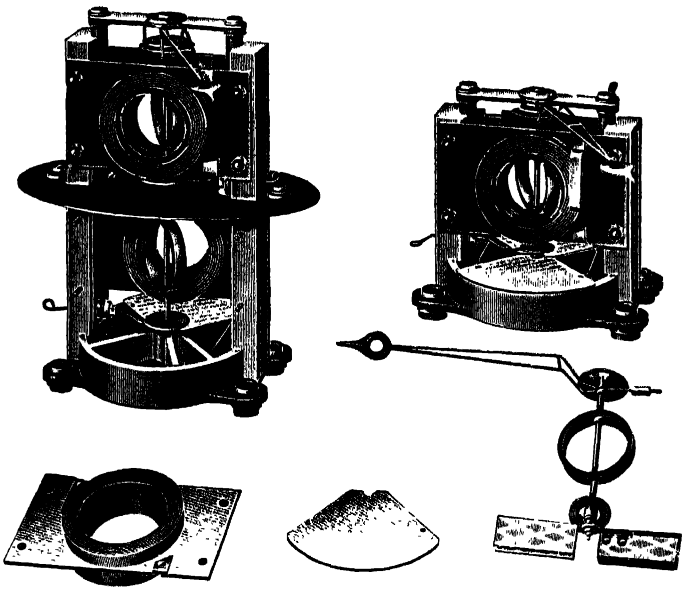


FIG. 234. MECHANISM OF ELECTRO-DYNAMIC WATTMETERS

(*Nalder Bros. and Thompson*)

(*a*) Assembled mechanism of single-phase wattmeter; (*b*) Moving system, showing damping vanes and control springs; (*c*) Mechanism of polyphase wattmeter (p. 381) with one of lower current coils and cover of one of damping chambers removed, these being shown separately at (*d*) and (*e*) respectively

with a non-inductive series resistance, is connected across the main circuit in order that the current in this coil shall be proportional to the voltage of the circuit.

The mechanism of a typical instrument is shown in Fig. 234.

Ranges. The maximum current range of *commercial electro-dynamic wattmeters* is from 100 to 200 A. For higher currents the fixed coils are usually wound for a maximum current of 5 A. and are supplied from the secondary winding of a current transformer of suitable ratio.

The moving coil is usually wound to carry a current of from 0·02 A. to 0·03 A., and therefore the resistance of the moving-coil

circuit must be of the order of from 30 to 50 ohms per volt of pressure range.

For pressures up to 600 V. a series resistance is employed, but for higher pressures an instrument having a 110 V. pressure circuit is employed in conjunction with a potential, or shunt, transformer.

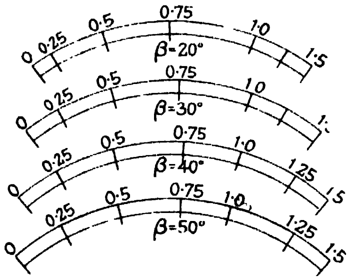


FIG. 235. THEORETICAL SCALE-FORMS FOR ELECTRO-DYNAMIC WATTMETER

Theory of the Electro-dynamic Wattmeter. Let the current and pressure in the circuit to which the wattmeter is connected be given by $i = I_m \sin(\omega t - \varphi)$ and $e = E_m \sin \omega t$, respectively. Then if the inductance of the moving-coil circuit is ignored and the resistance of this circuit is denoted by R, the current in the moving-coil will be given by $i_2 = (E_m/R) \sin \omega t$.

If this current is small in comparison with the current in the main circuit, the current in the fixed coil will be equal to and in phase with, the latter. Hence, with an air-path magnetic circuit and no eddy currents in the coils or their supports, the flux due to the current in the fixed coil will be proportional to, and in phase with, the current in the circuit Moreover, in the case of a *deflectional instrument*, if the flux density is assumed to be uniform throughout the space occupied by the moving coil, the torque at any instant will be given by

$$\begin{aligned}\mathfrak{T}_{d(inst)} &= k_2 M_{\mathrm{M}} i\, i_2 \sin(\beta + \theta) \\ &= (k_2 M_{\mathrm{M}} \sin(\beta + \theta)\, I_m E_m/R) \sin \omega t \sin(\omega t - \varphi) \\ &= (k_2 M_{\mathrm{M}} \sin(\beta + \theta) IE/R)\ [\cos\varphi - \cos(2\omega t - \varphi)]\end{aligned}$$

where k_2 is a constant, M_{M} the maximum mutual inductance of the fixed and moving coils, and $(\beta + \theta)$ the deflection of the moving coil from the position corresponding to the coincidence of the magnetic axes of the coils.

Therefore the mean torque during a period is

$$\begin{aligned}\mathfrak{T}_d = \frac{1}{T}\int^{T} \mathfrak{T}_{d(inst)}\, dt &= \frac{k_2 M_{\mathrm{M}} \sin(\beta + \theta)}{R} I E \cos\varphi \\ &= \frac{k_2 M_{\mathrm{M}} \sin(\beta + \theta)}{R} P\end{aligned}$$

where P ($= EI \cos\varphi$) is the power in the circuit.

Form of Scale. Since spring control is employed, the controlling torque is $\mathfrak{T}_c = k_1\theta$, where θ is the deflection from the zero position. Hence, for equilibrium, we have $\mathfrak{T}_d = \mathfrak{T}_c$, or, for a *deflectional instrument*,

$$\frac{k_2 M_{\mathrm{M}} \sin(\beta + \theta)}{R} P = k_1\theta$$

Whence $$P = k\theta/\sin(\beta + \theta) \quad . \quad . \quad . \quad . \quad (179)$$

where $k = k_1 R/k_2 M_{\mathrm{M}}$.

Typical scales calculated from this equation are shown in Fig. 235.

Correction for Power Expended in Wattmeter. If the current (i_2) in the moving-coil circuit is not small in comparison with the current in the circuit, and the wattmeter is connected in the manner shown in Fig. 236*a*, then if i

denotes the current in the "load," the current i_1 in the fixed coil is $i_1 = i + i_2$. Hence the torque is now given by

$$\begin{aligned}\mathfrak{T}_{d(inst)} &= k_2 M_M (i + i_2)\, i_2 \sin(\beta + \theta) \\ &= k_2 M_M \sin(\beta + \theta) \left\{ \frac{I_m E_m}{R} \sin \omega t \sin(\omega t - \varphi) + \frac{E_m^2}{R^2} \sin^2 \omega t \right\}\end{aligned}$$

and the mean torque by

$$\mathfrak{T}_d = \frac{1}{T} \int_0^T \mathfrak{T}_{d(inst)} \cdot dt = \frac{k_2 M_M}{R} \sin(\beta + \theta) \left(IE \cos\varphi + \frac{E^2}{R} \right)$$

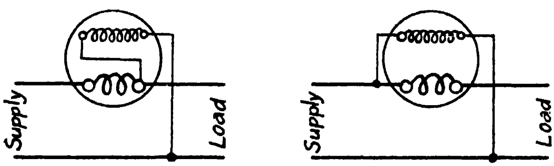
Whence, equating deflecting and controlling torques, we have

$$\frac{k_2 M_M}{R} \sin(\beta + \theta) \left(P + \frac{E^2}{R} \right) = k_1 \theta'$$

and
$$P = P_1 - E^2/R, \quad . \quad . \quad . \quad . \quad (180)$$

where $P_1 (= \theta' k_1 R / k_2 M_M)$ is the reading (watts) of the instrument.

Therefore, to obtain the power supplied to the load, the power expended in the pressure-coil circuit must be subtracted from the power (P_1) indicated



(a) (b)

FIG. 236. ALTERNATIVE CONNECTIONS FOR WATTMETER

by the wattmeter. If the wattmeter is connected as shown in Fig. 236*b*, the power expended in the current coils must be subtracted from the power indicated by the instrument.

In practice, this correction need only be applied to cases where small powers are to be measured. For example, if a wattmeter, the pressure circuit of which has a resistance of 3333 ohms, is connected as in Fig. 236*a*, and indicates a power of 15 watts, the power actually supplied to the load, assuming the pressure to be 100 volts, is

$$P = 15 - 100^2/3333 = 12 \text{ watts.}$$

Correction for Inductance of Moving-coil Circuit. The error due to inductance in the moving-coil circuit is most easily calculated in instruments of the non-deflectional, or torsion-head, type, as the moving coil always occupies a standard position relative to the fixed coil, and, therefore, the mutual inductance, if any, is constant. If L, R denote the inductance and resistance, respectively, of the moving-coil circuit, the current in this circuit, when the impressed E.M.F. is $e = E_m \sin \omega t$, is now

$$i_2' = [E_m / \sqrt{(R^2 + \omega^2 L^2)}] \sin(\omega t - \alpha),$$

where $\tan \alpha = \omega L/R = \omega\tau$. Thus, the current in the moving coil is not in phase with the impressed E.M.F.; moreover, its value and phase difference are both affected by frequency.

The deflecting torque at any instant t is

$$\begin{aligned}\mathfrak{T}_d'{}_{(inst)} &= k_2 M_M\, i\, i_2' \\ &= k_2 M_M I_m \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \alpha) \sin(\omega t - \varphi) \\ &= \frac{k' I_m E_m}{1 + \omega^2 \tau^2} \sin \omega t \sin(\omega t - \varphi) - \frac{k' I_m E_m \omega\tau}{1 + \omega^2\tau^2} \cos \omega t \sin(\omega t - \varphi)\end{aligned}$$

where $k' = k_2 M_M / R$, $1/\sqrt{(1 + \omega^2\tau^2)} = \cos\alpha$, and $\omega\tau/\sqrt{(1 + \omega^2\tau^2)} = \sin\alpha$

Whence the mean deflecting torque during a period is

$$\mathfrak{T}_d' = \frac{1}{T}\int_0^T \mathfrak{T}_d'{}_{(inst)}\, dt = \frac{k'IE}{1+\omega^2\tau^2}\cos\varphi + \frac{k'IE\omega\tau}{1+\omega^2\tau^2}\sin\varphi$$

$$= k'\left\{IE\cos\varphi + \left(\frac{\omega\tau}{1+\omega^2\tau^2}IE\sin\varphi - \frac{\omega^2\tau^2}{1+\omega^2\tau^2}IE\cos\varphi\right)\right\}$$

Since the term $\omega^2\tau^2$ is usually very small indeed in comparison with unity, we have, to a very close approximation,

$$\mathfrak{T}_d' = k'(IE\cos\varphi + \omega\tau\, IE\sin\varphi)$$
$$= k'(P + \omega\tau\, IE\sin\varphi)$$

If $k_1\theta''$ is the controlling torque, we have, for equilibrium,

$$k'(P + \omega\tau\, IE\sin\varphi) = k_1\theta''$$

Whence $\quad P = k\theta'' - \omega\tau\, IE\sin\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$

where $\quad k = k_1/k' = k_1R/k_2M_M$

Thus the effect of inductance in the moving-coil circuit is to cause the wattmeter to read high (on lagging power factor) by the amount $\omega\tau\, IE\sin\varphi$, and therefore this quantity must be subtracted from the wattmeter reading to obtain the true power. For leading power factors the wattmeter will read low, and the quantity $\omega\tau\, IE\sin\varphi$ must then be added to the wattmeter reading to obtain the true power.

It should be noted that the correction factor is zero at unity power factor (on the assumption that $\omega^2\tau^2$ is negligible in comparison with unity), and approaches its maximum value ($\omega\tau\, IE$) as the power factor approaches zero, so that when the wattmeter is used on circuits of very low power factor, the correction factor may require careful consideration. The percentage error

$$= 100\,\frac{\omega\tau\, IE\sin\varphi}{IE\cos\varphi} = 100\,\omega\tau\,\tan\varphi,$$

which, for a power factor of 1 per cent ($\varphi = 89{\cdot}4°$), becomes equal to $9550\,\omega\tau$, and, for a power factor of 0·175 per cent ($\varphi = 89{\cdot}9°$), becomes $57300\,\omega\tau$. For example, if the inductance of the moving-coil circuit is 6 milli-henries and the resistance is 3000 ohms, $\tau = L/R = 2 \times 10^{-6}$, and, for 50 frequency, $\omega\tau = 314 \times 2 \times 10^{-6} = 6{\cdot}28 \times 10^{-4}$. Hence, if this instrument is used on circuits having a power factor of 1 per cent, the error in the wattmeter reading is $(9550 \times 6{\cdot}28 \times 10^{-4} =)$ 6 per cent, and, if used on circuits having a power factor of 0·175 per cent, the error is $(57300 \times 6{\cdot}28 \times 10^{-4} =)$ 36 per cent.

Induction Wattmeters. The induction principle may be applied to indicating wattmeters, but such instruments are only suitable for circuits in which the frequency and voltage are constant.

The travelling magnetic field is produced by the joint action of two electromagnets, one being series wound and the other shunt wound (Fig. 237). The exciting current of the series magnet is proportional to the current in the circuit in which the instrument is connected, and that of the shunt magnet is proportional to the voltage of this circuit.

In the hypothetical case when the resistance of the shunt magnet is zero and both magnetic circuits are free from saturation and hysteresis, the fluxes will have a phase difference of $(90 \mp \varphi)$ degrees (φ being the phase difference between voltage and current of the

main circuit), and will be proportional to the current and voltage, respectively, of the main circuit. Hence, if these magnets act upon a spring-controlled pivoted disc or drum (assumed to have no inductance), the deflecting torque (p. 367) will be proportional to the product of the fluxes and the sine of their phase displacement, i.e. to

$$EI \sin(90 \mp \varphi) = EI \cos \varphi,$$

and the angular deflections will be proportional to $EI \cos \varphi$, or to the power in the circuit.

If, however, the shunt winding possesses appreciable resistance, the flux of this magnet will not lag 90° with respect to the applied voltage, and, in consequence, the phase difference between the fluxes will be smaller than $(90 \mp \varphi)$ degrees, as in the hypothetical case. Some means of increasing the natural lag of the flux of the shunt magnet is therefore necessary in a commercial instrument.

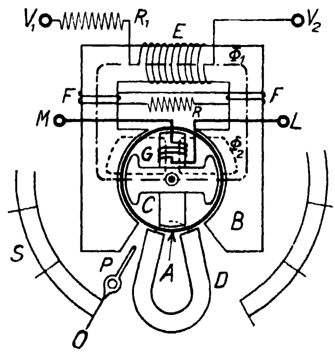


FIG. 237. ARRANGEMENT OF ELECTRIC AND MAGNETIC CIRCUITS OF DRUM-INDUCTION WATTMETER

(*Nalder Bros. and Thompson*)

Construction of Drum-type Induction Wattmeter. The Lipman form of operating magnet for a drum-type induction instrument is readily adaptable to wattmeters. Thus the only changes required are in the windings. The winding, G, on the inner electromagnet, C, (Fig. 237) is arranged for direct series excitation, and the winding, E, on the external electromagnet, B, is arranged for shunt excitation (with suitable series resistance to "swamp" the effects of resistance changes due to temperature).

The large air path in the magnetic circuit of the wound pole of the inner magnet, causes the flux produced by the series winding, G, to be proportional to, and in phase with, the current in the circuit.

On the other hand, the comparatively low-reluctance magnetic path of the flux of the outer magnet, together with the low resistance of the excitation winding, E, cause the flux to have a phase difference of nearly 90° from the voltage at the terminals of this winding (which is proportional to the voltage of the circuit).

To obtain the exact value of 90° for this phase difference, auxiliary coils, F, are wound on the limbs of the outer magnet and are connected in a closed circuit with an adjusting resistance, R, in series.

The transformer action between the windings E and F causes a modification of the resultant ampere-turns acting on the outer magnet, and the resistance, R, enables the phase of these ampere-turns to be adjusted, so that the flux lags exactly 90° from the supply voltage. The vector diagram (Fig. 76) of the transformer is directly applicable.

Theory of Induction Wattmeter. The theory of the uncompensated wattmeter is developed in a manner similar to that (p. 366) of the induction ammeter. Thus, if magnetic saturation and hysteresis are absent, the flux due to the series winding is directly proportional to, and in phase with, the current in the main circuit, i.e. $\Phi_1 = k_1 I_m \sin(\omega t - \varphi)$, where φ is the phase difference between voltage and current in the main circuit. Similarly, the flux due to the shunt winding is

$$\begin{aligned}\Phi_2 &= [k_2 E_m/\sqrt{(R_2^2 + \omega^2 L_2^2)}]\sin(\omega t - (\tfrac{1}{2}\pi - \beta)) \\ &= -[k_2 E_m/\sqrt{(R_2^2 + \omega^2 L_2^2)}]\cos(\omega t + \beta)\end{aligned}$$

where R_2, L_2 denote the resistance and inductance, respectively, of the shunt winding and $(\frac{1}{2}\pi - \beta)$ is the phase difference between the impressed E.M.F. and the flux in the shunt magnet; β being equal to $\tan^{-1} R_2/\omega L_2$.

The E.M.F. induced in the disc by the flux Φ_1 is

$$e_1 = -10^{-8} d\Phi_1/dt = -k_1 \omega I_m \cos(\omega t - \varphi),$$

and that induced by the flux Φ_2 is

$$e_2 = -10^{-8}\, d\Phi_2/dt = -[k_2 \omega E_m/\sqrt{(R_2^2 + \omega^2 L_2^2)}]\sin(\omega t + \beta)$$

The currents in the disc are deduced in the manner given on pp. 366 to 367, from which follow the expressions for the instantaneous forces due to the interaction of these currents and the fluxes. Thus

$$\begin{aligned}F_1 &= k\, i_2'\Phi_1 \\ &= k\left(\frac{-k_2'\omega E_m}{\rho(1 + \omega^2\tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}}\sin(\omega t + \beta)\right)(k_1 I_m \sin(\omega t - \varphi)) \\ F_2 &= k\, i_1'\Phi_2 \\ &= k\left(\frac{-k_1'\omega I_m}{\rho(1 + \omega^2\tau_3^2)}\cos(\omega t - \varphi)\right)\left(\frac{-k_2 E_m}{\sqrt{(R_2^2 + \omega^2 L_2^2)}}\cos(\omega t + \beta)\right)\end{aligned}$$

Whence the torque acting upon the disc is

$$\begin{aligned}\mathfrak{T}_d &= k'(F_2 - F_1) \\ &= \frac{K\omega E_m I_m}{\rho(1 + \omega^2\tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}}[\cos(\omega t + \beta)\cos(\omega t - \varphi) \\ &\qquad + \sin(\omega t + \beta)\sin(\omega t - \varphi)] \\ &= \frac{K\omega E_m I_m}{\rho(1 + \omega^2\tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}}\cos(\varphi + \beta) \\ &= \frac{K_1}{\rho L_2(1 + \omega^2\tau_3^2)\sqrt{(1 + 1/\omega^2\tau_2^2)}} EI\cos(\varphi + \beta)\end{aligned}$$

where $\tau_2 = L_2/R_2$.

Therefore, if the torque is to be proportional to the power in the main circuit, β must be zero. Under these conditions we have

$$\mathfrak{T}_d = \frac{K_1}{\rho L_2(1 + \omega^2\tau_3^2)\sqrt{(1 + 1/\omega^2\tau_2^2)}} EI\cos\varphi \quad . \quad . \quad . \quad (182)$$

This equation shows that the indications of the induction wattmeter are less affected by variations of frequency than those of the induction ammeter.

Polyphase, or Double, Wattmeters. These instruments are intended for measuring power in polyphase circuits by the two-wattmeter method (p. 195), and consist of two single-phase wattmeter mechanisms combined into one case and having a single pointer and scale. The instruments may be of either the electro-dynamic or the induction type.

In the *electro-dynamic instrument* the two moving coils are fixed to a common spindle which carries the pointer, control springs, and damping vanes. The two systems are shielded from each other by means of a flat shield of laminated iron fixed between the upper and lower systems perpendicularly to the spindle (Fig. 234).

The *induction type* of polyphase wattmeter consists usually of two single-phase instruments, the two discs being fixed to a common spindle which carries the pointer and control spring. Each disc is provided with its operating and braking magnets, as in a single-phase instrument.

WATT-HOUR OR ENERGY METERS

General Requirements. In all cases of electric supply for power and lighting the charge to the consumer must be based upon the energy (in kilowatt hours) supplied. The measurement of this energy is effected by an integrating wattmeter operating on the induction principle, which principle is particularly suitable for house service meters on account of the lightness and robustness of the rotating element and the absence of rotating or moving contacts. Moreover, on account of the smallness of the variations of voltage and frequency in commercial supply systems the accuracy of the induction meter is unaffected by such variations. The accuracy, however, is affected if the wave-form of the supply is badly distorted.

Theory. The induction energy meter may be derived from the induction wattmeter by substituting for the spring control and pointer an eddy current brake and a counting train, respectively. For the meter to read correctly, the speed of the disc must be proportional to the power in the circuit in which the meter is connected, and to fulfil this condition: (1) The torque due to the current generated in the disc by its rotation in the magnetic field of the operating magnets must be negligible in comparison with the operating torque; (2) the friction must be compensated at all speeds; and (3) the braking torque must be directly proportional to the speed of the disc.

Condition (1) is satisfied if the angular speed of the disc is very low in comparison with the angular speed of the travelling magnetic

field, and in commercial meters the speed of the disc is of the order of 30 revolutions per minute at full load.

On account of the low speed of rotation of the moving system, the friction after starting remains constant, and may be compensated by a constant torque acting in the same direction as the main driving torque. This compensating torque is obtained by producing a slight dissymmetry of the shunt flux, by means of an unsymmetrically placed shielding plate or magnetic shunt.

With the friction compensated, and with correctly adjusted operating magnets, the resultant torque acting upon the disc will be proportional to the power in the circuit, and if the speed of the disc is to be proportional to this quantity, a braking torque varying directly as the speed, must be applied to the disc. The braking torque is produced by eddy-currents induced in the disc by its rotation in a magnetic field of constant intensity, the magnetic field being provided by one or two permanent magnets, so placed as to be unaffected by the alternating-current operating magnets.

For a given disc and brake magnet, the braking torque varies with the distance of the poles from the centre of the disc, the maximum torque occurring when the distance of the centre of the pole faces from the centre of the disc is equal to 83 per cent of the radius of the disc.* This feature is utilized when testing the meter to obtain a final adjustment of the speed to a definite value, corresponding to a given power.

Construction. Numerous forms of construction for house-service induction meters have been devised, the principal differences in construction being confined chiefly to the arrangement of the magnetic circuits of the operating magnets, the method of obtaining the correct phase difference between the fluxes of the series and shunt magnets, and the method of compensating for friction.

In all cases the rotating disc, of aluminium, is fixed to a vertical spindle, the lower end of which is supported by a jewelled footstep bearing and the upper end is supported by a guide bearing. To this end is fitted either a pinion or a worm, from which the counting train is driven. When a pinion is fitted to the spindle, the worm is fitted to an intermediate spindle, this arrangement possessing the advantage of reduced friction compared with a direct worm drive. The worm drive to the counting train is necessary, because the spindles of the latter are horizontal.

Two forms of magnets, in which alternative methods of compensation are employed, are shown in Fig. 238.

* *Electrical Measuring Instruments*, Drysdale and Jolley, Part I, p. 105 The theory of the eddy-current brake is given on p. 102 of this volume.

The friction compensating torque, in one case, Fig. 238 (*a*), is produced by an unsymmetrical magnetic shunt, C, at the pole-face of the shunt magnet, A; and, in the other case, Fig. 238 (*b*), by an *unsymmetrically-placed* loop of brass or copper, G, which surrounds the pole-face of the shunt magnet at the air-gap, and is adjustable

(*a*)

(*b*)

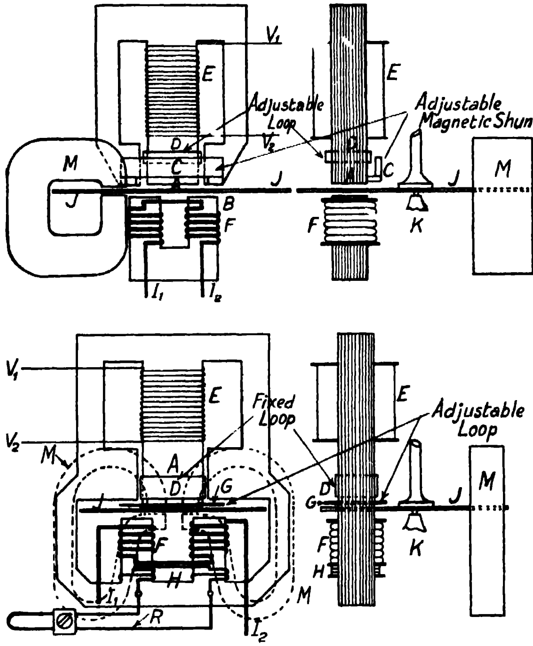


FIG. 238. TYPICAL ARRANGEMENTS OF OPERATING MAGNETS, DISC, BRAKE MAGNETS, AND COMPENSATING DEVICES IN WATT-HOUR METERS

(*Aron Electricity Meter, and Metropolitan-Vickers Electrical Co.*)

in a direction parallel to the pole-face. These devices are called "light load adjustments."

The correct phase difference (viz. $90 - \varphi°$) between the shunt and series fluxes is obtained in one case, Fig. 238 (*a*), by a loop of brass or copper, D, *symmetrically placed* over the shunt pole and adjustable in a direction perpendicular to the pole-face; and, in the other case, Fig. 238 (*b*), by shunting the series windings, or, alternatively, by providing a secondary winding, H, of a few turns on the series magnet and closing this through a loop of resistance wire, R, having an adjustable slider.

These devices are called "inductive load adjustments."

Theory of Shielded-pole Shunt Magnet. Consider the simple case of the magnet shown in Fig. 239, in which the shielding coil B surrounds the pole face of the core which is magnetized by the shunt-excited coil. Then if Φ is the flux at the pole face, and the vector, $O\Phi$, of this quantity is taken as the reference vector in the vector diagram, Fig. 240, the saturation ampere-turns for the magnetic circuit are represented by OA, which is slightly in advance of the flux owing to hysteresis and eddy currents in the core.

The E.M.Fs. induced in the exciting and shielding coils are represented by OE_1 and OE_2, respectively, both lagging 90° with respect to the flux.

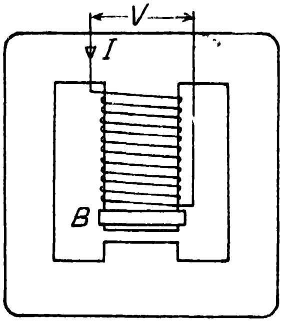


FIG. 239

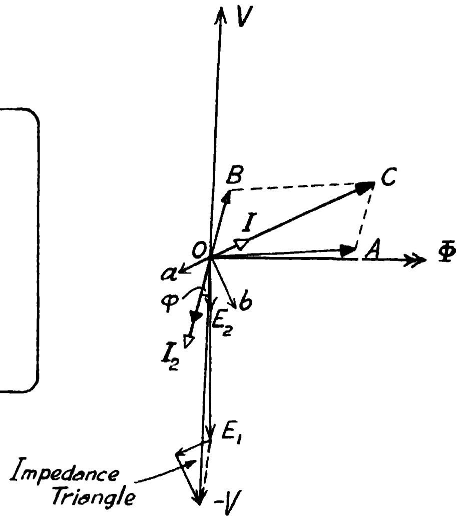


FIG. 240

CIRCUIT AND VECTOR DIAGRAMS FOR SHIELDED-POLE SHUNT-EXCITED ELECTROMAGNET

The current in the shielding coil is represented by OI_2, which also represents the ampere-turns due to this coil. Hence the ampere-turns to be provided by the exciting coil are represented by OC, which is equal to the vector sum of the saturation ampere-turns, OA, and the ampere-turns, OB, balancing the ampere-turns due to the shielding coil. The exciting current is therefore represented by OI.

The voltage to be applied to the exciting coil is obtained by determining the internal E.M.Fs. in this coil. These E.M.Fs. comprise the E.M.F., OE_1, induced by the alternations of the flux Φ; the E.M.F., Oa, due to the resistance of the coil; and the E.M.F., Ob, due to leakage reactance. The resultant internal E.M.F. is therefore represented by O-V, and the applied E.M.F.—which balances the resultant internal E.M.F.—is represented by OV.

Now, in the induction energy meter, OV must lead the flux vector by 90°, and an inspection of the vector diagram will show that this result can be obtained by suitably adjusting the ampere-turns of the shielding coil, this adjustment being effected either by alteration of the resistance of the shielding coil, or by altering its axial position.

Adjustment of Meter. The adjustments are effected when the meter is first tested, the method being as follows: Assuming correct

voltage and frequency to be maintained, the meter is run at its full load current alternatively on loads of unity power factor and a low lagging power factor, and the speed is adjusted to the correct value by varying the positions of the brake magnet and the shielding loop. For example, if the meter runs fast on inductive load and correctly on non-inductive load, the shielding loop must be moved towards the disc. Again, if the meter runs slow on non-inductive load, the brake magnet is moved towards the centre of the disc.

The meter is next run on a light load (i.e. about $\frac{1}{20}$ of full load) and the friction compensating device is adjusted to give the correct speed. In making this adjustment, care must be exercised to see that friction is not over-compensated, otherwise the meter will run when only the shunt magnet is excited.* The adjustment should be such that: (1) The meter will start with a non-inductive load of 0·5 per cent of full-load; and (2) with a non-inductive load of $\frac{1}{50}$th of full-load the speed is within 2 per cent of the correct speed at this load.

Accuracy. In commercial meters manufactured to the specifications of the British Standards Institution, the permissible error is $\pm$ 2 per cent for all loads between $\frac{1}{20}$ th of full load and 25 per cent overload, and at any power factor between unity and 0·5, lagging and leading.

Effects of Temperature Variations. Energy meters are inherently almost free from errors due to temperature variations, as changes in the resistance of the disc affect both the driving and braking torques in the same manner. Moreover, the error which would be caused by the diminution of the flux of the shunt magnet due to the increase in resistance of the shunt coil with increase of temperature is, to some extent, balanced by the decrease in the flux of the brake magnet. The resultant error at full-load, unity power factor, may be of the order + 0·05 per cent per 1° C. rise in temperature.

Polyphase Watt-hour Meters for Unbalanced Loads.† For the measurement of energy in polyphase circuits with unbalanced loads, a double-element watt-hour meter is usually employed. This meter consists essentially of two single-element meters with a common spindle and a single counting train. The series and

* With some meters this is prevented by a small iron tongue, or vane, fitted to the disc in such a position that when the tongue is adjacent to the brake magnet the attractive force between tongue and magnet is just sufficient to prevent rotation of the disc with full shunt excitation and no current in the series coil. In other cases two holes are pierced in the disc.

† For the measurement of energy in three-phase circuits with balanced loads, a single-phase meter may be employed. See *Power Wiring Diagrams*, Third Edition, pp. 116. 118.

shunt magnets of each element are connected according to the two-wattmeter method of measuring power in polyphase circuits, and each set of magnets acts upon a separate disc to avoid mutual interference. Each element is provided with the same adjustments as in a single-phase meter, and, in addition, one element is provided with an adjustment (usually a magnetic shunt fitted to the shunt magnet) for obtaining equality between the torques of the elements when the meter is operating with a balanced three-phase load of unity power-factor.

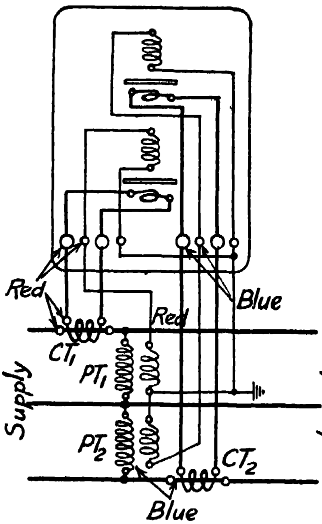


Fig. 241. Connections for Polyphase Watt-hour Meter when Used with Current and Potential Transformers

Marking of Terminals of Polyphase Watt-hour Meters. The following scheme is employed for marking the terminals of meters and the phases of the supply system, in order that no ambiguity shall exist about the manner in which the meter is to be connected in service. Moreover, the scheme also removes any ambiguity concerning the connections of a meter when it is calibrated at the manufacturer's testing department.

The standard direction of phase rotation shall be counter-clockwise. The phase, or line wire, of reference of the supply system shall be numbered 1 and coloured white; the phase which lags 120° shall be numbered 2 and coloured blue, and the other phase (which lags 240°) shall be numbered 3 and coloured red.

In connection with power and energy measurements by the two-wattmeter method, the current coils of the instruments shall be connected in the "blue" and "red" phases, and the common terminal of the potential circuits shall be connected to the "white" phase. The "blue" phase is then the "leading" phase and the "red" phase is the "lagging" phase.*

The terminals of the element (of the meter) which is to be

* This designation of the "leading" and "lagging" phases follows from a consideration of the phase differences between the line currents and the voltages impressed upon the potential circuits of the instruments. For example, for the standard conditions and unity power factor, these phase differences are 30°, leading, for the element connected in the "blue" phase, and 30°, lagging, for the element connected in the "red" phase.

connected in the "leading" phase shall be numbered 2 and coloured blue; those of the element which is to be connected to the "lagging" phase shall be numbered 3 and coloured red.

When a meter has been calibrated with current and potential transformers, the terminals of these transformers are marked in a manner similar to those of the meter, to ensure that when the meter is put into service the connections between transformers and elements may be the same as when the meter was calibrated. A diagram of connections for a typical case is given in Fig. 239(*b*).

Power-factor Meters

General. The term power-factor meter (or phase meter) refers to an instrument for indicating directly, by a single reading, the power factor of the circuit to which it is connected. Instruments are available for indicating the power factor of either single-phase or polyphase circuits, under balanced and unbalanced loads.

Principles of Operation. Power-factor meters operate on the electromagnetic principle, and are of either the moving-coil (electrodynamic) or the moving-iron types. In general, both types have two electromagnetic systems; one system being supplied with current equal, or proportional, to the current in the circuit of which the power factor is to be measured, and the other being supplied with current proportional to the voltage of the circuit. The coils of one system are arranged to produce either a rotating magnetic field, or a number of alternating magnetic fields having a time-phase difference with respect to one another, the axes of the fields being displaced in space by angles equal to the time-phase angles. The coil, or coils, of the other system usually produce an alternating magnetic field, the axis of which is fixed in space.

The direction of the resultant magnetic field in space depends upon the relative positions of the axes of the rotating and the (single) alternating fields at the instant when the latter (i.e. the alternating field) attains its maximum value. Hence, since these fields are excited by currents proportional to the current and pressure of the circuit, the position of the resultant field in space will depend upon, and vary with, the phase difference between these quantities, i.e. with the power factor of the circuit.

The position of the resultant field is indicated by a pointer and scale, the pointer being fixed to a spindle, which, in the moving-iron instrument, carries a set of moving-iron vanes, and in the electro-dynamic instrument carries the coils which produce the rotating magnetic field.

With both types of instrument the moving system is perfectly balanced and is free from controlling forces. Hence, when an instrument is disconnected from a circuit the pointer remains in the position which it occupied at the instant of disconnection. With these general remarks we will consider the theory of the moving-coil and moving-iron instruments.

Electro-dynamic Power-factor Meters. In instruments for single-phase and balanced three-phase circuits, the alternating magnetic

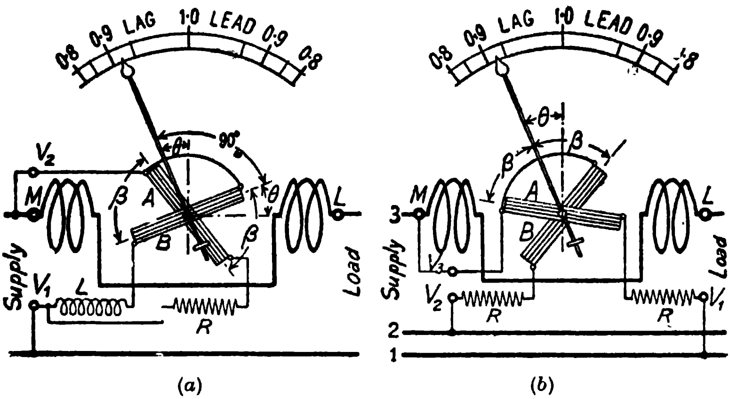


FIG. 242. PRINCIPLE OF ELECTRO-DYNAMIC POWER-FACTOR METERS
(a) Single-phase instrument; (b) Three-phase instrument.

field is produced by a pair of coaxial coils supplied with current proportional to the current in the circuit.

The rotating magnetic field is produced by a pair of intersecting coils, which are fixed to a common spindle. The coils are supplied with equal currents, proportional to the voltage of the circuit, the phase difference between these currents being approximately 90° in the single-phase instrument and 120° in the three-phase instrument.

The two moving coils should be exactly similar, and the angular displacement of their axes should be equal to the (time) phase difference between their currents.

In a single-phase instrument the requisite equality in, and phase difference between, the currents is obtained by connecting a non-inductive resistance in one circuit and a reactance, of the same ohmic value, in the other circuit, as shown in Fig. 242 (*a*).

In a three-phase instrument, Fig. 242 (*b*), the angular spacing, 2β, of the coils may be chosen according to the type of scale required. For example, with a spacing of 120° a normal cosine-law scale is

obtained, as with a single-phase instrument; but if the angular spacing is increased, certain parts of the scale are opened out (i.e. the range in power factor for a given angular deflection of the pointer is diminished).

The current is led into and out of the moving coils by fine silver ligaments, which are so arranged as to exert no controlling force on the moving system.

Theory of Electro-dynamic Power-factor Meter for Balanced Three-phase Circuits. Let 2β denote the angular spacing between the two moving coils and θ the angle of displacement of the pointer (which is symmetrically placed with respect to the moving coils) from the central plane perpendicular to the axis of the fixed coil (Fig. 242*b*) when the moving system is in equilibrium. Then, if the supply system is symmetrical, and the line voltages are given by

$$v_{1\text{-}2} = V_m \sin(\omega t + \tfrac{1}{6}\pi),\; v_{2\text{-}3} = V_m \sin(\omega t - \tfrac{1}{2}\pi),\; v_{3\text{-}1} = V_m \sin(\omega t - \tfrac{7}{6}\pi)$$

the voltages impressed upon the moving-coil circuits will be given by

$$v_{3\,1} = V_m \sin(\omega t - \tfrac{7}{6}\pi)\,; \qquad v_{3\text{-}2} = V_m \sin(\omega t - \tfrac{3}{2}\pi).$$

If R is the resistance of each of the moving-coil circuits, and self and mutual inductances are negligible, the currents in the coils will be

$$i_A = (V_m/R)\sin(\omega t - \tfrac{7}{6}\pi), \qquad i_B = (V_m/R)\sin(\omega t - \tfrac{3}{2}\pi).$$

The current in the fixed coil is given by $i = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$, where $\cos\varphi$ is the power factor of the three-phase circuit.

Hence the instantaneous torques acting on the moving coils are

$$\begin{aligned}
\mathfrak{T}_{A(inst)} &= k\,i\,i_A \sin(\beta + \theta)\\
&= k\,I_m\,(V_m/R)\sin(\beta + \theta)\sin(\omega t - \tfrac{4}{3}\pi - \varphi)\sin(\omega t - \tfrac{7}{6}\pi)\\
\mathfrak{T}_{B(inst)} &= -k\,i\,i_B \sin(\beta - \theta)\\
&= -k\,I_m\,(V_m/R)\sin(\beta - \theta)\sin(\omega t - \tfrac{4}{3}\pi - \varphi)\sin(\omega t - \tfrac{3}{2}\pi)
\end{aligned}$$

and the mean torques are

$$\begin{aligned}
\mathfrak{T}_A &= k\,I\,(V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)\\
\mathfrak{T}_B &= -k\,I\,(V/R)\sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)
\end{aligned}$$

Since the condition for equilibrium of the moving system is that $\mathfrak{T}_A + \mathfrak{T}_B = 0$, we have

$$\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi) = \sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)$$

i.e. $\quad \sin(\beta + \theta)\,[\sqrt{\tfrac{3}{4}}\cos\varphi - \tfrac{1}{2}\sin\varphi] = \sin(\beta - \theta)\,[\sqrt{\tfrac{3}{4}}\cos\varphi + \tfrac{1}{2}\sin\varphi]$

or $\quad \sqrt{3}\cos\varphi\,[\sin(\beta + \theta) - \sin(\beta - \theta)] = \sin\varphi\,[\sin(\beta - \theta) + \sin(\beta + \theta)]$

$\therefore \quad \sqrt{3}\cos\varphi\cos\beta\sin\theta = \sin\varphi\sin\beta\cos\theta$

Whence $\quad \tan\theta = (1/\sqrt{3})\tan\beta\tan\varphi \quad . \quad . \quad . \quad . \quad . \quad (183)$

If the angular displacement of the coils is $120°$, $\beta = 60°$ and $\tan\beta = \sqrt{3}$. Hence

$$\tan\theta = \tan\varphi$$

and $\quad \theta = \varphi$

Thus the angular deflection of the pointer from the plane of reference is equal to the phase difference between the phase E.M.F. and current in the circuit to which the instrument is connected.

Observe that the indications of the instrument are unaffected by variations of frequency: they are also unaffected by wave-form distortion.

Comparison of Theoretical and Practical Scale Shapes. The form of scale deduced theoretically in the preceding section refers to an ideal instrument in which: (1) The magnetic field produced by

the fixed coil is (*a*) in phase with the current (i.e. there are no eddy currents in the conductors or coil supports), (*b*) uniform over the space traversed by the moving coils when the latter are deflected over the full scale; (2) the moving coils are identical in dimensions and number of turns, and their self and mutual inductances are negligible in comparison with the resistances of the circuits; (3) the friction is zero; (4) the reactance employed in the single-phase instrument has no losses and produces no distortion of the wave-form of the current.

The agreement between the form of scale of a commercial instrument with the theoretical form of scale will depend upon the closeness with which the actual instrument approaches the ideal. For example, in a typical high-class instrument (such as that manufactured by the Weston Electrical Instrument Co.), the several layers of the moving coils are interlaced at the diametrical crossing points, so that each coil has the same mean diameter. The coils are extremely short and of light weight, and the moving system is pivoted in jewelled bearings. Moreover, in the polyphase instruments the values of the series resistances in the moving-coil circuits are such that the self and mutual inductances of these circuits are negligible. Again, the supports for the fixed coils are constructed of a material having a high specific resistance, and the amount of metal located in the magnetic field is reduced to a minimum, so that the eddy-current losses are very small.

In consequence of these features, the scale is almost identical with that for an ideal instrument, as is shown by the following data—

Deflection $\theta°$	0	5	10	15	20	25	30	35	40	43
Actual scale marking ($\cos\varphi$)	1·0	0·985	0·942	0·88	0·808	0·735	0·665	0·599	0·537	0·5
Theoretical scale marking* ($\cos\varphi'$)	1·0	0·987	0·9505	0·895	0·8275	0·77	0·682	0·605	0·54	0·5

Calibration. Since the indications of the polyphase electrodynamic power-factor meter are unaffected by frequency variations, the instrument may be calibrated on a direct-current circuit. The normal current is passed through the fixed coil, and such voltages are applied to the moving-coil circuits that the torques corresponding to a given position of the moving system are the same as the mean torques under normal operating conditions. The mean torques under normal operating conditions are given by the equations on p. 389, thus

$$\mathfrak{T}_A = kI(V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)$$
$$\mathfrak{T}_B = -kI(V/R)\sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)$$

* Calculated from $\tan\varphi' = \sqrt{3}\tan\theta/\tan 43°$.

and the torques with direct current passing through the coils are

$$\mathfrak{T}_A' = kI'I_A' \sin(\beta + \theta) = kI'(V_A'/R)\sin(\beta + \theta)$$

$$\mathfrak{T}_B' = -kI'I_A' \sin(\beta - \theta) = kI'(V_B'/R)\sin(\beta - \theta)$$

where I' is the current in the fixed coil and I_A', I_B', the currents in the moving coils, and V_A', V_B', the voltages applied to these circuits. The constant k has the same value for both direct- and alternating-current operation.

Hence, with normal current in the fixed coil, we have

$$(V_A'/R)\sin(\beta + \theta) = (V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)$$

or $$V_A' = V\cos(\tfrac{1}{6}\pi + \varphi) \quad . \quad . \quad . \quad (184)$$

and similarly $$V_B' = V\cos(\tfrac{1}{6}\pi - \varphi) \quad . \quad . \quad . \quad (185)$$

where V is the normal (alternating) operating voltage.

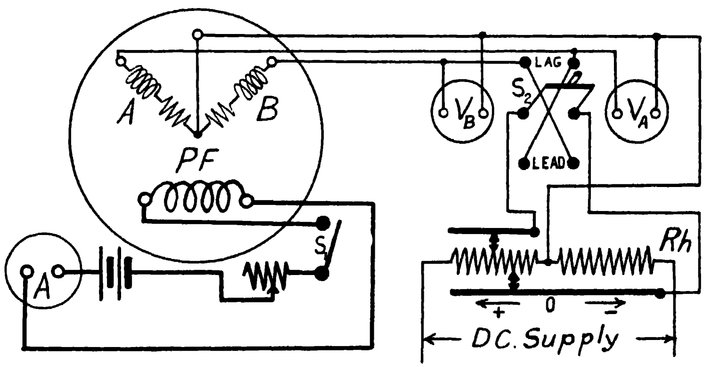


FIG. 243. CONNECTIONS FOR CALIBRATING ELECTRO-DYNAMIC THREE-PHASE POWER-FACTOR METER

Thus, if the normal operating voltage is 100, the values of V_A' and V_B' to be applied to the moving-coil circuits to obtain the deflection corresponding to a given power factor (lagging) are

$\cos\varphi$	1·0	0·95	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
V_A'	86·6	81·16	56·13	39·33	24·61	11·9	0	−11·13	−21·74	−31·63	−41
	86·6	91·17	99·74	99·29	96·35	91·93	86·6	80·4	73·65	66·34	58·43

To obtain the deflections corresponding to leading power factors, the moving-coil circuits must be interchanged with respect to V_A', V_B'. The connections are arranged as shown in Fig. 243; the moving-coil circuits are supplied—*via* a change-over switch and suitable rheostats—from a source having a voltage equal to twice

the normal operating voltage of the instrument, and the fixed coil is supplied with current from a low-voltage battery.*

Moving-iron Power-factor Meters. These instruments may be divided into two sub-classes, according to whether the operation depends upon a rotating magnetic field or a number of alternating fields. Since, in both forms of instruments, all coils are stationary, the moving system may be given complete freedom of movement, and a scale extending over the full 360° of arc may, therefore, be employed as shown in Fig. 244.

The essential features of a *rotating field instrument* for balanced polyphase loads are shown in Fig. 245. The electrical portion of the instrument consists of (1) a set of coils, A, capable of producing a uniformly rotating magnetic field when supplied with suitable polyphase currents ; and (2) a single coil, B, which is fixed coaxially inside A and is excited from one of the phases of the (polyphase) system. The outer coils, A, are surrounded by a laminated iron ring and are usually supplied, by means of current transformers, with currents proportional to the line currents in the polyphase system. The inner coil, B, together with a series resistance, is connected across two line wires of the system.

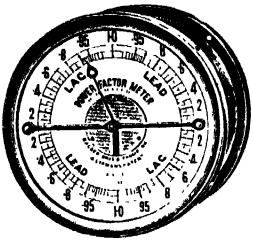


FIG. 244. MOVING-IRON FORM OF POWER-FACTOR METER WITH SCALE EXTENDING OVER 360° (*Nalder Bros. and Thompson*)

The moving system consists of a pair of light iron vanes, C, C, fixed to the ends of an iron core and projecting perpendicularly from the latter in opposite directions, as shown in Fig. 245. The core and spindle are coaxial with the coils A, B and the vanes project over the ends of coil B. The iron core is fixed to a spindle which is pivoted in jewelled bearings and carries the pointer P and the light mica damping vanes D

The moving system is perfectly balanced and is free from controlling forces. Hence, when the coils A, B, are excited, the iron vanes C, C, set themselves along the direction of the resultant M.M.F. due to these coils.

The *theory* of the instrument may be developed in a similar manner to that of the electro-dynamic instrument by considering

* If calibration at power factors below 0·5 is not required the right-hand rheostat, *Rh*, Fig. 243 (which is necessary for obtaining the reversal of the voltage V_A'), may be dispensed with, in which case the voltage of the direct-current supply should be equal to the normal operating voltage of the instrument.

the moving core and vanes to be magnetized, by the inner coil B, with current which is proportional to, and in phase with, the line voltage of the system. Then, if the effects of hysteresis and eddy currents are ignored, the core, vanes, and inner coil are equivalent electromagnetically to a rectangular moving coil pivoted within

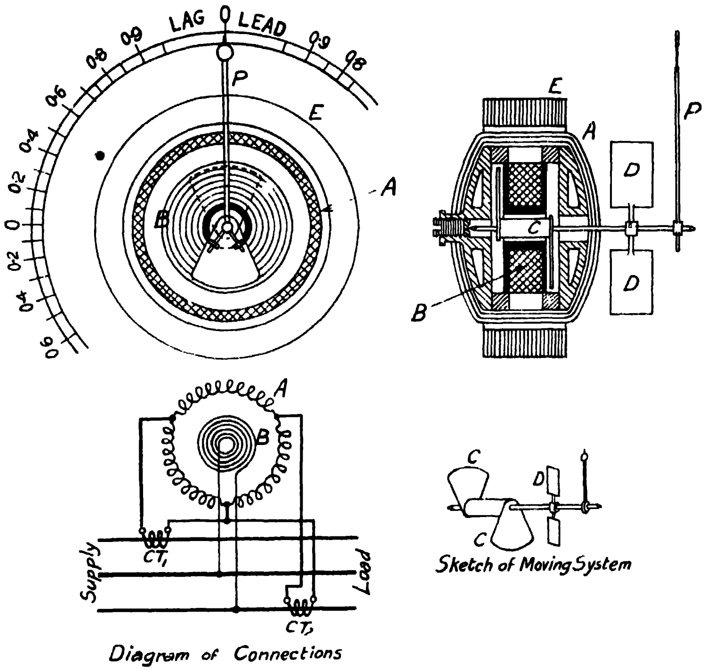


FIG. 245. MOVING-IRON FORM OF POWER-FACTOR METER IN WHICH A ROTATING MAGNETIC FIELD IS EMPLOYED

the outer coils, A, the centre line of the moving coil being coincident with the axis of the iron vanes.

Expressions for the torque acting upon such a coil are readily obtained, and it is easy to show that the angular deflection of the coil from a given position is equal to the phase difference between the voltage and current in the polyphase system.

In the actual instrument, however, the effects of hysteresis and eddy currents in the moving core and iron vanes, and the inductance of the inner magnetizing coil, B, cause the indications to be affected by variations of frequency and wave-form. Hence the instruments

must be calibrated at the particular frequency at which they will be used.

The essential features of *alternating-field* instruments (due to Lipman, and manufactured by Messrs. Nalder Bros. and Thompson) for three-phase balanced and unbalanced loads are shown in Figs. 246, 247.*

In the instrument for *balanced* loads (Fig. 246) the moving system comprises three pairs of iron vanes and cores, C_1, C_2, C_3, which are fixed to a common spindle pivoted in jewelled bearings, the spindle also carrying the pointer, P, and mica damping vanes, D. The vanes are sector-shaped—the arc subtending an angle of 120°—and the vanes forming each pair (which are magnetically connected together by the core) are fixed 180° apart (as in the rotating-field instrument, Fig. 245). The cores are separated on the spindle by distance pieces, S, of non-magnetic material, and the axes of symmetry of the three pairs of vanes are displaced 120° with respect to one another.

The cores and vanes are magnetized by the fixed coaxial pressure coils B_1, B_2, B_3, which are mounted coaxially with the spindle and are excited with currents proportional to the phase voltages of the three-phase system.

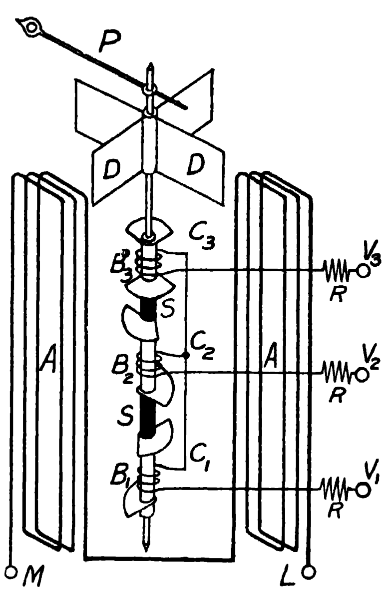
The current coil, A, is wound in two equal sections, which are mounted parallel to each other on opposite sides of the spindle, with their magnetic axes lying in a plane passing through the pivotal axis of the moving system. The two sections are connected in series or parallel, according to the current range of the instrument, and are supplied with current proportional to the current in one of the line wires of the three-phase system.

In the position of equilibrium of the moving system the resultant mean torque is zero, i.e. the mean torque acting on one pair of vanes is balanced by the mean torques due to the other pair of vanes. Under these conditions, and if the effects of hysteresis, eddy-currents, and inductance of the pressure-coil circuits are ignored, it can be shown that, for the pair of vanes which are magnetized from the same phase as the fixed coil, the angle of displacement of the axis of symmetry of these vanes is equal to the phase difference between the phase voltage and current of the three-phase system.

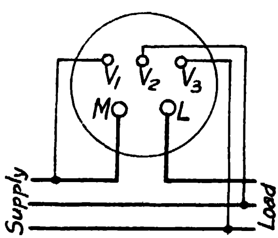
The instrument for *unbalanced* loads operates on the same principle as, but the construction differs in a number of features from, the instrument for balanced loads, since the moving system

* See also "Power-Factor Meters," by F. E. J. Ockenden, *Electrical Review*, vol. 93, p. 164.

must be acted upon by the currents in all line wires. The essential features are shown in Fig. 247.

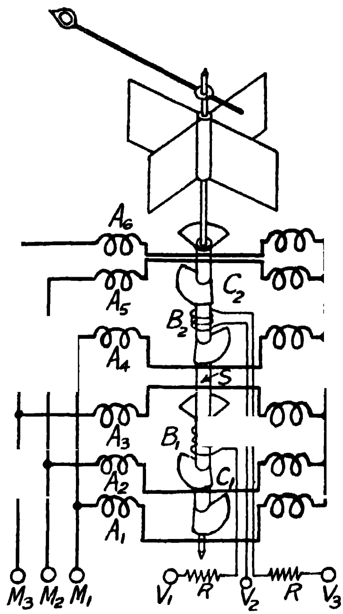


(*a*)

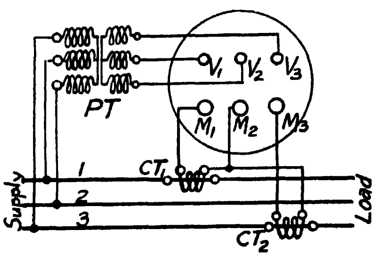


(*b*)

FIG. 246



(*a*)



(*b*)

FIG. 247

(*a*) PRINCIPLE OF MOVING-IRON, ALTERNATING FIELD, THREE-PHASE POWER-FACTOR METERS FOR BALANCED AND UNBALANCED LOADS
(*b*) DIAGRAMS OF EXTERNAL CONNECTIONS

The moving system consists of two sets of vanes, C_1, C_2, each set comprising three 120° sectors fixed with their axes of symmetry 120° apart and magnetically connected together by iron cores;

the two sets of cores being separated on the spindle by a distance piece of non-magnetic material, S. The cores and vanes are magnetized by the two coaxial pressure coils B_1, B_2, which, together with suitable non-inductive series resistances, are connected across the line wires of the three-phase system.

Six current coils, A_1–A_6 are employed, three coils being provided for each set of vanes. Each group of (three) coils is star-connected, and the two groups are connected in parallel and supplied, by means of current transformers, with currents proportional to the currents in the line wires of the three-phase system. Alternatively, for low-voltage circuits and currents not exceeding 30 A., the star connections may be omitted, in which case the corresponding coils of each group are connected in parallel with each other and inserted directly in the line wires of the three-phase system.

The current coils are of the flat type ; they are mounted parallel to one another and adjacent to the vanes, the magnetic axes of the coils being perpendicular to the spindle and coplanar with the vanes.

The mean resultant torque acting on one set of vanes is due to the combined effects of the currents in all line wires and the voltage across one pair of line wires, while the mean resultant torque acting on the other set of vanes is due to the combined effect of the currents in all line wires and the voltage across the other pair of line wires. These torques balance each other in the position of equilibrium of the moving system, and this position, therefore, corresponds to the average phase difference between the phase voltages and currents for the three-phase system.

In the *single-phase instrument* the moving system consists of two sets of vanes arranged as shown in Fig. 249. The two magnetizing coils are supplied, from the single-phase system, with currents having a phase difference of 90°, a reactance being connected in series with one coil and a non-inductive resistance in the other coil. The current coil is similar to that of the three-phase balanced-load instrument (Fig. 246).

Synchroscopes

A synchroscope, or synchronism indicator, is a special form of phase meter for indicating the phase coincidence of the E.M.Fs. of two alternators, or two alternating-current supply systems, which are to be operated in parallel.

Requirements. When an alternator is to be operated in parallel with other alternators already in service, the switch connecting the incoming machine to the bus-bars must be closed at the instant

when the E.M.F. of this machine is equal to, and is in phase-opposition with, the E.M.F. at the bus-bars, and the frequency of the two E.M.Fs. has the same value. With a polyphase machine the direction of phase rotation must agree with that of the bus-bars.

The equality in the magnitudes of the E.M.Fs. is indicated either by two voltmeters or a differential voltmeter (called a "paralleling" voltmeter). The latter consists of two moving systems fitted to a common spindle and acted upon differentially by operating coils excited from the bus-bars and the incoming machine respectively.

The coincidence of phase* may be indicated in a number of ways, such as by incandescent lamps suitably connected; by a combination of vibrating reeds; and by a special form of phase meter. Two important requirements—especially in large generating stations where the consequences resulting from incorrect closing of the switch are very serious—are that the coincidence of phase shall be indicated accurately, and that accurate indications shall be given of small phase differences in the E.M.Fs. of the incoming machine and bus-bars. These requirements are satisfied only with the phase-meter type of indicator, in which precautions have been taken to secure sensitiveness and low inertia of the moving system. Moreover, this type of indicator also indicates whether the frequency of the incoming machine is lower than, equal to, or higher than, the frequency at the bus-bars.

With polyphase alternators and systems, the phase coincidence is indicated by a single-phase instrument connected across corresponding line wires of the bus-bars and incoming machine. The direction of phase rotation of the incoming machine must agree with that of the bus-bars, and this must be tested before any attempt is made to parallel the machine. Methods of testing the phase rotation are given in Chapter XVIII.

Principles of Operation. The electromagnetic principle of operation is employed in the phase-meter type of synchroscope. Both the moving-iron and the electro-dynamic forms of construction are employed in practice, but the moving-iron form of instrument is in more general use, as the moving system can be given complete freedom of movement, and the pointer can travel over the full 360° of arc, as shown in Fig. 248. To obtain accurate indications from a moving-iron instrument, however, the effects of hysteresis and

* Although the E.M.Fs. are actually in phase-opposition relative to each other, it is convenient to consider that they are in phase but that one is reversed relatively to the other. The term "phase coincidence," as employed in connection with synchroscopes, is to be interpreted in this manner.

eddy currents in the moving-iron elements must be reduced to a minimum, and the inertia and friction of the moving system must be as small as practicable.

The moving-iron instrument may operate with either a rotating magnetic field or an alternating magnetic field. When a rotating field is employed, the moving-iron vanes are liable to be acted upon by the rotating field, as in an induction instrument, and precautions must be taken to reduce to a negligible amount the torque due to this effect. Instruments which operate with an alternating field are free from this defect.

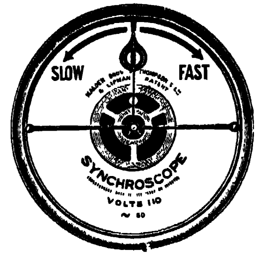


FIG. 248. EXTERNAL VIEW OF MOVING-IRON SYNCHROSCOPE (*Nalder Bros. & Thompson*)

Construction. A diagram showing the connections and essential parts of the moving-iron synchroscope is given in Fig. 249.* This instrument is of the Lipman alternating-field type. The moving system is identical with that of a single-phase power factor meter, and the magnetizing coils B_1, B_2 and the method of obtaining the required phase difference in their exciting currents are also identical in the two cases. The magnetizing coils are excited from the incoming machine.

The field coil, A, however, is wound with fine wire and, together with a non-inductive series resistance, is connected to the bus-bars.

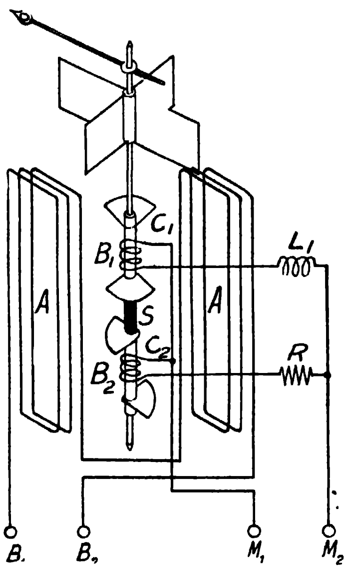


FIG. 249. PRINCIPLE OF MOVING-IRON ALTERNATING-FIELD SYNCHROSCOPE

Theory of Action of Moving-iron Instrument. When the frequency of the incoming machine is equal to that at the bus-bars, the action of the moving-iron synchroscope is identical with that of the corresponding form of power-factor meter, the angular displacement of the pointer from the central position being proportional to the phase difference between the two E.M.Fs. When, however, the E.M.Fs. have different frequencies, the moving system rotates in one direction or the other, according to whether the frequency of the

* For other examples, including lamp synchroscopes, see *Power Wiring Diagrams*, Third Edition, pp. 124–128.

incoming machine is higher or lower than that at the bus-bars. The speed of rotation, in revolutions per second, is equal to the difference between the two frequencies.

Hence if E_1, E_2 denote the E.M.Fs. applied to the fixed and moving-coil circuits, f the frequency at the bus-bars, $f \pm f'$ the frequency of the incoming machine, the instantaneous torques are

$$\mathfrak{T}_{A(inst)} = k_1 E_{1m} E_{2m} \cos(\theta - 90°) \sin[2\pi(f + f')t \pm a] \sin 2\pi ft$$
$$= k_1 E_1 E_2 \cos(\theta - 90°)\ [\cos(\pm 2\pi f't \pm a) - \cos(2\pi(2f + f')t \pm a)]$$
$$\mathfrak{T}_{B(inst)} = -k_1 E_{1m} E_{2m} \cos\theta \sin[2\pi(f \pm f')t - \tfrac{1}{2}\pi \pm a] \sin 2\pi ft$$
$$= -k_1 E_1 E_2 \cos\theta [\cos(\pm 2\pi f't \pm a - \tfrac{1}{2}\pi) - \cos(2\pi(2f + f')t \pm a - \tfrac{1}{2}\pi)]$$

where a is the phase difference between the two E.M.Fs.

If $f' = 0$, i.e. the frequency of the incoming machine is equal to that at the bus-bars, the above expressions reduce to

$$\mathfrak{T}_{A(inst)} = k_1 E_1 E_2 \sin\theta [\cos \pm a - \cos(4\pi ft \pm a)]$$
and $$\mathfrak{T}_{B(inst)} = -k_1 E_1 E_2 \cos\theta\ [\sin \pm a - \sin(4\pi ft \pm a)]$$

Hence the mean torques are

$$\mathfrak{T}_A = k_1 E_1 E_2 \sin\theta \cos \pm a$$
and $$\mathfrak{T}_B = -k_1 E_1 E_2 \cos\theta \sin \pm a$$

The condition for equilibrium is that $\mathfrak{T}_A + \mathfrak{T}_B = 0$,

i.e. $$\sin\theta \cos \pm a = \cos\theta \sin \pm a \qquad (186)$$
or $$\theta = \pm a$$

Thus the pointer is stationary, and its deflection from the "zero" position (which is vertically upwards) represents the phase difference between the E.M.Fs.

If the frequency of the incoming machine differs slightly from that at the bus-bars, the torques tending to deflect the moving system will be given by

$$\mathfrak{T}_A = k_1 F_1 E_2 \sin\theta \cos(\pm 2\pi f't \pm a)$$
and $$\mathfrak{T}_B = -k_1 E_1 E_2 \cos\theta \sin(\pm 2\pi f't \pm a)$$

since the component torques having a frequency of $2f \pm f'$ will have no effect upon the moving system.

Hence, for equilibrium $\mathfrak{T}_A + \mathfrak{T}_B = 0$, or

$$\sin\theta \cos(\pm 2\pi f't \pm a) = \cos\theta \sin(\pm 2\pi f't \pm a) \qquad (187)$$
i.e. $$\theta = \pm 2\pi f't \pm a$$

Thus the moving system rotates with an angular velocity equal to that corresponding to the difference in the two frequencies (i.e. one revolution corresponds to a difference of one cycle in these frequencies), and the direction of rotation depends upon whether the frequency of the incoming machine is higher (f' positive) or lower (f' negative) than the frequency at the bus-bars.

Frequency Meters

Principles of Operation. A number of principles may be employed, such as (1) mechanical resonance, (2) electrical resonance, (3) variation of electrical characteristics with frequency of two parallel circuits containing resistance and inductance. Instruments operating on electrical principles form interesting examples of the application of the elementary principles discussed in the earlier chapters.

Mechanical Resonance or Vibrating-reed Frequency Meters. In these instruments a number of steel reeds, which are so tuned that

the natural frequencies of successive reeds differ slightly from one another, are acted upon by an electromagnet excited from the supply system. Only those reeds having a natural frequency approximately equal to that of the exciting current will show visible vibration, and, if this frequency coincides with the natural frequency of any particular reed, that reed will have a large amplitude of vibration. If the frequency of the exciting current lies between the natural frequencies of two adjacent reeds, these reeds will vibrate with approximately equal amplitudes, which, however, will be less than the maximum. Thus the accuracy with which the frequency can be measured depends upon the interval between the natural frequencies to which the successive reeds are tuned. This disadvantage has led to the disuse of this form of frequency meter, the deflectional type being now always employed.

Electrical Resonance Frequency Meters. The operation of these instruments depends upon the principle that both the power and factor of, and the current in, a series resonant circuit supplied at constant voltage change with the frequency. At resonance frequency the power factor is unity and the current is a maximum; when the frequency is decreased or increased, the power factor becomes lagging or leading, respectively, and the current diminishes rapidly (see Fig. 46, p. 87).

Construction. The instruments are similar in construction to the moving-coil and moving-iron forms of single-phase power-factor meters. The operation of the moving-coil form of instrument may be made to depend upon either the variation of power factor, with frequency, of a single resonant circuit, or the variation of current, with frequency, in two resonant circuits which have different resonance frequencies and are connected in parallel. With moving-iron instruments the operation depends upon the variation of power factor, with frequency, of a single resonant circuit.

Moving-iron, Resonant Circuit, Frequency Meter. The circuits of an instrument of the Lipman type are shown in Fig. 250. The moving system is practically identical with that of the synchroscope (Fig. 249), and the two magnetizing and field coils are also identical in the two cases.

The resonant circuit consists of the field coil, A, and a condenser of suitable capacity connected in series, the field coil itself providing the required inductance. This circuit is adjusted so that its resonance frequency corresponds to the mid-point of the frequency range of the instrument.

The deflection of the moving system from the central, or mid-scale, position is proportional to the phase difference between the

voltage impressed upon, and the current in, the resonant circuit, and the action of the instrument is similar to that of a single-phase power-factor meter.

An inductance, L_2, is connected in series with the instrument for the purpose of rendering its indications practically independent of the wave-form of the supply circuit.

Frequency Meters with Parallel Circuits Having Dissimilar Electrical Characteristics. Frequency meters depending for their operation upon the variation of current, with change of frequency, in

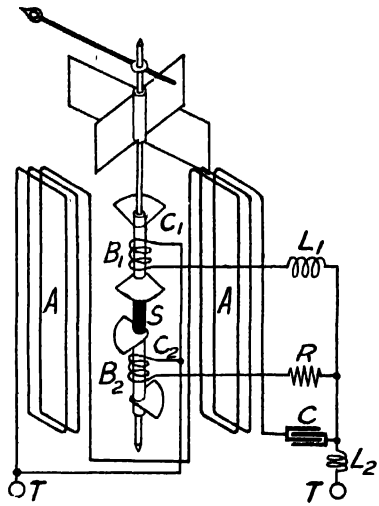


FIG. 250. PRINCIPLE OF MOVING-IRON, ALTERNATING-FIELD, RESONANT-CIRCUIT FREQUENCY METER

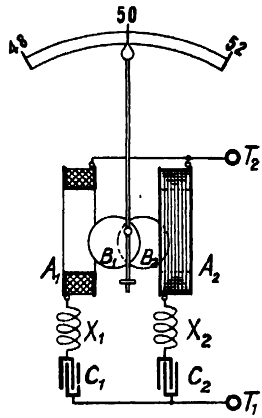


FIG. 251. PRINCIPLE OF MOVING-IRON PARALLEL-CIRCUIT FREQUENCY METER

(Everett-Edgcumbe & Co.)

parallel-connected circuits containing resistance and inductance, or resistance, inductance and capacitance, are constructed in both the induction and moving-iron forms.

In *induction instruments* two travelling magnetic fields act upon a pivoted aluminium disc or drum which is free from controlling forces. The torques due to these fields oppose each other, and follow different laws with respect to the frequency. Stability between the torques for a definite frequency is obtained by shaping the disc or drum, so that one field acts upon a diminishing surface, and the other field acts upon an increasing surface.

The principle of a moving-iron instrument (due to Coleman) is shown in Fig. 251. Two assemblies of soft-iron discs, B_1, B_2.

are mounted eccentrically on the spindle of the instrument, and are spaced sufficiently far apart to avoid mutual interference. The magnetizing coils, A_1, A_2, are connected in the branches of a parallel circuit, the constants of which are chosen according to the frequency range desired.

Thus for a wide range of frequency, e.g. 40 to 60 cycles per sec., one circuit is principally resistive and the other is inductively reactive. For example, a resistance is connected in series with the magnetizing coil A_1, and an inductive reactance is connected in

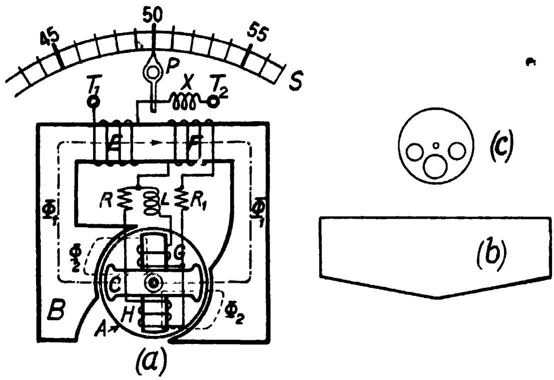


FIG. 252. ARRANGEMENT OF ELECTRIC AND MAGNETIC CIRCUITS (a) OF INDUCTION FREQUENCY METER (DRUM-TYPE)

The development of the cylindrical surface of the drum is shown at (b), and an end view of the drum is shown at (c)

(*Nalder Bros. & Thompson*)

series with the magnetizing coil A_2. The current in the former circuit is therefore independent of the frequency, while the current in the latter circuit is inversely proportional to the frequency. In order to reduce errors due to distorted wave-form, a choking coil is connected in series with the parallel circuits.

For a medium range of frequency, e.g. 45 to 55 cycles per sec., one circuit contains a condenser and a choking coil, which are adjusted to give a current which increases with the frequency over the range of the instrument. The other circuit is inductively reactive, as in the previous case.

For a narrow range of frequency, e.g. 48 to 52 cycles per sec., both circuits contain condensers and choking coils, as shown in Fig. 251. One circuit, X_1C_1, is adjusted to give a current which rises rapidly with the frequency (over the range of the instrument), and the other circuit, X_2C_2, is adjusted to give a current which falls rapidly with the frequency.

Construction of Drum-type Induction Frequency Meter. The arrangement of a Lipman type of instrument is shown in Fig. 252. The operating magnets resemble those of other Lipman induction instruments, but in the present case the poles of the outer magnet are cut away so as to reduce the pole arc.

The outer magnet is wound with primary and secondary windings, E, F, respectively, as for a voltmeter, except that the secondary winding is designed for a higher voltage. A reactance coil, X, is connected in series with the primary winding to reduce errors due to harmonics if present in the wave-form of the supply voltage.

The inner magnet has two independent exciting windings, G and H. Both are supplied from the secondary winding F, but a reactance, L, is connected in the circuit of G, and a resistance, R, is connected in the circuit of H. Another resistance, R, common to both circuits, is provided for adjusting purposes during the calibration of the instrument.

The exciting windings on the inner magnet produce, in combination with the winding on the external magnet, travelling fields in the air-gap, which move in opposite directions relative to the drum. These fields are of equal strength at the frequency corresponding to the central scale position of the pointer.

A change of frequency alters the relative values of these fluxes (e.g. an increase in frequency will cause the current in coil G to decrease and that in coil H to increase), and the drum will, therefore, take up a new position of equilibrium.

CHAPTER XVII

INSTRUMENT TRANSFORMERS

Use of Instrument Transformers in Practice. The range of alternating current switchboard instruments is usually extended by means of transformers—in preference to series resistances and shunts—as, in addition to extending the range, the transformers also insulate the instruments from the main circuit, and therefore enable low-voltage instruments (supplied through transformers) to be used with safety on high-voltage circuits. In such cases the current circuits of the instruments are designed for a maximum current of 5 A., and the pressure circuits are designed for a maximum pressure of about 100 V. Again, with transformers, the range of an instrument may be extended almost indefinitely with only a small increase in the power consumption, as the losses in the transformer are extremely small.

The use of transformers with instruments, however, introduces slight errors into the instrument readings, and these errors must be taken into consideration when accurate results are required. Usually the errors are only important in connection with wattmeter readings at low power factors, and the method of applying the corrections is discussed later.

Construction. *Current Transformers.* The primary winding of a current, or series, transformer is connected in series with the main circuit, and the secondary winding is connected to the current coil of the measuring instrument, the function of the transformer being to supply the instrument with current proportional to that in the main circuit and in phase opposition thereto.* The current range of the instrument connected to the secondary winding is usually 5 A.

The number of turns in the primary winding vary with the type of magnetic circuit and the magnitude of the current in the main circuit, and when this current is large the primary winding consists of a single conductor.

The insulation of the primary winding is of extreme importance in transformers for high-voltage circuits. After being wound and insulated, the windings are impregnated, in vacuo, with insulating

* The phase relationship of the currents is important only in connection with power measurements.

compound, and the entire transformer may be either immersed in oil or fitted into a case and filled with insulating compound applied in vacuo.

The magnetic circuit is constructed of high quality laminations and may take the form of a ring, a rectangle, or a double rectangle, as shown in Fig. 253. For the highest accuracy the laminations must be of good quality alloyed iron, and the magnetic circuit must be without joints, thereby necessitating hand-wound coils. In commercial transformers, considerations of cost require the use of

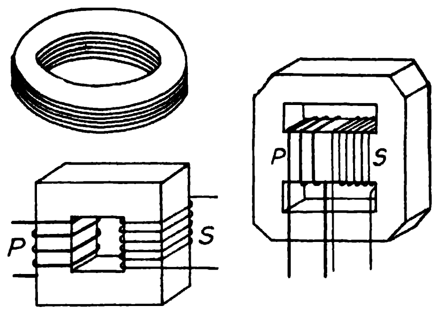


FIG. 253. FORMS OF CORE CONSTRUCTION FOR CURRENT TRANSFORMERS

machine wound coils, and therefore a jointless magnetic circuit cannot be employed. But the laminations at the joints must be interleaved so as to reduce the reluctance.

Potential Transformers. In these transformers the primary winding is connected across the supply system and the secondary winding is connected to the pressure circuit of the measuring instrument, the function of the transformer being to supply the latter with a potential difference proportional to the voltage of the supply system and in phase opposition thereto.*

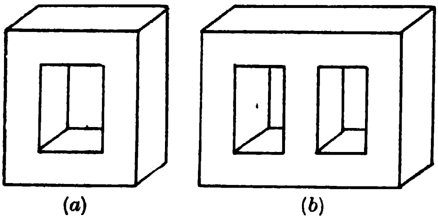
The primary winding must therefore consist of a large number of turns and must be adequately insulated. Moreover, the terminals must be separated from one another and must be adequately insulated from the frame by porcelain or other bushings. Fuses are necessary as a protection against breakdown of the insulation, internal short-circuits, etc., and these are usually mounted on bushings adjacent to the terminal bushings.

The magnetic circuit is constructed of high-quality laminations

* The phase relationship of the voltages is important only in connection with power measurements.

and takes the form of a single rectangle (Fig. 254*a*) for single-phase transformers, and a double rectangle (Fig. 254*b*) for three-phase transformers. As former wound coils are only permissible in the present case, the laminations forming the cores and yokes must be assembled after the coils are in position and the joints must be interleaved.

The coils forming the secondary and primary windings are located symmetrically one over the other on each core of the magnetic circuit, the primary coils being outside the secondary coils. For high-voltage circuits the primary coils are wound in sections, the several sections being separately insulated and connected in series. The windings are impregnated in vacuo in the same manner as those of current transformers.

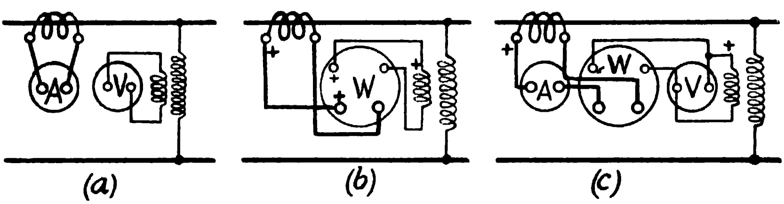


(*a*) (*b*)

FIG. 254. FORMS OF CORE CONSTRUCTION FOR POTENTIAL TRANSFORMERS

Connections of Transformers and Instruments. The diagrams of Fig. 255 show the connections of an ammeter, a voltmeter, and a single-phase wattmeter, or power-factor meter, when used with instrument transformers.

If the ammeter, voltmeter, and wattmeter are scaled as straight-through (i.e. low voltage) instruments, the current in the main



(*a*) (*b*) (*c*)

FIG. 255. CONNECTION DIAGRAMS FOR INSTRUMENTS USED WITH INSTRUMENT TRANSFORMERS

circuit will be given by: Ratio of current transformer × ammeter reading; the voltage by: Ratio of potential transformer × voltmeter reading; and the power by: Wattmeter reading × ratio of current transformer × ratio of potential transformer.

When extreme accuracy is required in connection with power measurements, the correction factors of the transformers must be known.

In the connections for the wattmeter and power-factor meter, a

knowledge of the *relative polarities* of the secondary terminals of the transformers with respect to the primary terminals is important, otherwise the instruments may be so connected as to read improperly.

The relative polarities of the terminals of the primary and secondary windings should be marked by the manufacturer, either by conventional signs or letters. In the event of a transformer having no markings the relative polarities of the terminals may be determined very simply by passing a small direct current through one winding and connecting a low reading permanent-magnet moving-coil voltmeter across the other winding. Then, if the voltmeter deflects *up* the scale on *closing* the direct-current circuit, the terminal of the transformer which is connected to the positive terminal of the voltmeter will have the same polarity as that which is connected to the positive terminal of the direct-current supply. Precautions should be taken to avoid leaving the magnetic circuit of the transformer in a highly magnetized condition, otherwise its accuracy will be impaired.

Precautions to be Observed with Current Transformers. When using current transformers it is important that the secondary circuit be *always closed* when current is passing through the primary winding, otherwise the magnetic circuit may become highly saturated. The iron losses and the heating of the core will then become excessive, and a relatively high voltage will be induced in the secondary winding which may cause a breakdown of the insulation between adjacent turns and a burning-out of the transformer. Even if the transformer is capable of successfully withstanding these abnormal conditions, it is highly probable that, when the primary current is switched off, or the secondary circuit is again closed, the magnetic circuit will be left in a highly-magnetized condition, which will impair the accuracy for future work. To restore the core to its normal magnetic state, it must be demagnetized by passing an alternating current through the primary winding (with the secondary winding open circuited) and gradually decreasing this current to zero.

When current transformers are used with portable instruments, precautions are necessary to avoid errors due to (1) the relatively large stray magnetic field which may exist with open-type transformers for high-voltage circuits owing to the separation of the primary and secondary coils ; (2) the inadvertent substitution of instruments having a range (e.g. 1, 2, or 3 A.) lower than the standard range (5 A.), as the former have a much higher impedance than the latter and may involve operating conditions in the

transformer which may be quite different from the normal conditions.

Theory of Current Transformer. In an ideal transformer the magnetizing ampere-turns and the losses are zero, and therefore, (1) the ratio of primary and secondary currents is constant for all loads; and (2) the phase difference between these currents is 180°.

The vector diagram representing these conditions is shown in Fig. 256*a*, in which $O\Phi$ represents the flux; OE_1, OE_2, the E.M.Fs. induced in the primary and secondary windings (which are directly proportional to the numbers of turns in these windings); $OV_1(= -OE_1)$, the potential difference at the terminals of the primary winding; OI_2, the current in the secondary circuit lagging $\varphi°$ with respect to OE_2; OB, the ampere-turns due to the secondary winding; $OA(= -OB)$, the ampere-turns due to the primary winding; and OI_1, the current in the primary circuit. Observe that, since the resultant ampere-turns are zero, the ratio of currents is equal to the inverse ratio of turns, i.e. $I_1/I_2 = N_2/N_1$, where I_1, I_2, denote the primary and secondary currents, respectively, and N_1, N_2, the number of turns in the primary and secondary windings, respectively.

In a practical transformer a definite number of ampere-turns are required for the magnetic circuit, there are losses in the core and windings, and magnetic leakage occurs between the windings. The vector diagram representing these conditions is shown in Fig. 256*b*, some of the vectors being exaggerated to obtain legibility. The diagram is drawn with the secondary current as the vector of reference, since when this current, together with data of the transformer, are known, the induced E.M.Fs., the flux, and the primary current may be readily determined.

The E.M.F. induced in the secondary winding is represented by OE_2, and balances the vector sum (Od) of the internal E.M.Fs. due to the resistance and reactance of the secondary circuit, of which Oa, bc—in phase opposition with respect to the secondary current—represent the E.M.Fs. due to the resistances of the secondary winding and load (i.e. ammeter or other instrument), respectively, and ab, cd—lagging 90° with respect to the current—represent the E.M.Fs. due to the reactance (inductive) of the load and the leakage reactance of the secondary winding, respectively.

The E.M.F. induced in the primary winding is represented by OE_1, the ratio OE_2/OE_1 being equal to the ratio of the numbers of secondary and primary turns.

The flux is represented by $O\Phi$, which leads the induced E.M.Fs. by 90°. The magnetizing ampere-turns are represented by OD, and the exciting ampere-turns by OC, the latter leading the flux by the angle α. The exciting ampere-turns are the resultant of the ampere-turns due to the primary and secondary windings. Hence, if the ampere-turns of the secondary winding are represented by OB, then the vector sum (OA) of OB reversed and OC will represent the primary ampere-turns. The primary current is, therefore, represented by OI.

The voltage at the terminals of the primary winding is represented by OV_1, and is equal to the vector sum of the induced E.M.F. (OE_1) and the E.M.Fs. due to the resistance and leakage reactance of the primary winding, these E.M.Fs. being represented by Oe and Of, respectively.

If this vector diagram, Fig. 256*b*, is compared with that, Fig. 256*a*, for an ideal transformer, we observe that, in the commercial transformer—

1. The ratio of primary and secondary currents is not strictly proportional to the ratio of the numbers of turns.
2. The phase difference between these currents is less than 180°.
3. The ratio and phase difference are not constant, but vary with the magnitudes of the currents and the impedance of the secondary circuit.

To approach ideal conditions in a commercial transformer, the exciting

ampere-turns must be small in comparison with the primary, or secondary, ampere-turns, and the resistance and reactance of the secondary circuit must be kept as low as practicable.

Expressions for Ratio and Phase Angle of Current Transformer. The ratio of the primary and secondary currents (called in practice the "ratio" of the transformer) may be calculated from the vector diagram, the vectors concerned being drawn separately in Fig. 256c, in which the secondary ampere-turn

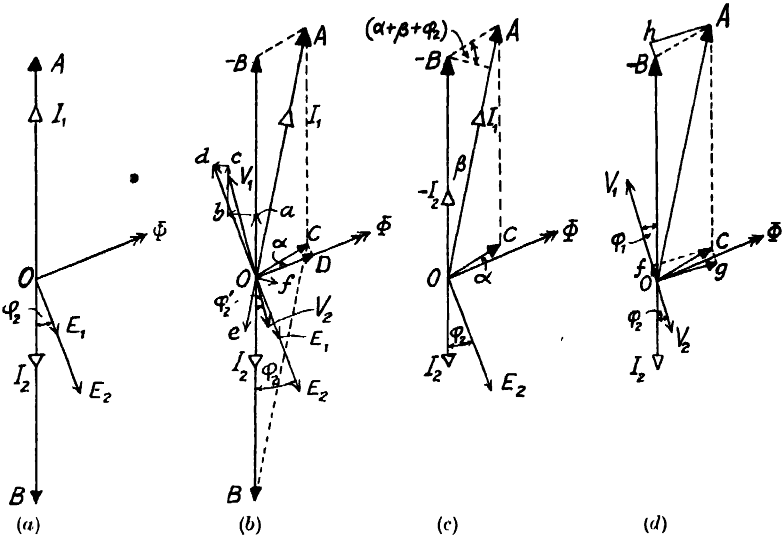


FIG. 256. VECTOR DIAGRAMS FOR CURRENT TRANSFORMERS

vector is reversed from its normal position. From this diagram we obtain the relation

$$OA = OB\cos\beta + OC\sin(\alpha + \varphi_2 + \beta)$$

where β is the angle between the primary ampere-turns vector and the reversed secondary ampere-turn vector. This angle (β) is called the "phase angle" of the transformer, and is usually very small (from about 0·5° to 3·0°).

Substituting ampere-turns in this expression, we have

$$I_1N_1 = I_2N_2\cos\beta + I_0N_1\sin(\alpha + \varphi_2 + \beta)$$

where I_0 is the fictitious exciting current (i.e. I_0 = exciting ampere-turns/N_1).

Whence $$\frac{I_1}{I_2} = \frac{N_2}{N_1}\cos\beta + \frac{I_0}{I_2}\sin(\alpha + \varphi_2 + \beta). \qquad (188)$$

or, since β is a small angle, we have, to a very close approximation,

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} + \frac{I_0}{I_2}\sin(\alpha + \varphi_2) \quad . \quad . \quad . \quad . \quad . \qquad (188a)$$

Hence, on account of magnetizing current, losses, and reactance in the secondary circuit, the ratio of turns (i.e. N_2/N_1) must be smaller than the ratio required for the currents. For example, a transformer designed for a ratio of 10 : 1 would have N_2/N_1 equal to between 9·8 to 9·9.

Observe that both the exciting current (I_0) and α depend upon the flux density and the magnetic qualities of the core. With a given transformer, the flux density is almost directly proportional to the E.M.F. induced in the

secondary winding, and with a given secondary current, this E.M.F. is proportional to the impedance of the secondary circuit.

An alternative expression for the ratio may be obtained by resolving the exciting ampere-turns into in-phase and quadrature components with respect to the primary voltage V_1. Thus, if these components are denoted by N_1I_{0P} and N_1I_{0Q} respectively, and are represented by the vectors Of, Og respectively, in Fig. 256 (d), then, since β is a small angle, we have to a very close approximation

$$N_1I_1 = N_2I_2 + N_1I_{0P}\cos\varphi_1 + N_1I_{0Q}\sin\varphi_1.$$

Whence

$$I_1 = \frac{N_2}{N_1}I_2 + I_{0P}\cos\varphi_1 + I_{0Q}\sin\varphi_1$$

and

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} + \frac{I_{0P}}{I_2}\cos\varphi_1 + \frac{I_{0Q}}{I_2}\sin\varphi_1 \quad . \quad . \quad . \quad . \quad . \quad (189)$$

The phase angle, β, is readily determined from the diagram, Fig. 256c. Thus

$$\tan\beta = \frac{OC\ \cos(a + \varphi_2 + \beta)}{OB\cos\beta} = \frac{I_oN_1\cos(a + \varphi_2 + \beta)}{I_2N_2\cos\beta} \quad . \quad (190)$$

Since β is a small angle, we have, to a close approximation,

$$\beta_{\text{radians}} = \frac{N_1}{N_2} \cdot \frac{I_o}{I_2}\cos(a + \varphi_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (190a)$$

Observe that the vector OI_2 of the secondary current is leading with respect to the reversed vector of the primary current.

Data of Current Transformer. A commercial current transformer, intended for a primary current of 50 A., a secondary current of 5 A., and a secondary load of 40 V.A., has a ring core wound with 30 primary turns and 294 secondary turns. The resistances (at 20° C.) of the windings are 0·018 ohm (primary) and 0·65 ohm (secondary). The mean diameter of the core is 10 cm. and the magnetic cross-section is 5 cm.^{2}

Theory of Potential Transformer. In an ideal potential transformer without losses, the voltages at the terminals of the primary and secondary windings are equal to the E.M.Fs. induced in these windings, and their ratio is constant and equal to the ratio of the numbers of turns in the windings. Thus, $V_1/V_2 = E_1/E_2 = N_1/N_2$. Moreover, with constant frequency the flux in the core is proportional to the voltage applied to the primary winding.

The vector diagram is shown in Fig. 257a, in which $O\Phi$ represents the flux in the core ; OE_1, OE_2, the E.M.Fs. induced in the primary and secondary windings, respectively ; $OV_1(= -OE_1)$, the impressed E.M.F. ; OI_2, the current in the secondary circuit ; OB, the secondary ampere-turns ; $OA(= -OB)$, the primary ampere-turns ; and OI_1, the primary current.

In the diagram for the practical transformer, Fig. 257b, the secondary terminal voltage is taken as the vector of reference, OV_2. The current in the secondary circuit is represented by OI_2, which, in practice, lags by a small angle with respect to OV_2. The E.M.F. induced in the secondary winding is represented by OE_2, and is obtained by compounding with OV_2 the pressure drops due to the resistance and leaking reactance of the secondary winding, these pressure drops being represented by V_2a and aE_2, respectively.

The flux leads OE_2 by 90°, and is represented by $O\Phi$. The exciting ampere-turns are represented by OC, and from this vector and the vector OB, representing the secondary ampere-turns, the vector OA, representing the primary ampere-turns, is obtained. The primary current is therefore represented by OI_1.

The E.M.F. induced in the primary winding is represented by OE_1, the ratio OE_1/OE_2 being equal to the ratio of the numbers of turns in the windings. The external voltage at the terminals of the primary winding is represented by OV_1, and balances the induced E.M.F., OE_1, together with the internal

E.M.Fs. due to the resistance and reactance of the primary winding, these E.M.Fs. being represented by Oc and Od, respectively.

Expressions for Ratio and Phase Angle of Potential Transformer. The ratio of the primary and secondary voltages (called in practice the "ratio" of the transformer) is easily calculated from the vector diagram if the diagram is re-drawn with all secondary voltage vectors increased in the ratio N_1/N_2, and the secondary current vector diminished in the ratio N_2/N_1; i.e. the scale for secondary voltages is N_1/N_2 times that for the primary voltages, and the scale for secondary currents is N_2/N_1 times that for primary currents. Thus the vector representing the E.M.F. induced in the secondary is now of

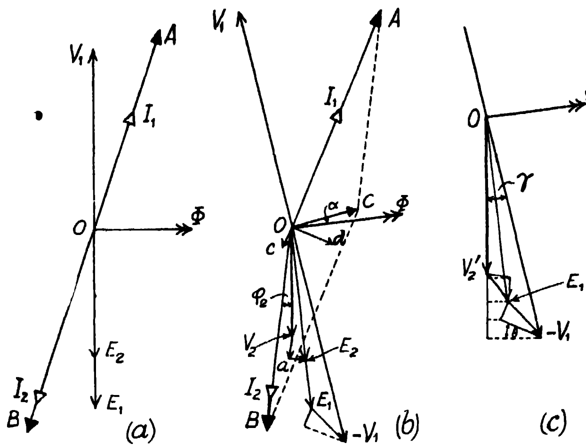


FIG. 257. VECTOR DIAGRAMS FOR POTENTIAL TRANSFORMER

equal length to that representing the E.M.F. induced in the primary winding. The new vector diagram for the quantities concerned is shown in Fig. 257c.

If the various voltage vectors are projected upon OV_2 produced, we have

$$V_1 \cos\gamma = V_2 \frac{N_1}{N_2} + \frac{N_1}{N_2}(I_2R_2 \cos\varphi_2 + I_2X_2 \sin\varphi_2) + I_1(R_1 \cos\theta + X_1 \sin\theta)$$

where γ is the angle between OV_2 and OV_1 reversed; θ is the angle between OV_2 reversed and I_1; R_2, X_2 are the resistance and leakage reactance, respectively, of the secondary winding; R_1, X_1, the resistance and leakage reactance, respectively, of the primary winding.

Now γ is usually much less than 1°, so that, to a very close approximation,

$$\cos\gamma = 1\cdot 0:$$

$$I_1 \cos\theta = (I_2N_2/N_1)\cos\varphi_2 + I_0 \sin a;$$

$$I_1 \sin\theta = (I_2N_2/N_1)\sin\varphi_2 + I_0 \cos a$$

Hence, substituting in the preceding expression, we have

$$V_1 = V_2\frac{N_1}{N_2} + I_2\frac{N_1}{N_2}(R_2\cos\varphi_2 + X_2\sin\varphi_2) + R_1\left(I_2\frac{N_2}{N_1}\cos\varphi_2 + I_0\sin a\right)$$
$$+ X_1\left(I_2\frac{N_2}{N_1}\sin\varphi_2 + I_0\cos a\right)$$

$$= V_2\frac{N_1}{N_2} + I_2\cos\varphi_2\left(\frac{N_1}{N_2}R_2 + \frac{N_2}{N_1}R_1\right) + I_2\sin\varphi_2\left(\frac{N_1}{N_2}X_2 + \frac{N_2}{N_1}X_1\right)$$
$$+ I_0(R_1\sin\alpha + X_1\cos\alpha)$$

$$= V_2\frac{N_1}{N_2} + I_2\frac{N_2}{N_1}\left[\left(R_1 + \left(\frac{N_1}{N_2}\right)^2R_2\right)\cos\varphi_2 + \left(X_1 + \left(\frac{N_1}{N_2}\right)^2X_2\right)\sin\varphi_2\right]$$
$$+ I_0(R_1\sin\alpha + X_1\cos\alpha)$$

Now I_2N_2/N_1 is the component of the primary current which balances the secondary current I_2, and is called the equivalent secondary current referred to the primary circuit. Similarly, $R_2(N_1/N_2)^2$ and $X_2(N_1/N_2)^2$ are called the equivalent resistance and reactance, respectively, of the secondary winding referred to the primary circuit. These equivalent quantities will be denoted by I_2', R_2', X_2'. Hence the preceding expression becomes

$$V_1 = V_2N_1/N_2 + I'_2[(R_1 + R_2')\cos\varphi_2 + (X_1 + X'_2)\sin\varphi_2]$$
$$+ I_0(R_1\sin\alpha + X_1\cos\alpha)$$

and the ratio (V_1/V_2) is given by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_2'[(R_1 + R_2')\cos\varphi_2 + (X_1 + X_2')\sin\varphi_2] + I_0(R_1\sin\alpha + X_1\cos\alpha)}{V_2} \quad (191)$$

or, since φ_2 and α are usually small angles, we have, approximately,

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_2'(R_1 + R_2') + I_0X_1}{V_2} \quad . \quad . \quad . \quad . \quad . \quad (191a)$$

Observe that, on account of losses, magnetic leakage between the primary and secondary windings (to which the leakage reactances X_1, X_2, are due), and magnetizing current, the ratio of turns (N_1/N_2) must be slightly smaller than the ratio required for the terminal voltages. Also, since I'_2 is usually small in comparison with I_0, and X_1 is greater than $(R_1 + R_2')$, the term I_0X_1 in equation (191*a*) is more important than the term $I_2'(R_1 + R_2')$. Thus the ratio is given very approximately by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_0X_1}{V_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191b)$$

The **phase angle**, γ, of the transformer is readily calculated from the vector diagram. Thus

$$\sin\gamma = \frac{1}{V_1}\left[\frac{N_1}{N_2}(I_2X_2\cos\varphi_2 - I_2R_2\sin\varphi_2) + I_1(X_1\cos\theta - R_1\sin\theta)\right]$$

or, since γ is a very small angle, $\sin\gamma = \gamma$ (in radians),

$$\gamma\text{ radians} = \frac{1}{V_1}[I_2'\{(X_1 + X_2')\cos\varphi_2 - (R_1 + R_2')\sin\varphi_2\}$$
$$+ I_0(X_1\sin\alpha - R_1\cos\alpha)] \quad . \quad . \quad . \quad . \quad . \quad (192)$$

Application of Correction Factors for Ratio and Phase Angle. The correction factors for ratio and phase angle of instrument transformers vary with the magnitude and nature of the load connected to the secondary winding. The correction factors for a given transformer, are therefore expressed in the form of curves, typical examples being given in Figs. 258, 259.

With measurements of current and voltage using instrument transformers, the instrument readings are multiplied by the ratio correction factors, the application of which in this case is quite straightforward.

With **power measurements in single-phase circuits** by the wattmeter method, correction factors for both ratio and phase angle must be applied to the instrument readings, together with the correction factors for the wattmeter itself.

In applying the correction factors for phase angle, we observe, from Fig. 256*b*, that, for the current transformer, the secondary current leads the reversed primary current, but that, for the potential transformer and the

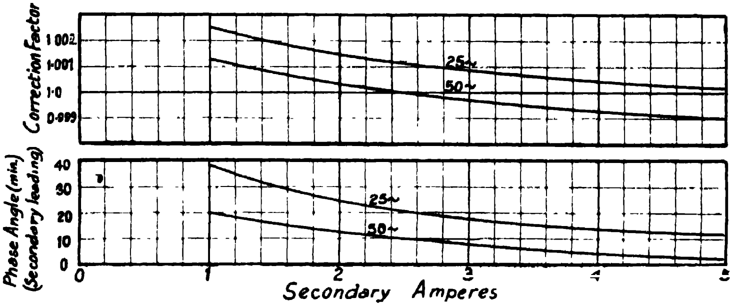


FIG. 258. CORRECTION FACTOR AND PHASE ANGLE CURVES FOR HIGH-CLASS CURRENT TRANSFORMER WITH AMMETER AND WATTMETER CONNECTED IN SECONDARY

(Silicon-steel core)

conditions represented in Fig. 257*b*, the secondary terminal voltage is lagging with respect to the reversed primary terminal voltage. With potential transformers, however, the secondary terminal voltage may be in phase with, or lagging, or leading the reversed primary voltage, according to the design of the transformer and the nature of the load.

Hence, if φ is the phase difference between line voltage and line current, the phase difference between the secondary terminal voltage of the potential

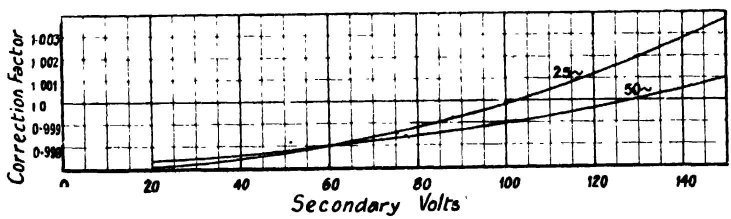


FIG. 259. CORRECTION FACTOR CURVES FOR POTENTIAL TRANSFORMER

(Silicon-steel core)

transformer and the current in the secondary circuit of the current transformer is $\varphi - (\beta \pm \gamma)$ when φ is lagging, and $\varphi + \beta \pm \gamma$ when φ is leading, the *plus* sign being used in connection with γ when the secondary terminal voltage is lagging, and the *minus* sign when this voltage is leading, with respect to the reversed primary terminal voltage.

If δ is the phase difference (lagging) between the voltage applied to the potential circuit of the wattmeter and the current in this circuit, the phase difference between the currents in the fixed and moving coils is $\varphi - (\beta \pm \gamma + \delta)$ when φ is lagging, and $\varphi + \beta \pm \gamma + \delta$ when φ is leading.

Hence, if $\cos \varphi_a$ is the apparent power factor determined from the ratio of

the uncorrected wattmeter reading and the readings of the ammeter and voltmeter, the power factor $\cos\varphi$ of the primary circuit is given by $\cos[\varphi_a \pm (\beta \pm \gamma + \delta)]$, the *plus* sign to be taken for lagging power factors and the *minus* sign for leading power factors.

Therefore the power in the primary circuit is given by

$$P = [\cos\varphi / \cos\{\varphi_a \pm (\beta \pm \gamma + \delta)\}]$$

× wattmeter reading × corrected ratio of current transformer × corrected ratio of potential transformer, when the power loss in the instruments is ignored.

Example. The following readings (corrected for calibration of instruments) were taken on a voltmeter, an ammeter, and a wattmeter connected (as in Fig. 255*c*) to instrument transformers on a high-voltage circuit, the load having a lagging power factor.

Volts 100·2; Amperes 3·4; Watts 280.

Data of the instrument transformers are as follow—

Current transformer—

Nominal ratio	15 : 1
Ratio correction factor for a secondary load of 3·4 A.	0·992
Phase angle at this load (β)	0·75°

Potential transformer—

Nominal ratio	20 : 1
Ratio correction factor	1·001
Phase angle (γ)	−0·25°

Hence,

Current in main circuit $= 3{\cdot}4 \times 15 \times 0{\cdot}992 = 50{\cdot}6$ A.

Voltage of main circuit $= 100{\cdot}2 \times 20 \times 1{\cdot}001 = 2006$ V.

Apparent power factor (from instrument readings)

$$= \cos\varphi_a = \frac{280}{100{\cdot}2 \times 3{\cdot}4} = 0{\cdot}822$$

Phase difference (φ_a) between voltage and current in instrument circuits $= \cos^{-1} 0{\cdot}822 = 34{\cdot}7°$

Actual phase difference (φ) between voltage and current in main circuit

$$= \varphi_a - (\beta - \gamma) = 34{\cdot}7 - 0{\cdot}75 + 0{\cdot}25 = 34{\cdot}2°$$

Power factor of main circuit

$$= \cos\varphi = 0{\cdot}827$$

Power in main circuit

$$= \frac{0{\cdot}827}{0{\cdot}822} \times 280 \times 15 \times 0{\cdot}992 \times 20 \times 1{\cdot}001 = 83{,}900 \text{ W.}$$

With **power measurements in three-phase circuits** by the two-wattmeter method the effect of the phase angles of the instrument transformers is to reduce the phase differences of the currents in the fixed and moving coils of the wattmeters by the angle $\beta \pm \gamma + \delta$. Hence, with balanced loads, the phase difference between the currents in the coils of wattmeter No. 1 is $[30° + \{\varphi - (\beta \pm \gamma + \delta)\}] = (30° + \varphi_a)$, and that between the currents in the coils of wattmeter No. 2 is $[30° - \{\varphi - (\beta \pm \gamma + \delta)\}] = (30° - \varphi_a)$, assuming that similar transformers are used in each phase.

If V_2, I_2 denote the voltage and current for the secondary circuits of the transformers, the readings of the wattmeters represent the quantities $V_2 I_2 \cos(30° + \varphi_a)$, and $V_2 I_2 \cos(30° - \varphi_a)$. Whence the algebraic sum of the readings represents the quantity $\sqrt{3} \, . \, V_2 I_2 \cos\varphi_a$.

Therefore the power in the primary circuit is given by

$$P = [\cos\varphi / \cos\{\varphi \pm \beta \pm \gamma + \delta\}]$$

× algebraic sum of wattmeter readings × corrected ratio of current transformer × corrected ratio of potential transformer.

CHAPTER XVIII

ALTERNATING-CURRENT MEASUREMENTS

In this chapter we shall discuss a few of the methods available for determining the principal constants of electric circuits and apparatus as well as methods of measuring current, potential difference, and power. The methods of determining the direction of the phase rotation of a polyphase system will also be considered.

Simple Measurement of Inductance and Capacitance by the "Impedance" Method. This method possesses the advantage of simplicity. It is very convenient for the test-room and laboratory when a high degree of accuracy is not required, and when the value of the impedance under test is within the range of the instruments available.

The general procedure is to measure the impedance of the apparatus (using the ammeter and voltmeter method) at a known frequency, employing preferably a source of sinusoidal E.M.F., and, if possible, an electrostatic voltmeter.

Then, for the inductive circuit, if I is the current, V the applied voltage, f the frequency, and R the resistance, the inductance is given by

$$L = \sqrt{(V^2 - R^2I^2)}/2\pi fI \text{ henries,}$$

or, if RI is negligible in comparison with V,

$$L = V/2\pi fI \text{ henries.}$$

For the capacitive circuit

$$C = I/2\pi fV \text{ farads.}$$

The **mutual inductance** of two circuits may be determined by measuring the induced E.M.F. in the secondary circuit when a known current, at a known frequency, is passing in the primary circuit. Then, if E_2 is the value of the induced E.M.F., I_1 the primary current, and f the frequency, we have $E_2 = 2\pi fMI_1$, whence

$$M = E_2/2\pi fI_1$$

It is to be observed that in all these measurements the accuracy of the results will be affected by any distortion of the wave-form from the sine wave, the greatest errors occurring in the measurement of capacitance. Examples, showing the magnitude of these errors for a particular case, are given in Chapter XIV.

The error due to non-sinusoidal wave-form may be calculated and allowed for when the equation to the E.M.F. wave is known. The method of doing so is explained in Chapter XIV.

Again, if an electromagnetic voltmeter is employed instead of an electrostatic voltmeter, the instrument should be so connected that its operating current passes through the ammeter, and the reading of the ammeter should be corrected to allow for this current.

Measurement of Power and Power Factor in Single-phase Circuits. We shall only consider the case in which the power to be measured is small and the measurements are to be made by means of a wattmeter, ammeter, and voltmeter. The connections should be made according to Fig. 260. The wattmeter reading then includes the losses in the pressure circuit of this instrument and the voltmeter; the ammeter reading includes the currents in the pressure circuit of the wattmeter and the voltmeter

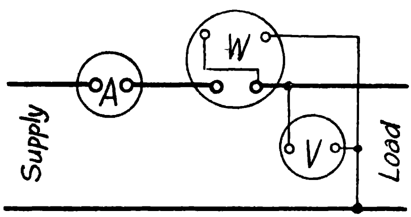


FIG. 260. CONNECTIONS OF INSTRUMENTS FOR MEASURING A SMALL AMOUNT OF POWER

If P, I, denote the true power and current, respectively, in the load; P', I', the readings of the wattmeter and ammeter, respectively; V the voltage at the load; R_v the resistance of the voltmeter; and R_w the resistance of the pressure circuit of the wattmeter then

$$P = P' - V^2/R_v - V^2/R_w$$

and
$$I = I' - V/R_v - V/R_w$$

The power factor is given by

$$\cos\varphi = P/VI$$

[NOTE. The corrections for the inductance of the pressure circuits are assumed to be negligible.]

Example. The following readings were taken on instruments connected as in Fig. 260—

Volts 100; Amperes 0·4; Watts 30.

The resistance of the voltmeter was 1200 ohms, and that of the pressure circuit of the wattmeter 4,000 ohms. Calculate the power supplied to the load and also the power factor.

Employing the expressions previously obtained, we have

$$P = 30 - 100^2/1200 - 100^2/4000 = 30 - 8{\cdot}33 - 2{\cdot}5 = 19{\cdot}17 \text{ W.}$$

$$I = 0{\cdot}4 - 100/1200 - 100/4000 = 0{\cdot}4 - 0{\cdot}0833 - 0{\cdot}025$$
$$= 0{\cdot}2917 \text{ A.}$$

$$\cos\varphi = 19{\cdot}17/100 \times 0{\cdot}2917 = 0{\cdot}657$$

Note that if the power factor is calculated from the uncorrected readings of the instruments, we obtain

$$\cos \varphi' = 30/100 \times 0{\cdot}4 = 0{\cdot}75$$

Measurement of Power in Single-phase, High-voltage Circuit, Using a Wattmeter without Instrument Transformers. When the power in a high-voltage circuit is to be measured by a " straight-through " wattmeter, a series resistance, of suitable value, must be connected in series with the moving coil of the instrument. If the moving-coil circuit is designed for a normal voltage V and the resistance

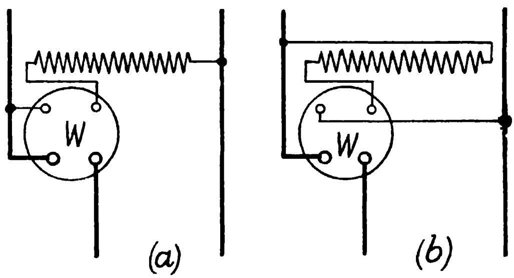


FIG. 261. CORRECT (*a*) AND INCORRECT (*b*) CONNECTIONS OF WATTMETER ON HIGH-VOLTAGE CIRCUIT

of this circuit is R, the additional resistance, R_1, to be connected in this circuit when the instrument is used on a circuit of voltage V_1 is $R_1 = R(V_1/V - 1)$. The readings of the wattmeter must then be multiplied by the quantity $(1 + R_1/R)$ to obtain the power in the main circuit, and, if necessary, correction factors must be applied (as explained in Chapter XVI) for the power expended in the instrument, and for the inductance (and distributed capacitance, if any) of this circuit.

Precautions must be taken when connecting the instrument to the circuit to (1) support the instrument upon an insulating stand (or, if one side of the system is earthed, to connect the instrument to this side of the circuit); (2) make a common connection between the fixed and moving coils as shown in Fig. 261*a*, to ensure that a high voltage cannot exist between these coils.

It is of the utmost importance that the connections shown in Fig. 261*b* be not inadvertently made, as in this case the full line voltage would exist between the coils, thereby causing a breakdown of the insulation and a burning-out of the instrument.

Measurement of Power in Single-phase Circuits without Using a Wattmeter. The wattmeter method of measuring power is described

in Chapter V, and the corrections which have to be applied to the wattmeter readings under certain circumstances are discussed in Chapter XVI. In cases where a wattmeter is not available, and the conditions are favourable, the power (and power-factor) may be measured by alternative methods, known as the " three-voltmeter " and " three-ammeter " methods.

Three-voltmeter Method of Measuring Power and Power Factor. A non-inductive resistance is connected in series with the apparatus under test, and the voltages across the apparatus, non-inductive resistance, and supply are measured, preferably by electrostatic instruments. The diagram of connections is shown in Fig. 262,

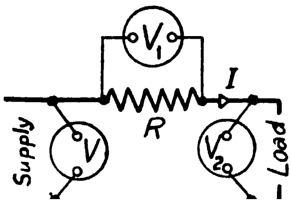


FIG. 262

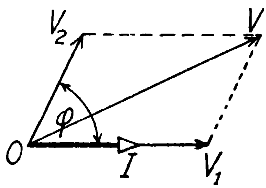


FIG. 263

CIRCUIT AND VECTOR DIAGRAMS FOR THE "THREE-VOLTMETER" METHOD OF MEASURING POWER

and the vector diagram for the circuit is given in Fig. 263. From the vector diagram we have

$$V^2 = V_1^2 + V_2^2 + 2V_1V_2 \cos \varphi$$

where V is the line voltage, V_1 the voltage across the non-inductive series resistance, and V_2 the voltage across the apparatus under test.

The power supplied to the apparatus is

$$P = V_2 I \cos \varphi = (V_2 V_1/R) \cos \varphi,$$

where R is the value of the non-inductive resistance.

Hence, substituting in the preceding expression and re-arranging, we obtain

$$P = \tfrac{1}{2}(V^2 - V_1^2 - V_2^2)/R \qquad . \quad . \quad . \quad . \quad (193)$$

Also $$\cos \varphi = (V^2 - V_1^2 - V_2^2)/2V_1V_2 \qquad . \quad . \quad . \quad (194)$$

These expressions are valid when the wave-form of the supply voltage is sinusoidal, and (in cases of non-sinusoidal wave-form) when the wave-form of the current is identical with that of the impressed E.M.F.

NOTE. Equation (194) may also be employed to calculate the phase difference between two E.M.Fs. which are acting in series.

Thus if V_1, V_2 are the R.M.S. values of the separate E.M.Fs., and V their resultant, or vector sum, then if φ is the phase difference between V_1 and V_2 we have

$$\varphi = \cos^{-1} [(V^2 - V_1^2 - V_2)/2V_1V_2].$$

Three-ammeter Method of Measuring Power and Power Factor. In this method a non-inductive resistance is connected in parallel with the apparatus under test, and the currents in this resistance, the apparatus, and the line are measured. The diagram of connections is shown in Fig. 264, and the vector diagram for the circuit is given in Fig. 265. From the vector diagram we have

$$I^2 = I_1^2 + I_2^2 + 2I_1I_2 \cos \varphi$$

where I is the line current, I_1 the current in the non-inductive resistance, and I_2 the current in the apparatus under test.

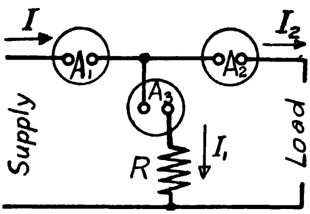


Fig. 264

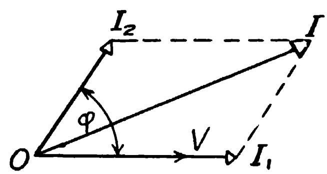


Fig. 265

Circuit and Vector Diagram for the "Three-ammeter" Method of Measuring Power

The power supplied to the apparatus is

$$P = VI_2 \cos \varphi = I_1R \, . \, I_2 \cos \varphi,$$

where R is the value of the non-inductive resistance. Hence substituting in the preceding expression and re-arranging, we obtain

$$P = \tfrac{1}{2}R(I^2 - I_1^2 - I_2^2) \quad . \quad . \quad . \quad . \quad . \quad (195)$$

Also $$\cos \varphi = (I^2 - I_1^2 - I_2^2)/2I_1I_2 \quad . \quad . \quad . \quad . \quad . \quad (196)$$

The validity of each of these expressions is governed by the same conditions as apply to the three-voltmeter method.

Electrostatic Method of Measuring Power. In this method a quadrant electrometer is employed in conjunction with a standard non-inductive resistance and a potential divider. The method is particularly suitable for research and standardizing laboratories, and has been perfected by the National Physical Laboratory for measurements of high accuracy.* It is the standard method of

* "The use of the electrostatic method for the measurement of power.' *Journal of the Institution of Electrical Engineers*, vol. 51, p. 294.

measuring power in the testing of watt-hour meters and the calibration of wattmeters at this laboratory. The method also possesses advantages over other methods for the measurements of small amounts of power at high voltages and low power factors, such as losses in dielectrics.

To measure power by the quadrant electrometer, a potential difference proportional to, and in phase with, the current in the circuit is applied to the quadrants, and the voltage of the circuit, or a definite fraction thereof, is applied between the needle and quadrants. The deflection is then proportional to the power expended between the point to which the needle is connected and

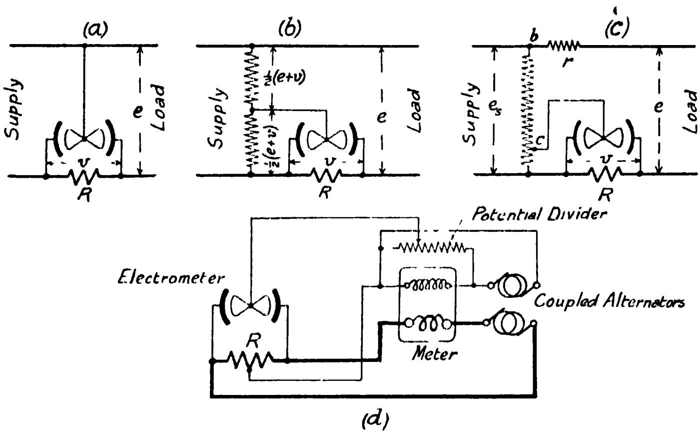


FIG. 266. CONNECTIONS FOR MEASUREMENT OF POWER BY QUADRANT ELECTROMETER

the point which has a potential midway between the potentials of the quadrants, as proved later.

Thus if the connections are arranged as shown in Fig. 266*a*—which refers to the case where the load is connected directly to the supply system—the deflection, of the electrometer is proportional to the power expended in the load *plus* one-half of the power expended in the non-inductive series resistance, or shunt, to which the quadrants are connected.

The correction for the power expended in one-half of the series resistance, R, may be eliminated either by connecting the needle to the mid-point of a non-inductive resistance (or a potential divider) connected across the supply system as shown in Fig. 266*b*, or, if this is impracticable, by inserting a second non-inductive resistance, r, in the main circuit as shown in Fig. 266*c*, and making

the value of this resistance equal to $\frac{1}{2}R(n-2)$, where R is the value of the resistance across which the quadrants are connected and n is the ratio: (supply voltage/voltage between needle and "line" quadrants).

If, however, the mid-point of the resistance R can be utilized, the wattmeter reading is directly proportional to power expended in the load. This is the case when a fictitious load is employed as shown in Fig. 266*d*, which refers to the test circuit for the calibration of a wattmeter or a watt-hour meter. The current coils of the meter are supplied through a transformer from an alternator, and the pressure coils are supplied from another alternator. The latter is direct-coupled to the first machine, and is so arranged that its frame may be given angular displacements with reference to this machine, in order to obtain any desired phase difference between the current and voltage supplied to the meter.

In cases where no correction is necessary for the power expended in the series resistance, R, the power is given by

$$P = kn\,\theta/2R \qquad \ldots \qquad (197)$$

where θ is the deflection, k the "constant" of the instrument (which is determined in the manner described later), and n, R refer to the quantities mentioned previously.

Theory of Electrostatic Method of Measuring Power. Let e denote the instantaneous value of the potential difference between the needle and one pair of quadrants, and v the instantaneous value of the potential difference between the quadrants. Then, from the law of the quadrant electrometer, the deflection, θ, is given by

$$k\theta = \frac{1}{T}\int_o^T (2ev + v^2)dt.$$

Now, with the connections arranged as in Fig. 266*a*, $v = iR$, where i is the instantaneous value of the current in the circuit. Hence

$$\begin{aligned} k\theta &= \frac{1}{T}\int_o^T (2Rei + i^2R^2)dt \\ &= \left[2R \cdot \frac{1}{T}\int_o^T ei\,dt + R \cdot \frac{1}{T}\int_o^T Ri^2\,dt\right] \end{aligned}$$

But $\frac{1}{T}\int_o^T ei\,dt$ represents the power (P) supplied to the circuit, and $\frac{1}{T}\int_o^T Ri^2\,dt$ represents the power (P_R) expended in the series resistance, R.

Therefore $\quad k\theta = 2R(P + \frac{1}{2}P_R)$

whence $\quad P = k\theta/2R - \frac{1}{2}P_R$

With the connections arranged as in Fig. 266*b*, the potential difference

between the needle and the pair of quadrants connected to the load is $[\frac{1}{2}(e + v) - v]$. Hence

$$k\theta = \frac{1}{T}\int^{T}(2v[\tfrac{1}{2}(e + v) - v] + v^2)dt = \frac{1}{T}\int_0^{T} ev\,dt$$
$$= R \cdot \frac{1}{T}\int_0^{T} ei\,dt = RP$$

Whence $\quad k\theta/R$

With the connections arranged as in Fig. 266c, the potential difference between the needle and the pair of quadrants connected to the load is $(e + ir + iR)/n - iR$, where n is the ratio of the voltage across the tapped

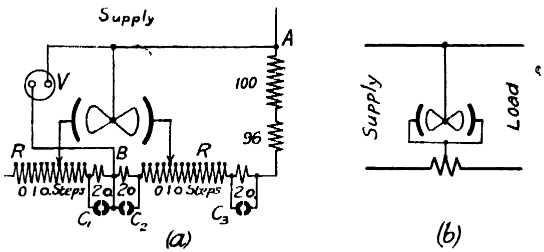


FIG. 267. CONNECTIONS FOR CALIBRATING QUADRANT ELECTROMETER FOR POWER MEASUREMENTS

portion of the potential divider to the supply voltage, i.e. $n =$ voltage across points $a\,c$/voltage across points $a\,b$. Hence

$$k\theta = \frac{1}{T}\int_0^{T}(2v[(e + ir + iR)/n + iR] + v^2)dt$$
$$= \frac{2R}{n}\cdot\frac{1}{T}\int_0^{T} ei\,dt + \frac{1}{T}\int_0^{T}\frac{i^2R}{n}(2r - R(n-2))dt$$

The first integral represents the power (P) supplied to the load. The second integral becomes zero when $2r = R(n-2)$, or when $r = \frac{1}{2}R(n-2)$. Under these conditions, we have

$$P = nk\theta/2R$$

When the instrument is used with a fictitious load, as in Fig. 266d, the potential difference between the needle and the pair of quadrants connected to the load is $(e/n - \frac{1}{2}v)$. Hence

$$k\theta = \frac{1}{T}\int_0^{T}[2v(e/n - \tfrac{1}{2}v) + v^2]dt$$
$$= \frac{2R}{n}\cdot\frac{1}{T}\int_0^{T} ei\,dt = \frac{2R}{n}\cdot P$$

whence $\quad P = nk\theta/2R$

Method of Determining the "Constant" of the Electrometer. The method employed by the National Physical Laboratory requires the use of standard non-inductive resistances and an electrostatic voltmeter, the calibration of which is accurately known. The connections are shown in Fig. 267.

The alternating supply voltage is adjusted to give the normal voltage (100 V.) between the needle and the neutral point of the quadrants (i.e. 100 V. between points A and B, Fig. 267), and the potential difference between the quadrants is adjusted to any desired value by means of selector switches

connected to standard non-inductive resistances. The resistance in circuit between the points A and B is maintained constant at 200 ohms, and, since, a constant potential difference of 100 V. is maintained between these points the current in the circuit is 0·5 A. Hence when the 2-ohm coils, C_1, C_2, are short-circuited the potential difference between the quadrants can be varied from 0·1 V. to 2 V. in 0·1 V. steps.

[NOTE. The positions of the selector switches must always be such that equal resistances are included between each quadrant and the neutral point A.]

This potential difference can be extended up to 4 V., in 0·1 V. steps by unplugging the 2-ohm coils, C_1, C_2, but when C_2 is inserted an equal resistance (C_3) must be cut out between the points A, B, in order to maintain the total resistance between these points constant at 200 ohms.

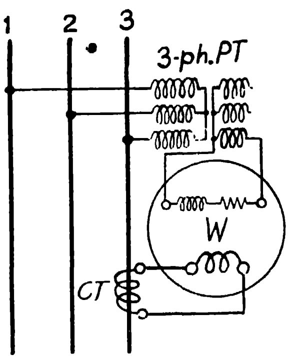


FIG. 268

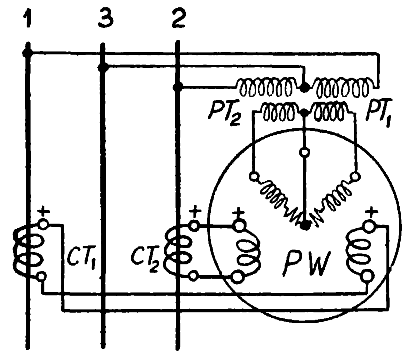


FIG. 269

CONNECTIONS OF WATTMETER AND INSTRUMENT TRANSFORMERS FOR MEASURING POWER IN BALANCED AND UNBALANCED LOADS

If θ_1 is the deflection from the "electrical zero," E the R.M.S. voltage between the points A, B; I the current in the circuit; R the resistance connected across the quadrants; and k the constant of the instrument corresponding to the deflection θ_1, then, from the law of the electrometer,

$$k\theta_1 = 2IR(E - \tfrac{1}{2}IR) + I^2R^2 = 2EIR$$

whence $$k = 2EIR/\theta_1$$

The "electrical zero" is obtained by connecting both quadrants to the point A (as shown in the diagram b, Fig. 267), with normal voltage between the points A, B.

Measurement of Power in Three-phase Circuits. The principles of the methods of measuring power in three-phase circuits have been discussed in Chapter IX, and diagrams of connections for low-voltage circuits are given on pp. 194, 195, 199.

With high-voltage circuits it is customary to use instrument transformers with the wattmeters, and connection diagrams for balanced and unbalanced loads are given in Figs. 268, 269.

In cases where it is desired to use straight-through wattmeters the connections must be made in accordance with Fig. 270, and two separate wattmeters (*not* a polyphase wattmeter) must be employed, both of which must be supported upon insulating stands.

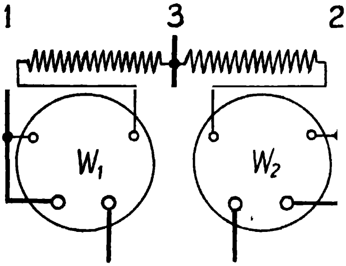


Fig. 270. Connections of Wattmeters for Measuring Power in High-voltage Circuit

The precautions already mentioned in connection with the measurement of power in single-phase, high-voltage circuits apply with equal force in the present case.

Alternating-current Potentiometer Methods of Measuring E.M.F. The potentiometer principle of comparing E.M.Fs. enables measurements of high accuracy to be performed, as the balance is obtained by a null method. In the direct-current potentiometer the balance is obtained between the magnitudes of the "unknown" E.M F. and the potential difference between certain points in the potentiometer wire. but in the alternating-current potentiometer the balance must be obtained between the phases of these quantities as well as between their magnitudes.

The theory of the alternating-current potentiometer is quite simple. Thus, if the "unknown" E.M.F. is represented by

$$\dot{E} = E\underline{/\varphi_e} = \pm \boldsymbol{e}_1 \pm j\boldsymbol{e}_2,$$

and the potential difference against which it is balanced is represented by

$$\dot{V} = V\underline{/\varphi_v} = \pm \boldsymbol{v}_1 \pm j\boldsymbol{v}_2,$$

then the condition of balance is

$$\dot{V} = \dot{E}.$$

Hence $\qquad V = E$, and $\varphi_v \doteq \varphi_e$.

Or, alternatively, $\pm \boldsymbol{v}_1 \pm j\boldsymbol{v}_2 = \pm \boldsymbol{e}_1 \pm j\boldsymbol{e}_2$

which requires $\qquad \pm \boldsymbol{v}_1 = \pm \boldsymbol{e}_1$

and $\qquad \pm j\boldsymbol{v}_2 = \pm j\boldsymbol{e}_2$

Hence, two adjustments, which must be carried out in succession, are necessary in obtaining the balance with an alternating-current potentiometer.

Two methods of effecting the double adjustment have been devised: In one method (due to Drysdale) a phase-shifting transformer is employed ; in the other method (due to Gall) the in-phase and quadrature components of the two quantities are balanced separately. In the Larsen A.C. potentiometer a standard variable mutual inductance is employed instead of a phase-shifting transformer.

Gall-Tinsley Co-ordinate Potentiometer. This instrument (Fig. 271) consists of two potentiometers (which are called the " in-phase " and " quadrature " potentiometers) fitted into a common case. The potentiometers are supplied

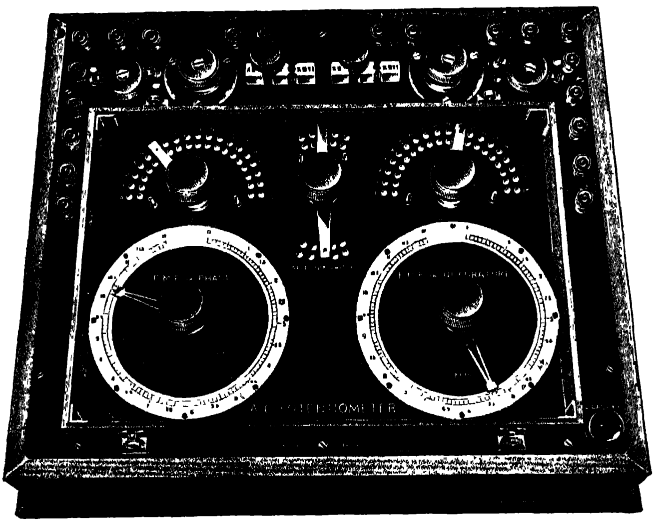


FIG. 271. CO-ORDINATE POTENTIOMETER
(*H. Tinsley & Co.*)

with equal currents (50 mA.) having a mutual phase difference of 90°, the currents being obtained from a single-phase supply system through a step-down isolating transformer, the latter being for the double purpose of isolating the potentiometer circuits from the supply system and for obtaining a suitable voltage (6 V.) for these circuits.

The correct value of the current in the " in-phase " potentiometer is indicated (to about 1 part in 10,000) by a sensitive reflecting electro-dynamic instrument which is connected in this circuit. The electro-dynamic instrument is not permanently calibrated (as is the case with the instrument used with the Drysdale potentiometer), but the current in the potentiometer wire is adjusted to its standard value (50 mA.) by means of direct current and a standard cell, in the same manner as if the potentiometer were being used on a direct-current circuit. The control (torsion head) of the electro-dynamic instrument is then so adjusted to bring the " spot " to a definite position on the scale, and this position therefore indicates the standard value of current

in the potentiometer wire when the supply current is either direct or alternating.

The adjustment of the current in the "quadrature" potentiometer to its correct value and phase difference is effected by connecting in this circuit the primary winding of a standard mutual inductance and balancing the E.M.F. induced in the secondary winding against a definite potential difference of the "in-phase" potentiometer. If the mutual inductance is so constructed as to be free from eddy currents, the E.M.F. induced in its secondary will have a phase difference of 90° with respect to the current in the primary. Therefore, when this E.M.F. is balanced against the potential difference of the "in-phase" potentiometer, the current in the "quadrature" potentiometer has a phase difference of 90° with respect to that in the in-phase potentiometer.

By suitably choosing the value of the mutual inductance in relation to the frequency of the supply circuit, a simple relationship may be obtained between induced E.M.F. and frequency. For example, if the mutual inductance is 0·03183 ($= 0{\cdot}1/\pi$) H., the E.M.F. (E_2) induced in the secondary winding when the standard current (50 mA.) is passing in the primary winding is given by—

$$E_2 = 2\pi f M I_1 = 2\pi f \times (0{\cdot}1/\pi) \times 50 \times 10^{-3} = f/100.$$

Thus $E_2 = 0{\cdot}5$ V. for 50 frequency.

Therefore, if the supply frequency is 50, the potential slides of the "in-phase" potentiometer are set to 0·5 V. and the magnitude and phase of the current in the "quadrature" potentiometer are adjusted in succession until no deflection is obtained on the vibration galvanometer, the (alternating) current in the "in-phase" potentiometer being maintained at the standard value.

The "unknown" E.M.F. to be determined is applied, *via* a vibration galvanometer, to both sets of potential slides (which are now connected in series with each other) and a balance is obtained by adjusting the potential slides, and, if necessary, changing the positions of the reversing switches, which are connected in series with them (see Fig. 272). The "in-phase" component of the E.M.F. is then given, in magnitude and sign, by the settings of the potential slides and reversing switch, respectively, of the "in-phase" potentiometer. Similarly, the "quadrature" component of the E.M.F. is given in magnitude and sign by the settings of the potential slides and reversing switch of the "quadrature" potentiometer. By means of shunts incorporated in the instrument, the "constants" of either set of potential slides may be changed to one-tenth normal, thereby enabling small magnitudes of either of the E.M.F. components to be accurately determined.

The magnitude of the "unknown" E.M.F. is, of course, given by

$$\sqrt{[(\text{in-phase E.M.F.})^2 + (\text{quadrature E.M.F.})^2]},$$

and its phase difference with respect to the current in the in-phase potentiometer wire (which is the quantity of reference for all E.M.F. measurements) is

$$\varphi = \tan^{-1}(\text{quadrature E.M.F.})/(\text{in-phase E.M.F.}).$$

Potentiometer Method of Measuring Impedance. The impedance to be measured has a standard non-inductive resistance connected in series with it, and is supplied with current from the same source as the potentiometer. The potential differences at the terminals of the standard resistances and the impedance are then determined. Let these quantities, as determined by the Drysdale potentiometer, be $V_1/\underline{\varphi_1}$ and $V_2/\underline{\varphi_2}$, respectively, where V_1, V_2 are the settings of the potential slides and φ_1, φ_2 the settings of the phase-shifting

transformer. Then, if R_s is the value of the standard resistance, the impedance of the apparatus under test is given by

$$Z = \frac{R_s V_2}{V_1} \underline{/(\varphi_2 - \varphi_1)}$$

Whence $\qquad Z = R_s V_2 / V_1.$

Observe that, although the phase angles φ_1, φ_2 are not involved in the determination of the numerical value of the impedance, a

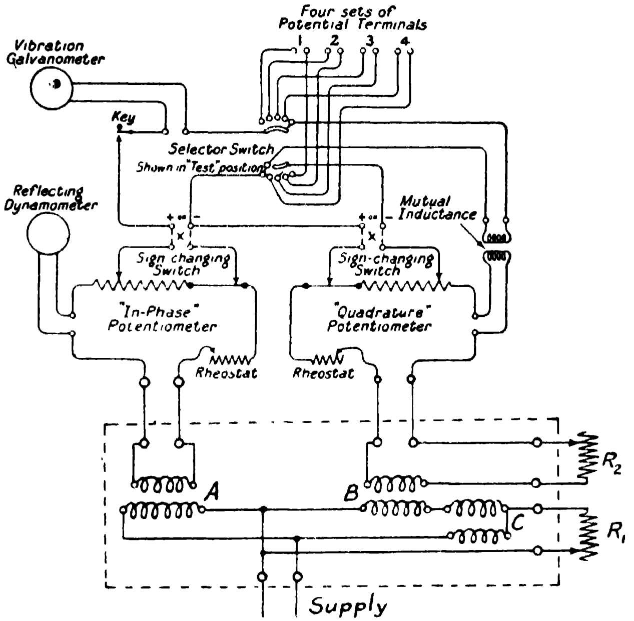


FIG. 272. CONNECTIONS OF CO-ORDINATE POTENTIOMETER

[NOTE.—The lower portion of the diagram (inside the dotted rectangle) refers to the isolating transformers A, B, and the phase-splitting transformer, C]

knowledge of these angles enables the resistance and reactance to be calculated.

If the measurements are made on the Gall co-ordinate potentiometer and are denoted by $\pm e_1 \pm je_2$, and $\pm e'_1 \pm je'_2$, where $\pm e_1 \pm e'_1$ represent the settings of the potential slides of the "in-phase" potentiometer, and $\pm e_2$, $\pm e'_2$, the settings of the "quadrature" potentiometer when determining the voltages across the standard resistance and impedance, respectively, then

$$Z = R_s \sqrt{[(e'^2_1 + e'^2_2)/(e_1^2 + e_2^2)]}$$

Method of Testing Watt-hour and Power-factor Meters at Varying Power Factors. Two methods are available for polyphase meters and three methods for single-phase meters, two of the latter being similar to the methods employed with three-phase meters. The methods common to single-phase and polyphase meters are—

1. Supplying the current and potential coils of the meters from separate alternators, which are mechanically coupled together and have the same number of poles. The stator of one alternator is

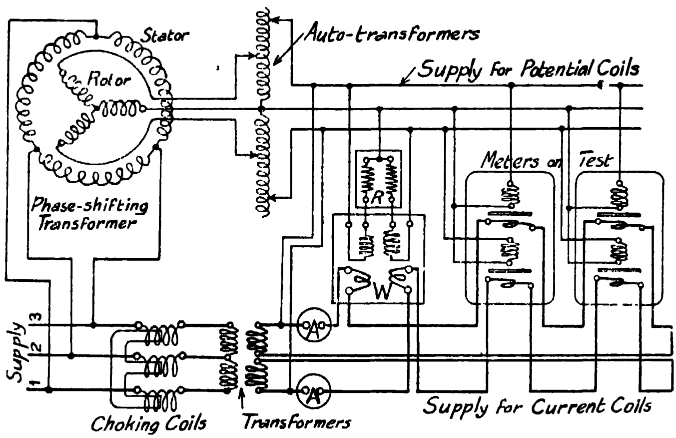


FIG. 273. CONNECTIONS FOR TESTING POLYPHASE WATT-HOUR METERS ON FICTITIOUS LOAD

so constructed that its magnetic axis can be displaced relatively to that of the other alternator, and by these means any desired phase difference may be obtained between the E.M.Fs. of the alternators. Thus, conditions equivalent to any desired power factor may be obtained in the meter circuits. The connections in the case of a single-phase meter have already been given in Fig. 266*d*.

2. Supplying the potential coils from the secondary winding of a phase-shifting transformer, the primary of which is excited from the system supplying the current coils.

A diagram of connections for this method is given in Fig. 273, from which it will be observed that the current coils of the meters are supplied through step-down transformers (and, if necessary, current transformers), and the potential circuits are supplied, through auto-transformers, from the secondary windings of the phase-shifting transformer. Regulation of current in the " current "

circuit of the meters is effected by means of a regulating choking coil connected in the primary circuit of the step-down transformer, and regulation of voltage for the potential circuits is effected by means of tappings on the auto-transformers.

The equivalent power in the meter circuits is determined by means of a standard electro-dynamic wattmeter, and the equivalent power factor is obtained either from the dial of the phase-shifting transformer or the volt-amperes in the meter circuits.

The equalizing connections between the current and potential circuits are for the purpose of ensuring that the fixed and moving coils of the wattmeter are at the same potential.

[NOTE. In both the above cases the meters are tested on a fictitious load, and the tests are effected with the expenditure of only a small amount of energy, viz. that necessary to supply the losses in the instruments and apparatus.]

Another method—which is applicable only to single-phase meters, and does not require the special apparatus which is necessary with the preceding methods—is based upon the phase relationship between the currents and voltages in a three-phase system. The connections are shown in Fig. 274*a*, and Fig. 274*b*, and vector diagrams are shown in Fig. 274*c*. The load should be non-inductive and may consist of banks of incandescent lamps.

The current coil of the meter under test, *M*, and that of the standard wattmeter, *W*, are connected in series with the line wire, No 3, which is common to the two loads. When a power factor between unity and 0·5 is required, the potential circuits of the instruments are connected between this line (No. 3) and one of the other lines (Fig. 274*a*), according to whether a lagging or leading power factor is required. But when a power factor between 0·5 and zero is required, the potential circuits are connected across lines 1 and 2, as shown in Fig. 274*b*.

The vector diagram for these conditions is shown in Figs. 274*c*. The current, *I*, is the current coils of the instruments in the vector sum of the currents in the loads. The latter are represented by the vectors OI_A, OI_B—which are in phase with the line voltage vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$, respectively—and the current in the instrument is represented by OI. Hence, if the loads are equal, OI bisects the angle between the vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$, and is perpendicular to both $OV_{1\text{-}2}$ and $OV_{2\text{-}1}$. These conditions represent power factors (in the meter circuit) of 0·866 (lagging), 0·866 (leading), zero (lagging), zero (leading).

With unequal loads, the vector OI will occupy a position intermediate between the vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$.

In the extreme case, when one load (B) is zero, OI is in phase with OV_{1-3}, and leads 60° with respect to OV_{2-3}. Hence these conditions represent power factors of unity and 0·5 (leading), according to whether the potential circuits are connected across lines 3 and 1, or across lines 3 and 2. In the other extreme case, when the load A is zero, the conditions are equivalent to power

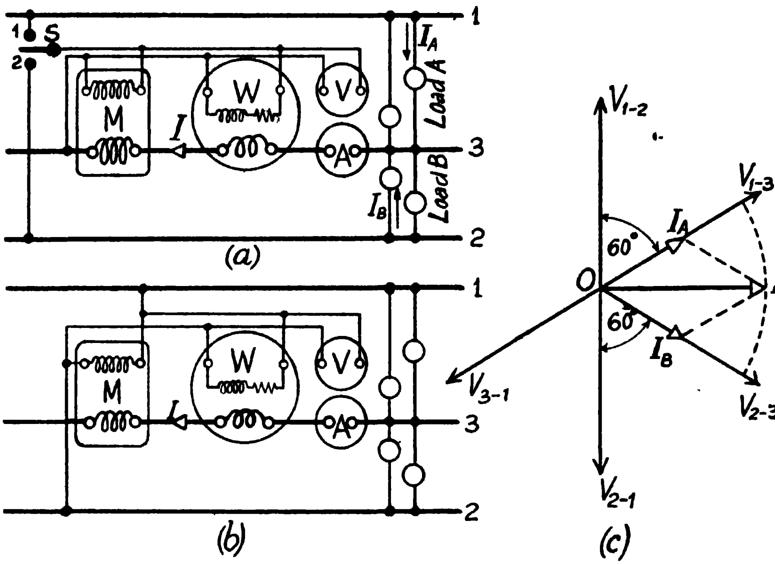


FIG. 274. CONNECTIONS FOR TESTING SINGLE-PHASE WATT-HOUR METERS AT VARIOUS POWER FACTORS

factors of unity and 0·5 (lagging), according to whether the potential circuits are connected across lines 3 and 2, or 3 and 1.

In all these cases it is assumed that the symmetry of the three-phase system is unaffected by the unbalanced loading.

If the direction of the phase rotation of the three-phase system is unknown, the phase of the current (i.e. whether lagging or leading) may be ascertained by a simple test, which consists of connecting an inductance temporarily in the pressure-coil circuit of the standard wattmeter, and noting the readings, with and without the inductance, for the same load conditions. If the reading with the inductance in circuit is larger than that without the inductance, the current is lagging; if the reading is smaller, the current is leading.

Determination of the Phase Sequence or Direction of Phase Rotation in a Three-phase System. A number of methods are available, some of which utilize the principles discussed in Chapter XX in connection with unbalanced star-connected circuits, while others—such as those employed in phase-rotation relays, and similar switchboard instruments—utilize the principle of the induction wattmeter.

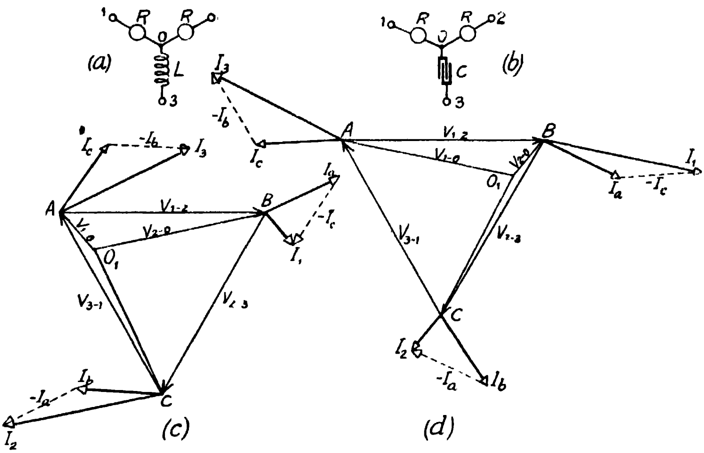


FIG. 275. CIRCUIT AND VECTOR DIAGRAMS FOR THE LAMP-RESISTANCE METHOD OF DETERMINING DIRECTION OF PHASE ROTATION

A simple phase-rotation indicator, in which an unbalanced star-connected circuit is employed, requires only two *similar* incandescent lamps (which form a visual indicator) and either an inductance (preferably iron-cored) or a condenser. These are connected in star as shown in Figs. 275*a*, 275*b*. When either of these circuits is connected to a three-phase system, the lamps will have unequal voltages impressed upon them, and this feature is utilized to determine the direction of the phase rotation, as with the *inductive* circuit (Fig. 275*a*), the phase, or line wire, connected to the *bright* lamp is *lagging* with respect to that connected to the dim lamp, and with the *capacitive* circuit (Fig. 275*b*) the phase connected to the *bright* lamp is *leading* with respect to that connected to the dim lamp.

The vector diagrams are shown in Figs. 275*c*, 275*d*.* The line voltages are represented by the vector triangle *ABC*, the line currents (which are obtained from the phase currents of the

* See p. 456 for the method of drawing these diagrams.

equivalent delta-connected circuit) are represented by the vectors OI_1, OI_2, OI_3.

Since the voltages across the lamps are in phase with the currents OI_1, OI_2, the potential of the neutral point O_1 is determined by drawing from the corners A and B of the triangle ABC, parallels AO_1, BO_1, to the vectors BI_1, CI_2, respectively, to meet at the point O_1.

[NOTE. The vectors BI_1, BI_2 represent the currents in the lamps A and B respectively.]

Then AO_1 represents the voltage across lamp A, BO_1 the voltage across lamp B, and CO_1 the voltage across the remaining branch (inductance or condenser) of the circuit.

Observe that with the inductive circuit (Figs. 275*a*, 275*c*), BO_1 is greater than AO_1; but that with the capacitive circuit (Figs. 275*b*, 275*d*), AO_1 is greater than BO_1. Hence, since the vector diagrams have been drawn for counter-clockwise phase sequence, the bright lamp is connected in the "*lagging* phase" when an *inductance* is employed (Fig. 275*a*), and in the "*leading* phase" when a *condenser* is employed (Fig. 275*b*); the terms "lagging phase" and "leading phase" referring to the phases in which the lamps are connected.

In a practical form of this indicator for 200/230 V., 50-cycle circuits, ordinary 25 W. or 40 W., 230 V., lamps are employed, and the condenser has a capacitance of from 1 to 2 μF. The voltage impressed upon one lamp is about 80 per cent of the line voltage, and that impressed upon the other lamp is about 25 per cent of the supply voltage.

Example. Two equal non-inductive resistances and a condenser are connected in star, as in Fig. 275, and the combination is connected to a three-phase supply system. The capacitance of the condenser is so chosen that the reactance of the condenser branch is equal to the resistance of one of the non-inductance branches. Determine the voltages across the lamps and condenser.

Let R denote the resistance in each of the non-inductive branches. Then $-jR$ denotes the reactance of the condenser branch.

The line currents may be determined either by the indirect method, which involves the conversion of the star-connected circuit into the equivalent delta-connected circuit, or by the direct application of Kirchhoff's laws to the star-connected circuit. As the former method introduces some features of interest for the circuit under consideration, it will be adopted in present case.

Thus, denoting the equivalent impedances between the line wires 1–2, 2–3 3–1 (Fig. 275*b*), as Z_a, Z_b, Z_c, taken in order, we have, from p. 450

$$Z_a = R + R + R \,.\, R/(-jR) = R(2 + j1)$$

$$Z_b = R - jR - R \,.\, jR/R \qquad = R(1 - j2)$$

$$Z_c = -jR + R - jR \,.\, R/R \qquad = R(1 - j2)$$

Hence, for counter-clockwise phase rotation, the currents in the equivalent delta-connected circuit are

$$I_a = \frac{V}{R}\frac{(1 + j0)}{(2 + j1)} = \frac{V}{R}(0{\cdot}4 - j0{\cdot}2)$$

$$I_b = \frac{V}{R}\frac{(-0{\cdot}5 - j0{\cdot}866)}{(1 - j2)} = \frac{V}{R}(0{\cdot}246 - j0{\cdot}373)$$

$$I_c = \frac{V}{R}\frac{(-0{\cdot}5 + j0{\cdot}866)}{(1 - j2)} = \frac{V}{R}(-0{\cdot}446 - j0{\cdot}027)$$

The currents in the star-connected circuit are then $I_1 = I_a - I_c$, $I_2 = I_b - I_a$, $I_3 = I_c - I_b$, and the voltages across the branches of this circuit are $V_{1\text{-}0} = RI_1$, $V_{2\text{-}0} = RI_2$, $V_{3\text{-}0} = -jRI_3$.

Whence $V_{1\text{-}0} = V\,(0{\cdot}846 - j0{\cdot}173)$

$$V_{1\text{-}0} = V\sqrt{(0{\cdot}846^2 + 0{\cdot}173^2)} = 0{\cdot}866\ V.$$

$$V_{2\text{-}0} = V\,(-0{\cdot}154 - j0{\cdot}173)$$

$$V_{2\text{-}0} = V\sqrt{(0{\cdot}154^2 + 0{\cdot}173^2)} = 0{\cdot}232\ V.$$

$$V_{3\text{-}0} = V\,(0{\cdot}346 + j0{\cdot}692)$$

$$V_{3\text{-}0} = V\sqrt{(0{\cdot}346^2 + 0{\cdot}692^2)} = 0{\cdot}775\ V.$$

Wattmeter Method of Determining the Direction of Phase Rotation in a Three-phase System. When the power in the system is being measured by the two-wattmeter method (using separate wattmeters) the phase rotation may be determined at the same time by taking two extra readings with a reactance (e.g. either a condenser or an inductance) inserted at the junction of the two pressure-coil circuits, as shown in Fig. 276. The star-connected circuit thus formed is similar to the circuit in Fig. 276, and, therefore, the changes in the magnitudes and phases of the voltages impressed upon the pressure coil circuits of the wattmeters, due to the insertion of the reactance, will depend upon the direction of the phase rotation of the three-phase system.

Vector diagrams representing the conditions when a condenser is connected in the pressure-coil circuits are shown in Fig. 276. A study of these diagrams will show that if A, B, denote the readings of the wattmeters when the connections are normal (A being the larger reading*), and A', B', denote the readings when the condenser is connected in the pressure-coil circuits, then if

$$A' \simeq A,\ B' < B;\ \text{or if } A' < A,\ B' > B,$$

* The special case when the two readings are equal is considered in the example which follows.

wattmeter B is connected in the "lagging" phase when the power factor is "lagging" and is in the "leading" phase when the factor is "leading."

To ascertain whether the power factor—or, more correctly, the

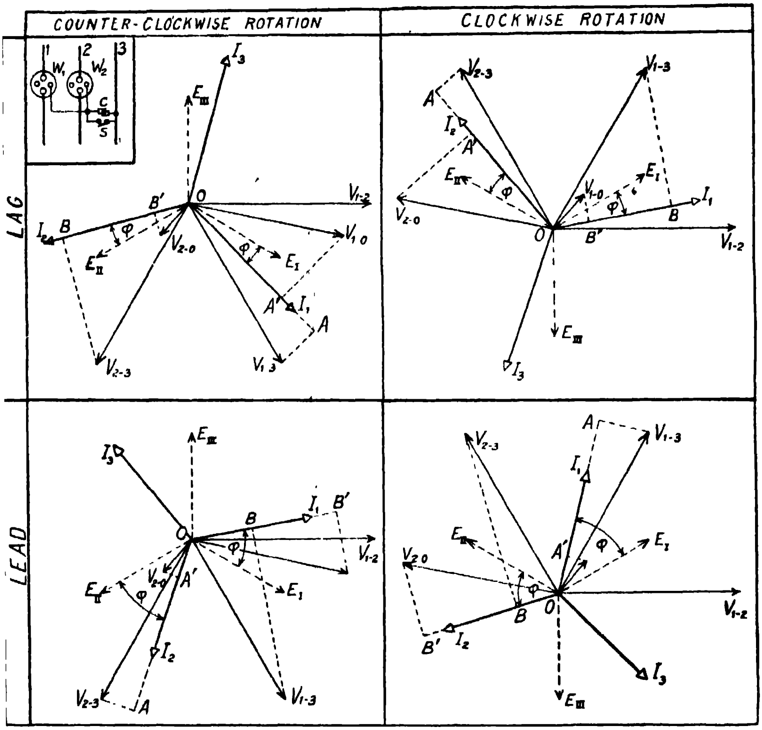


FIG. 276. VECTOR AND CIRCUIT DIAGRAMS FOR TWO-WATTMETER METHOD OF DETERMINING DIRECTION OF PHASE ROTATION

current in the main circuit—is lagging or leading, the following simple test is applied*—

Open the common connection between current and pressure coils of the wattmeter which gives the larger reading when the connections

* This test depends upon the phase relationship between the line voltages and currents of a symmetrical three-phase system. Thus, with normal connections the phase difference between the currents in the coils of the wattmeter giving the larger reading is $(30° - \varphi)$, and when the common end of the pressure coil is transferred to the other line the phase difference becomes $(90° - \varphi)$ when the power factor is lagging and $(90° + \varphi)$ when the power factor is leading.

are normal. Connect this end of the pressure coil to the line wire in which the second wattmeter is connected. If the new reading is *positive*, the current is *lagging*; if *negative*, the current is *leading*.

Experimental Verification. The following readings were obtained on two wattmeters connected, in accordance with the diagram in Fig. 276, to a three-phase circuit, the power factor of which could be varied as desired. The power factor was indicated by a power factor meter, and the direction of phase rotation was determined by an independent test—

Reading of Wattmeter Connected in Line No. 1	Reading of Wattmeter Connected in Line No. 2.	Power Factor.	Phase Rotation.
20	47	lagging	2–1–3
2*	35*	lagging	2–1–3
48	21	leading	2–1–3
17*	44*	leading	2–1–3
24	48	leading	1–2–3
46*	17*	leading	1–2–3
49	20	lagging	1–2–3
37*	3*	lagging	1–2–3

* These readings were taken with a condenser inserted in the potential circuit of wattmeters.

Example. Two similar wattmeters are connected in the usual manner for measuring the power in a three-phase circuit and provision is made for inserting a condenser in the pressure-coil circuits in the manner shown in Fig. 276. The reactance of the condenser is equal to the resistance of one of the pressure-coil circuits of the wattmeters.* Determine, for power factors of 1·0, 0·707, 0·5, 0·2, and both directions of phase rotation, the relationship between the readings of the wattmeters (i) when the connections are normal; (ii) when the condenser is connected in the pressure-coil circuits. The line voltage, current, and frequency are constant.

The voltages impressed upon the potential coils of the wattmeters when the condenser is in circuit are calculated as in the above example, and are given by

$\dot{V}_{1-0} = V\,(\mathbf{0{\cdot}846 - j0{\cdot}173})$, $\dot{V}_{2-0} = V\,(-\,\mathbf{0{\cdot}154 - j{\cdot}0173})$, for counter-clockwise phase rotation, and by

$\dot{V}'_{1-0} = V\,(\mathbf{0{\cdot}154 + j0{\cdot}173})$, $\dot{V}'_{2-0} = V\,(-\,\mathbf{0{\cdot}846 + j{\cdot}0173})$ for clockwise phase rotation, the voltage between lines 1 and 2 being the quantity of reference.

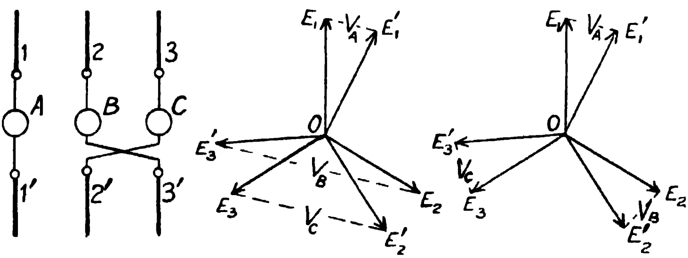
For counter-clockwise phase rotation and with the condenser in the pressure-coil circuits the reading (W'_1) on the wattmeter connected in line 1 is equal to $I_1 V_{1-0} \cos\,(18{\cdot}4° + \varphi)$, and that ($W'_2$) of the wattmeter connected in line 2 is $I_2 V_{2-0} \cos\,(18{\cdot}4° + \varphi)$, the plus sign being taken for lagging power factors. Similarly, for counter-clockwise phase rotation, the readings are: $W''_1 = I_1 V'_{1-0} \cos\,(18{\cdot}4° + \varphi)$, $W''_2 = I_2 V'_{2-0} \cos\,(18{\cdot}4° + \varphi)$.

* If R is the resistance of the pressure-coil circuit, the capacitance required is given by $C = 10^6/\omega R$ μF. If the frequency is 50 and $R = 3{,}180\ \Omega$., $C = 1\ \mu$F.

The results are best expressed in tabular form—

Power Factor.	Phase Rotation.	W_1 (without Condenser).	W'_1 (with Condenser).	W_2 (without Condenser).	W'_1 (with Condenser).
1·0	CC	0·866 VI_1	0·712 VI_1	0·866 VI_2	0·22 VI_2
1·0	C	0·866 VI_1	0·22 VI_1	0·866 VI_2	0·712 VI_2
0·707 (lag)	CC	0·966 VI_1	0·398 VI_1	0·259 VI_2	0·104 VI_2
0·707 (lead)	CC	0·259 VI_1	0·774 VI_1	0·966 VI_2	0·207 VI_2
0·707 (lag)	C	0·259 VI_1	0·104 VI_1	0·966 VI_2	0·398 VI_2
0·707 (lead)	C	0·966 VI_1	0·207 VI_1	0·259 VI_2	0·774 VI_2
0·5 (lag)	CC	0·866 VI_1	0·174 VI_1	0	0·0466 VI_2
0·5 (lead)	CC	0	0·65 VI_1	0·866 VI_2	0·174 VI_2
0·5 (lag)	C	0	0·0466 VI_1	0·866 VI_2	0·174 VI_2
0·5 (lead)	C	0·866 VI_1	0·174 VI_1	0	0·65 VI_2
0·2 (lag)	CC	0·664 VI_1	− 0·102 VI_1	− 0·315 VI_2	− 0·0274 VI_2
0·2 (lead)	CC	− 0·315 VI_1	0·433 VI_1	0·664 VI_2	0·116 VI_2
0·2 (lag)	C	− 0·315 VI_1	− 0·0274 VI_1	0·664 VI_2	− 0·102 VI_2
0·2 (lead)	C	0·664 VI_1	0·116 VI_1	− 0·315 VI_2	0·433 VI_2

Watt-hour Meter Method of Determining Direction of Phase Rotation in a Three-phase System. For this method, an induction-type polyphase watt-hour meter with the terminals marked according to the B.S.I. specification is required. The tests applied



(*a*) (*b*) (*c*)

FIG. 277. CIRCUIT AND VECTOR DIAGRAMS FOR THE COMPARISON OF PHASE ROTATION BY LAMPS

depend upon the power factor of the load connected to the meter, and are as follows—

1. When the load is non-inductive. Disconnect the " red " element of the meter from the circuit and connect a non-inductive resistance* in series with the potential or shunt circuit of the " blue " element. Take a reading. Next disconnect the " blue " element from the circuit and connect the non-inductive resistance in series with the shunt circuit of the " red " element and take a reading with the same load as before. Then the element which gives the higher reading is connected in the " leading " phase of the supply system.

* The value of the resistance should be of the order of 500 ohms per 100 volts of the normal operating voltage.

2. When the current is lagging. The test is similar to that for case (1), but the non-inductive resistance is not required.

Comparison of Phase Rotation of Two Three-phase Systems. A simple method of comparing the phase rotation of two three-phase systems—e.g. two alternators which are to be operated in parallel—is to connect three similar lamps to the systems according to the scheme shown in Fig. 277*a*. Then, assuming the frequencies to be equal, if the systems have the same phase rotation, the lamps A and B—which are cross-connected in relation to the supply systems—will glow brightly, and lamp C will be either very dim or completely dark. On the other hand, if the phase rotations are not in the same direction, *all* the lamps will either glow with equal brightness or be completely dark.

Vector diagrams for these cases are shown in Figs. 277*b*, 277*c*, respectively. The vectors OE_1, OE_2, OE_3, represent the equivalent phase voltages of one system and OE'_1, OE'_2, OE'_3, represent the equivalent phase voltages of the other system. Then, if the two systems have the same phase rotation and equal phase voltages (E), and θ is the phase difference between the systems, the voltages, V_A, V_B, V_C, impressed upon the lamps are

$$V_A = 2E \sin \tfrac{1}{2}\theta, \quad V_B = 2E \sin \tfrac{1}{2}(120° \pm \theta),$$
$$V_c = 2E \sin \tfrac{1}{2}(120° \mp \theta).$$

In the special case, when $\theta = 0$,

$$V_A = 0, \quad V_B = 2E \sin 60° = V, \quad V_c = 2E \sin 60° = V,$$

where $V (= \sqrt{3} \,.\, E)$ is the line voltage of each system.

If the phase rotations are not the same, then

$$V_A = 2E \sin \tfrac{1}{2}\theta, \quad V_B = 2E \sin \tfrac{1}{2}\theta, \quad V_c = 2E \sin \tfrac{1}{2}\theta$$

If the phase rotations are the same, but the frequencies differ slightly, then θ varies with respect to time and passes through a cycle of 360° during the time that one system is gaining or losing a cycle with respect to the other system. Hence, during the interval, the lamps A, B, C will each complete in succession the cycle dark-bright-dark-bright-dark.

The order in which the lamps B and C light up depends upon which system has the higher frequency. This feature may be utilized in connection with the synchronizing of small alternators, and was formerly applied by Siemens to synchroscopes.

Determination of Ratio and Phase Angle of Instrument Transformers. These quantities may be determined directly by means of the Gall co-ordinate potentiometer (Fig. 271). In the case of a current transformer, the secondary winding is connected to the normal

instrument load, in which is included a standard non-inductive low resistance. A similar resistance is connected in the primary circuit. The potential differences across these resistances due to the primary and secondary currents are measured by the potentiometer, and from these measurements and a knowledge of the values of the standard resistances both the ratio and phase angle can be calculated.

In the case of a potential transformer, the secondary winding is connected to an instrument load and also to a potential divider from which a suitable voltage is obtained for the potentiometer. A similar potential divider is connected across the primary winding. The voltages are measured by the potentiometer, and the ratio and phase angle may be calculated from these measurements and a knowledge of the ratios of the potential dividers.

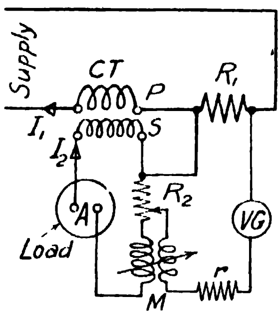


FIG. 278. METHOD OF DETERMINING RATIO AND PHASE ANGLE OF CURRENT TRANSFORMER

When a co-ordinate potentiometer is not available, the methods now to be described may be adopted, the principles of which are based upon the potentiometer method of comprising E.M.Fs. A standard variable mutual inductance, standard non-inductive resistances, and a vibration galvanometer are required for the tests.

Ratio and Phase Angle of Current Transformer. The connections are shown in Fig. 278, in which R_1, R_2 represent standard non-inductive low resistances, M a standard variable mutual inductance, and VG a vibration galvanometer. R_2 and M are adjusted successively until there is no deflection of the galvanometer. Then, if the standard resistances are entirely non-inductive, the ratio is given by

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}\sqrt{(1 + \omega^2 M^2/R_2^2)} \quad . \quad . \quad . \quad . \quad (198)$$

and the phase angle β is given by

$$\beta = \tan^{-1} \omega M/R_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

Proofs. In the proofs we shall consider that the standard resistances R_1, R_2 have slight inductances L_1, L_2 respectively. Then, when the current in the vibration galvanometer is zero, we have

$$I_2\,[(R_2 + j\omega L_2) - j\omega M\,] = I_1(R_1 + j\omega L_1)$$

i.e. $$\frac{I_1}{I_2} = \frac{R_2 + j\omega(L_2 - M)}{R_1 + j\omega L_1} = \frac{R_1R_2 + \omega^2 L_1(L_2 - M)}{R_1^2 + \omega^2 L_1^2} - j\omega\left[\frac{L_1R_2 - R_1(L_2 - M)}{R_1^2 + \omega^2 L_1^2}\right]$$

Whence
$$\frac{I_1}{I_2} = \sqrt{\left[\frac{R_2^2 + \omega^2(L_2 - M)^2}{R_1^2 + \omega^2 L_1^2}\right]}$$
$$= \frac{R_2}{R_1}\sqrt{\left[\frac{1 + \omega^2 (L_2 - M)^2/R_2^2}{1 + \omega^2 L_1^2/R_1^2}\right]} \quad . \quad . \quad . \quad . \quad . \quad (198a)$$

which, when $L_1 = 0$, $L_2 = 0$, reduces to
$$\frac{I_1}{I_2} = \frac{R_2}{R_1}\sqrt{(1 + \omega^2 M^2/R_2^2)}$$

Similarly, the phase angle, β, is given by
$$\beta = \tan^{-1}\frac{\omega L_1 R_2 - \omega R_1(L_2 - M)}{R_1 R_2 + \omega^2 L_1(L_2 - M)} \quad . \quad . \quad . \quad . \quad . \quad (199a)$$

which, when $L_1 = 0$, $L_2 = 0$, reduces to
$$\beta = \tan^{-1}\omega M/R_2$$

Ratio and Phase Angle of Potential Transformer. The connections are shown in Fig. 279 (*a*). A non-inductive high resistance R_1

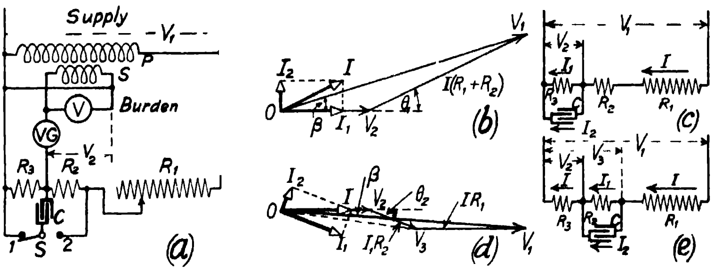


FIG. 279. METHOD OF DETERMINING RATIO AND PHASE ANGLE OF POTENTIAL TRANSFORMER

(*a*) Connection diagram; (*b*), (*c*), Vector and circuit diagrams for lagging phase-angle; (*d*), (*e*), Vector and circuit diagrams for leading phase-angle.

(*Note.* In the vector diagrams the angle β is shown much exaggerated. In practice, this angle may be only a small fraction of 1°.)

together with two smaller non-inductive resistances, R_3, are connected across the primary winding and the supply. A condenser, C, of variable capacitance, and a two-way switch, S, are connected to the resistances R_2, R_3, in such a manner that the condenser may be shunted across either R_2 or R_3. The secondary winding, to which the instrument load is connected, is reversed and connected through a vibration galvanometer across the resistance R_3. Zero deflection on the galvanometer is obtained by successive adjustment of R_1 and C, after a preliminary trial has been made to ascertain whether switch S should be in position 1 or 2. Then the ratio is given to a close approximation by
$$\frac{V_1}{V_2} = \frac{R_1 + R_2 + R_3}{R_3} \quad . \quad . \quad . \quad . \quad . \quad (200)$$

The phase angle, β, is given by

$$\beta = \tan^{-1}\omega CR_3(R_1 + R_2)/(R_1 + R_2 + R_3) \quad . \quad . \quad (201)$$

when lagging (i.e. when the reversed secondary voltage lags with reference to the supply voltage), and by

$$\beta = \tan^{-1}\omega CR_2{}^2/[R_2 + (R_1 + R_3)(1 + \omega CR_2)^2] \quad . \quad (201a)$$

when leading.

Proofs. The resistances R_1, R_2, R_3 are assumed to be without reactance and the condenser without losses. Then for the case when the reversed secondary voltage lags the supply voltage (diagrams (*b*), (*c*), Fig. 279), we have from the circuit and vector diagrams

$$\dot{V}_1 = \dot{V}_2 + \dot{I}(R_1 + R_2); \qquad \dot{V}_2 = \dot{I}_1R_3.$$

Hence, $$\frac{\dot{V}_1}{\dot{V}_2} = \frac{\dot{I}_1R_3 + (\dot{I}_1 + \dot{I}_2)(R_1 + R_2)}{\dot{I}_1R_3} = \frac{R_3 + (R_1 + R_2)(1 + \dot{I}_2/\dot{I}_1)}{R_3}$$
$$= (R_1 + R_2 + R_3)/R_3 + j\omega C(R_1 + R_2).$$

Whence to a close approximation,

$$V_1/V_2 = (R_1 + R_2 + R_3)/R_3.$$

The phase angle is given accurately by

$$\beta = \tan^{-1}\omega C(R_1 + R_2)/[(R_1 + R_2 + R_3)/R_3]$$
$$= \tan^{-1}\omega CR_3(R_1 + R_2)/(R_1 + R_2 + R_3).$$

For the case when the reversed secondary voltage leads the supply voltage (diagrams (*d*), (*e*), Fig. 279), we have from the circuit diagram

$$\dot{V}_1 = \dot{V}_2 + \dot{I}_1R_2 + \dot{I}R_1; \qquad \dot{V}_2 = \dot{I}R_3.$$

The ratio (V_1/V_2) is best determined trigonometrically. Thus from the vector diagram,

$$V_1 = \sqrt{\{[I(R_1 + R_3) + I_1R_2\cos\theta_2]^2 + (I_1R_2\sin\theta_2)^2\}}$$
$$V_1/V_2 = \sqrt{\{[R_1 + R_3 + R_2(I_1/I)^2]^2 + (R_2I_1I_2/I^2)^2\}}/R_3$$

Now,

$$I_1/I = 1/(1 + \omega CR_2); \; I_1I_2/I^2 = \omega CR_2/(1 + \omega CR_2)^2;$$

and since ωCR_2 may be ignored in comparison with unity, we have to a close approximation

$$V_1/V_2 = (R_1 + R_2 + R_3)/R_3.$$

The phase angle is given accurately by

$$\beta = \tan^{-1}\{I_1R_2\sin\theta_2/[I_1(R_1 + R_3) + I_1R_2\cos\theta_2]\}$$
$$= \tan^{-1}\omega CR_2{}^2/[R_2 + (R_1 + R_3)(1 + \omega CR_2)^2].$$

CHAPTER XIX

CALCULATION OF THREE-PHASE CIRCUITS

In this chapter, some of the simpler methods of calculating the currents and voltages in three-phase circuits with balanced and unbalanced loads will be discussed. Consideration will be given, first, to circuits in which the loads are supplied at constant voltage, and then to circuits in which the voltage at the load is affected by the voltage drop in the generator and line wires. In all cases it is to be understood that the currents and E.M.Fs. vary sinusoidally with respect to time, so that vector diagrams and complex algebra are applicable to the solutions.

The simpler examples are calculated by graphic, trigonometric, and complex algebraic methods, so as to bring out the salient features of each. But with the more complicated cases of unbalanced star-connected three-wire circuits, only complex algebraic methods of solution are given. In such cases, and also in the calculation of unbalanced delta-connected loads, the phase sequence of the generator has to be taken into account. It will, therefore, be appropriate now to consider this quantity and the manner in which the symbolic expressions of E.M.Fs. are affected by the phase sequence.

Phase Sequence. A knowledge of the phase sequence, or phase rotation, of a polyphase system is necessary before a vector diagram can be drawn or the symbolic equations written down. The vector diagrams previously given have all been drawn for counterclockwise phase sequence, which is the standard direction, and the diagrams of the simple two-phase and three-phase alternators, shown on p. 175, have been drawn to give this sequence. A reference to Fig. 100 (p. 175) will show that the E.M.Fs. of the three-phase alternator reach their positive maximum values in the order E_{I}, E_{II}, E_{III}. Thus in the vector diagram (*a*), Fig. 100, if the vector OE_{I} is the vector of reference, the vector OE_{II} lags 120 degrees and the vector OE_{III} lags 240 degrees.

Observe that if the direction of rotation of the armature is reversed, the E.M.Fs. will now reach their positive maximum values in the order I, III, II. Hence the vector diagram will now have to be drawn with OE_{II} lagging 240 degrees and OE_{III} lagging 120 degrees; or, alternatively, OE_{II} leading 120 degrees and OE_{III} leading 240 degrees.

Standard Phase Sequence and Generator Terminal Markings. The standard phase sequence is counter-clockwise, as explained above. Hence, if the terminals of the generator are marked A, B, C, it is understood that—(1) the E.M.F. between terminal B and the neutral point lags 120 degrees with respect to the E.M.F. between terminal A and the neutral point; and (2) the E.M.F. between terminal C and the neutral point leads 120 degrees with respect to the E.M.F. between terminal A and the neutral point.

When a colour scheme of terminal or cable markings is employed, the standard colours are red, blue, and white (or yellow). The colour white (or yellow) is assigned to terminal A and all cables connected to it; blue is assigned to terminal B and all cables connected to it; red is assigned to terminal C and all cables connected to it. Hence, with this scheme the "blue" phase lags 120 degrees, and the "red" phase leads 120 degrees, with respect to the "white" phase.

Symbolic Representation of Phase and Line E.M.Fs. in a Three-phase, Star-connected System. With a symmetrical system the phase E.M.Fs. may be represented by either of the symbolic expressions (202), (202a), according to the phase sequence,

Phase E.M.Fs. for Counter-clockwise Phase Sequence—

$$\left.\begin{aligned} \dot{E}_{\text{I}} &= E(1 + j0) \\ \dot{E}_{\text{II}} &= E(\cos - 120^\circ + j \sin - 120^\circ) = E(-0{\cdot}5 - j0{\cdot}866) \\ \dot{E}_{\text{III}} &= E(\cos - 240^\circ + j \sin - 240^\circ) = E(-0{\cdot}5 + j0{\cdot}866) \end{aligned}\right\} \quad (202)$$

Phase E.M.Fs. for Clockwise Phase Sequence—

$$\left.\begin{aligned} \dot{E}_{\text{I}} &= E(1 + j0) \\ \dot{E}_{\text{II}} &= E(\cos 120^\circ + j \sin 120^\circ) = E(-0{\cdot}5 + j0{\cdot}866) \\ \dot{E}_{\text{III}} &= E(\cos 240^\circ + j \sin 240^\circ) = E(-0{\cdot}5 - j0{\cdot}866) \end{aligned}\right\} \quad (202a)$$

Hence the line E.M.Fs. are given by the expressions,

Counter-clockwise Phase Sequence—

$$\begin{aligned} \dot{V}_{1\cdot2} &= \dot{E}_{\text{I}} - \dot{E}_{\text{II}} = E(1 + 0{\cdot}5 + j0{\cdot}866 = E(1{\cdot}5 + j0{\cdot}866) \\ \dot{V}_{2\cdot3} &= \dot{E}_{\text{II}} - \dot{E}_{\text{III}} = E(-0{\cdot}5 - j0{\cdot}866 + 0{\cdot}5 - j0{\cdot}866) = -j\sqrt{3}E \\ \dot{V}_{3\cdot1} &= \dot{E}_{\text{III}} - \dot{E}_{\text{I}} = E(-0{\cdot}5 + j0{\cdot}866 - 1 - j0) \\ &= E(-1{\cdot}5 + j0{\cdot}866) \end{aligned}$$

Clockwise Phase Sequence—

$$\begin{aligned} \dot{V}_{1\cdot2} &= \dot{E}_{\text{I}} - \dot{E}_{\text{II}} = E(1 + 0{\cdot}5 - j0{\cdot}866) = E(1{\cdot}5 - j0{\cdot}866) \\ \dot{V}_{2\cdot3} &= \dot{E}_{\text{II}} - \dot{E}_{\text{III}} = E(-0{\cdot}5 + j0{\cdot}866 + 0{\cdot}5 + j0{\cdot}866) \\ &= j\sqrt{3}E \\ \dot{V}_{3\cdot1} &= \dot{E}_{\text{III}} - \dot{E}_{\text{I}} = E(-0{\cdot}5 - j0{\cdot}866 - 1 - j0) \\ &= E(-1{\cdot}5 - j0{\cdot}866) \end{aligned}$$

These expressions and the vector diagrams of Fig. 280 show that the phase sequence of a polyphase system affects the relative positions of the E.M.F. and current vectors of the system, but has no effect upon the magnitudes of the E.M.Fs. or currents when the system is symmetrical and balanced. The phase sequence, however, may have a considerable effect upon these currents when the system is unbalanced, as is shown in the examples which follow.

Calculation of Line and Load Currents for Star- and Delta-connected Loads

Calculation of Load and Line Currents for Delta-connected Balanced Load. This represents the simplest case of a three-phase circuit, and

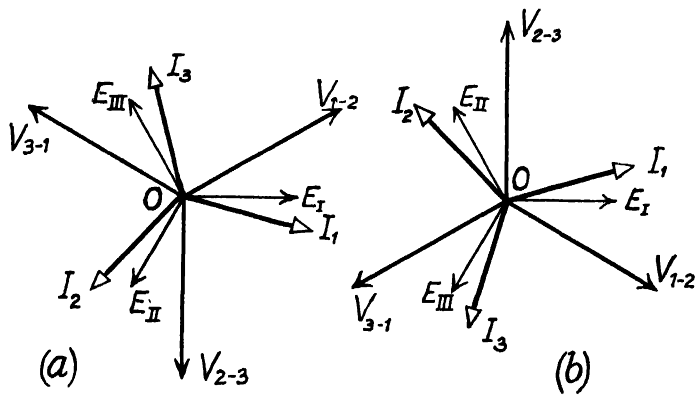


FIG. 280. VECTOR DIAGRAMS OF THREE-PHASE SYSTEMS
(a) Counter-clockwise (standard) phase sequence; (b) Clockwise phase sequence.

the solution follows readily from the vector diagram of Fig. 117 (p. 187). Thus the phase current is calculated first, and the line current is obtained from the relationship

$$\text{line current} = \sqrt{3} \times \text{phase current.}$$

For example, if V is the line voltage and Z is the impedance of each phase of the load, then the phase current is $I_{ph} = V/Z$, and the line current is $I = \sqrt{3}I_{ph}$.

Calculation of Load Currents for Star-connected Balanced Load. The solution is simple in this case, because, as the load is balanced, the phase voltages are symmetrical. The appropriate vector diagram is shown in Fig. 113 (p. 185), from which

$$\text{phase voltage} = \text{line voltage}/\sqrt{3}.$$

Hence, if V is the line voltage and Z is the impedance of each phase of the load, the load (and line) current (I) is given by $I = V/(\sqrt{3}Z)$.

Calculation of Load and Line Currents for Star-connected Unbalanced Load Supplied from a Symmetrical Four-wire System (Voltage Drops in Lines and Neutral Ignored). The assumption that no voltage drops occur in any of the lines or the neutral enables a simple solution to be obtained, as the phase voltages at the load are symmetrical and are equal to

$$\text{line voltage}/\sqrt{3}.$$

Hence, if V is the line voltage and Z_1, Z_2, Z_3 are the impedances of the several branches or phases of the load, the load currents are

$$I_1 = V/\sqrt{3}Z_1;\ I_2 = V/\sqrt{3}Z_2;\ I_3 = V/\sqrt{3}Z_3.$$

The currents in the line wires are equal to the respective load currents, and the current in the neutral is equal to the vector sum of the load currents.

The current in the neutral may, therefore, be determined either graphically—by drawing to scale a vector diagram—or, analytically, using either the symbolic method or the component method. In the component method the load currents are resolved into components along two axes perpendicular to one another, and the resultant is calculated. This method usually involves simpler working than the symbolic method, especially if a suitable choice of axes is made. The worked example given below illustrates how the methods are applied to the solution of a particular problem.

Example. An unbalanced star-connected load—the branches of which have the following impedances—$Z_1 = 2{\cdot}5\underline{/10^\circ}$, $Z_2 = 3{\cdot}0\underline{/15^\circ}$, $Z_3 = 3{\cdot}5\underline{/5^\circ}$ ohms—is supplied from a three-phase, four-wire symmetrical system, the line voltage being 400 V., and the phase sequence counter-clockwise. Determine the currents in each line wire, ignoring the voltage drop in all line wires.

The phase voltage of the system $= 400/\sqrt{3} = 231$ V. Hence, if the line currents are denoted by I_1, I_2, I_3, we have

$$I_1 = 231/2{\cdot}5 = 92{\cdot}4 \text{ A.}$$
$$I_2 = 231/3 = 77 \text{ A.}$$
$$I_3 = 231/3{\cdot}5 = 66 \text{ A.}$$

These currents are lagging with respect to the phase voltages by the angles 10°, 15°, and 5°, respectively.

The *graphical solution* for the current in the neutral wire is shown in Fig. 281, the vector diagram being drawn to a scale of 1 cm. = 10 A. By measurement, the vector OI_0, representing the sum of the vectors OI_1, OI_2, OI_3, is 1·38 cm., and therefore the current in the neutral wire is $1{\cdot}38 \times 10 = 13{\cdot}8$ A. The angle between OI_0 and OI_1 is 41°.

The *analytical solution* for the current in the neutral wire using the component method is as follows—

One of the axes AOA_1 (Fig. 281) is taken along the direction of the current

vector OI_1, in order that this vector shall have no component along the perpendicular axis BOB_1.

The components along the axis AOA_1 are therefore

$$I_1, \quad I_2 \cos(120 + 15 - 10)^\circ \text{ and } I_3 \cos(240 + 5 - 10)^\circ$$

their sum is given by

$$I_x = 92{\cdot}4 - 77 \times 0{\cdot}5736 - 66 \times 0{\cdot}5736 = 10{\cdot}4$$

The components along the perpendicular axis BOB_1 are

$$I_2 \sin(120 + 5 - 10)^\circ \text{ and } I_3 \sin(240 + 5 - 10)^\circ$$

their sum is given by

$$I_y = -77 \times 0{\cdot}819 + 66 \times 0{\cdot}819 = -9{\cdot}1$$

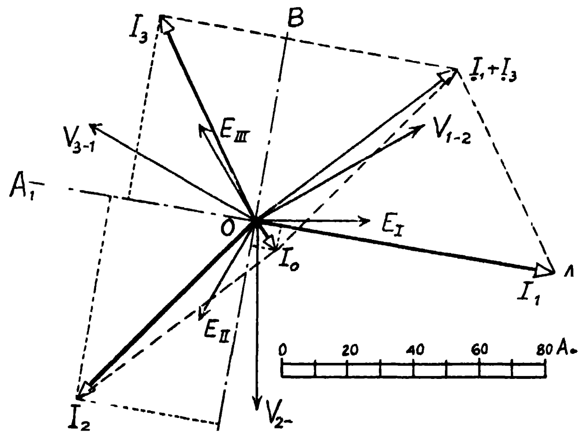


FIG. 281. GRAPHIC SOLUTION FOR CURRENT IN NEUTRAL WIRE OF THREE-PHASE FOUR-WIRE SYSTEM

OI_0 should be reversed in direction

Whence the current in the neutral wire is given by

$$I_0 = \sqrt{(I_x^2 + I_y^2)} = \sqrt{(10{\cdot}4^2 + 9{\cdot}1^2)} = 13{\cdot}8 \text{ A.}$$

and the phase difference of this current with respect to I_1 is given by

$$\varphi_0 = \tan^{-1} I_y/I_x = \tan^{-1} - 9{\cdot}1/10{\cdot}4 = -41{\cdot}2^\circ$$

Analytical solution for the current in the neutral using the symbolic method—

The vector of reference is the phase voltage E_I.

Hence the expressions for the load currents are

$$\dot{I}_1 = 92{\cdot}4(\cos - 10^\circ + j \sin - 10^\circ) = \mathbf{91 - j16}$$

$$\dot{I}_2 = 77 \ [\cos - (120 + 15)^\circ + j \sin - (120 + 15^\circ)] = \mathbf{-54{\cdot}4 - j54{\cdot}4}$$

$$\dot{I}_3 = 66 \ [\cos - (240 + 5^\circ) + j \sin - (240 + 5^\circ)] = \mathbf{-27{\cdot}9 + j59{\cdot}8}$$

Whence $\dot{I}_0 = -(\dot{I}_1 + \dot{I}_2 + \dot{I}_3) = \mathbf{-8{\cdot}7 + j10{\cdot}6}$.

Hence, $\quad \dot{I}_0 = \sqrt{(8{\cdot}7^2 + 10{\cdot}6^2)} = 13{\cdot}8$ A,

and its phase difference with respect to E_I is

$$\varphi_0 = \tan^{-1} - 10{\cdot}6/8{\cdot}7 = 50{\cdot}6^\circ \text{ (lagging).}$$

Calculation of Line and Load Currents for Three-phase, Delta-connected, Unbalanced Load Supplied at Constant Voltage. The simplest solution is a graphic one and is shown in Fig. 282. The line voltages are represented by the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\text{-}1}$. The

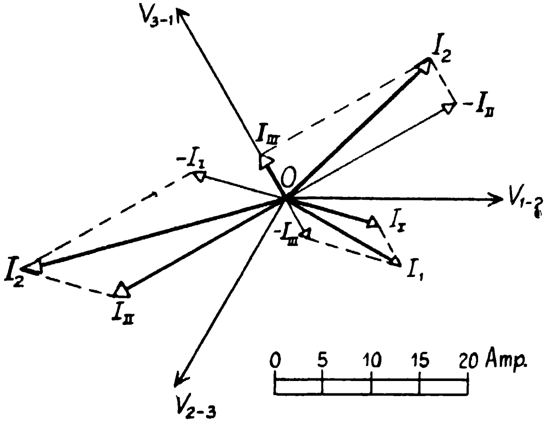


FIG. 282. GRAPHICAL SOLUTION FOR CURRENT IN DELTA-CONNECTED LOAD

currents in the branches of the load are calculated from the respective line voltages and impedances, and are represented by the vectors OI_{I}, OI_{II}, OI_{III}, which have phase differences of φ_1, φ_2, φ_3, respectively, with respect to the line voltages. The line currents are represented by the vectors OI_1, OI_2, OI_3; OI_1 being the vector difference of OI_I and OI_{III}, OI_2 the vector difference of OI_{II} and OI_I, and so on.

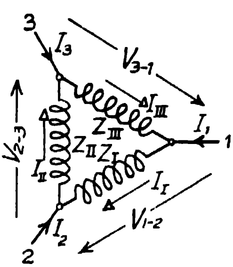


FIG. 283. CIRCUIT DIAGRAM FOR DELTA-CONNECTED LOAD

The problem, however, may be easily solved by the symbolic method, the polar form described on p. 27 being preferable to the rectangular form. Thus, if the *phase rotation is counter-clockwise*, the line E.M.Fs. are given by the expressions

$$\dot{V}_{1\text{-}2} = VJ^0, \quad \dot{V}_{2\text{-}3} = VJ^{-120/90} \quad \dot{V}_{3\text{-}1} = VJ^{-240/90} = VJ^{120/90}$$

and if the load impedances are arranged in the order Z_{I}, Z_{II}, Z_{III}, as shown in Fig. 283, the load currents will be given by

$$\dot{I}_{I} = \dot{V}_{1\text{-}2}/\dot{Z}_{I} = I_{I}J^{-\varphi_1°/90}$$

$$\dot{I}_{II} = \dot{V}_{2\text{-}3}/\dot{Z}_{II} = I_{II}J^{-(120+\varphi_2°)/90}$$

$$\dot{I}_{III} = \dot{V}_{3\text{-}1}/\dot{Z}_{III} = I_{III}J^{-(240+\varphi_3°)/90} = I_{III}J^{(120-\varphi_3°)/90}$$

Hence the line currents are given by

$$\begin{aligned}
\dot{I}_1 &= \dot{I}_{\mathrm{I}} - \dot{I}_{\mathrm{III}} = I_{\mathrm{I}} J^{-\varphi_1{}^\circ/90} - I_{\mathrm{III}} J^{(120-\varphi_3{}^\circ)/90} \quad . \\
&= I_{\mathrm{I}} J^{-\varphi_1{}^\circ/90} + I_{\mathrm{III}} J^{-(60+\varphi_3{}^\circ)/90} \quad . \\
\dot{I}_2 &= \dot{I}_{\mathrm{II}} - \dot{I}_{\mathrm{I}} = I_{\mathrm{II}} J^{-(120+\varphi_2{}^\circ)/90} - I_{\mathrm{I}} J^{-\varphi_1{}^\circ/90} \quad . \\
&= I_{\mathrm{II}} J^{-(120+\varphi_2{}^\circ)/90} + I_{\mathrm{I}} J^{-(180+\varphi_1{}^\circ)/90} \\
\dot{I}_3 &= \dot{I}_{\mathrm{III}} - \dot{I}_{\mathrm{II}} = I_{\mathrm{III}} J^{(120-\varphi_3{}^\circ)/90} - I_{\mathrm{II}} J^{-(120+\varphi_2{}^\circ)/90} \\
&= I_{\mathrm{III}} J^{(120-\varphi_3{}^\circ)/90} + I_{\mathrm{II}} J^{(60-\varphi_2{}^\circ)/90}
\end{aligned} \tag{203}$$

or

$$\begin{aligned}
I_1 &= \sqrt{[I_{\mathrm{I}}{}^2 + I_{\mathrm{III}}{}^2 + 2I_{\mathrm{I}}I_{\mathrm{III}} \cos(60 + \varphi_3{}^\circ - \varphi_1{}^\circ)]} \quad . \\
I_2 &= \sqrt{\{I_{\mathrm{II}}{}^2 + I_{\mathrm{I}}{}^2 + 2I_{\mathrm{I}}I_{\mathrm{II}} \cos[(180 + \varphi_1{}^\circ) - (120 + \varphi_2{}^\circ)]\}} \\
&= \sqrt{[I_{\mathrm{II}}{}^2 + I_{\mathrm{I}}{}^2 + 2I_{\mathrm{I}}I_{\mathrm{II}} \cos(60 + \varphi_1{}^\circ - \varphi_2{}^\circ)]} \quad . \\
I_3 &= \sqrt{\{I_{\mathrm{III}}{}^2 + I_{\mathrm{II}}{}^2 + 2I_{\mathrm{II}}I_{\mathrm{III}} \cos[(120 - \varphi_3{}^\circ) - (60 - \varphi_2{}^\circ)]\}} \\
&= \sqrt{[I_{\mathrm{III}}{}^2 + I_{\mathrm{II}}{}^2 + 2I_{\mathrm{II}}I_{\mathrm{III}} \cos(60 + \varphi_2{}^\circ - \varphi_3{}^\circ)]} \quad .
\end{aligned} \tag{204}$$

The angles α_1, α_2, α_3 by which these currents lag with respect to the voltage ($\dot{V}_{1-2}$) between lines 1 and 2 are given by

$$\begin{aligned}
\alpha_1 &= \tan^{-1} \frac{I_{\mathrm{I}} \sin -\varphi_1{}^\circ + I_{\mathrm{III}} \sin -(60 + \varphi_3{}^\circ)}{I_{\mathrm{I}} \cos -\varphi_1{}^\circ + I_{\mathrm{III}} \cos -(60 + \varphi_3{}^\circ)} \\
\alpha_2 &= \tan^{-1} \frac{I_{\mathrm{II}} \sin -(120 + \varphi_2{}^\circ) + I_{\mathrm{I}} \sin -(180 + \varphi_1{}^\circ)}{I_{\mathrm{II}} \cos -(120 + \varphi_2) + I_{\mathrm{I}} \cos -(180 + \varphi_1{}^\circ)} \\
\alpha_3 &= \tan^{-1} \frac{I_{\mathrm{III}} \sin -(240 + \varphi_3{}^\circ) + I_{\mathrm{II}} \sin -(300 + \varphi_2{}^\circ)}{I_{\mathrm{III}} \cos -(240 + \varphi_3{}^\circ) + I_{\mathrm{II}} \cos -(300 + \varphi_2{}^\circ)}
\end{aligned} \tag{205}$$

If, however, the phase rotation is clockwise, the line voltages for the circuit diagram of Fig. 283 will now be represented by

$$\dot{V}'_{1\text{-}2} = VJ^0, \quad \dot{V}'_{2\text{-}3} = VJ^{120/90}, \quad \dot{V}'_{3\text{-}1} = VJ^{240/90},$$

and the load currents will be given by

$$\begin{aligned}
\dot{I}_{\mathrm{I}}' &= \dot{V}'_{1\text{-}2}/\dot{Z}_{\mathrm{I}} = I_{\mathrm{I}} J^{-\varphi_1{}^\circ/90} \\
\dot{I}_{\mathrm{II}}' &= \dot{V}'_{2\text{-}3}/\dot{Z}_{\mathrm{II}} = I_{\mathrm{II}} J^{(120-\varphi_2{}^\circ)/90} \\
\dot{I}'_{\mathrm{III}} &= \dot{V}'_{3\text{-}1}/\dot{Z}_{\mathrm{III}} = I_{\mathrm{III}} J^{(240-\varphi_3{}^\circ)/90}
\end{aligned}$$

Hence the line currents will be given by

$$\begin{aligned}
\dot{I}_1' &= \dot{I}'_{\mathrm{I}} - \dot{I}'_{\mathrm{III}} = I_{\mathrm{I}} J^{-\varphi_1{}^\circ/90} + I_{\mathrm{III}} J^{(60-\varphi_3{}^\circ)/90} \\
\dot{I}_2' &= \dot{I}'_{\mathrm{II}} - \dot{I}'_{\mathrm{I}} = I_{\mathrm{II}} J^{(120-\varphi_2{}^\circ)/90} + I_{\mathrm{I}} J^{(180-\varphi_1{}^\circ)/90} \quad . \\
\dot{I}_3' &= \dot{I}'_{\mathrm{III}} - \dot{I}'_{\mathrm{II}} = I_{\mathrm{III}} J^{(240-\varphi_3{}^\circ)/90} + I_{\mathrm{II}} J^{(60-\varphi_2{}^\circ)/90}
\end{aligned} \tag{203a}$$

or

$$\begin{aligned}
I_1' &= \sqrt{[I_{\mathrm{I}}{}^2 + I_{\mathrm{III}}{}^2 + 2I_{\mathrm{I}} I_{\mathrm{III}} \cos(60 - \varphi_3{}^\circ + \varphi_1{}^\circ)]} \\
I_2' &= \sqrt{[I_{\mathrm{II}}{}^2 + I_{\mathrm{I}}{}^2 + 2I_{\mathrm{II}} I_{\mathrm{I}} \cos(60 - \varphi_1{}^\circ + \varphi_2{}^\circ)]} \\
I_3' &= \sqrt{[I_{\mathrm{III}}{}^2 + I_{\mathrm{II}}{}^2 + 2I_{\mathrm{III}} I_{\mathrm{II}} \cos(60 - \varphi_2{}^\circ + \varphi_3{}^\circ)]}
\end{aligned} \tag{204a}$$

Comparing these equations with those for counter-clockwise phase sequence, we find that the two sets of equations are similar except for the signs of the phase-angles φ_1, φ_2, φ_3. Hence in the special case where the branches of the unbalanced load have the same power-factor (i.e. $\varphi_1 = \varphi_2 = \varphi_3$), the *magnitudes* of the line currents will be unaffected by a change of phase sequence.

Example. An unbalanced delta-connected load, the branch-circuit impedances of which are $Z_{\mathrm{I}} = 10\underline{/15^\circ}$, $Z_{\mathrm{II}} = 5\underline{/30^\circ}$, $Z_{\mathrm{III}} = 20\underline{/0^\circ}$, is supplied from a symmetrical three-phase system in which the line pressure is 100 volts. Calculate the line currents.

Assuming the phase rotation to be counter-clockwise and the load impedances to be connected in the order Z_{I}, Z_{II}, Z_{III}, the impedance Z_{I} being connected between line wires 1 and 2; and denoting the load currents by I_{I}, I_{II}, I_{III}, and the line currents by I_1, I_2, I_3, we have,

$$I_{\mathrm{I}} = \frac{100}{10} J^{-15/90} = 10 J^{-15/90}$$

$$I_{\mathrm{I}} = 10\text{A}.$$

$$I_{\mathrm{II}} = \frac{100}{5} J^{-(120+30)/90} = 20 J^{-150/90}$$

$$I_{\mathrm{II}} = 20\text{A}.$$

$$I_{\mathrm{III}} = \frac{100}{20} J^{-240/90} = 5 J^{-240/90}$$

$$I_{\mathrm{III}} = 5\text{A}.$$

$$I_1 = \sqrt{[10^2 + 5^2 + 2 \times 10 \times 5 \cos(60 + 0 - 15)^\circ]} = 14\text{A}.$$

$$I_2 = \sqrt{[20^2 + 10^2 + 2 \times 20 \times 10 \cos(60 + 15 - 30)^\circ]} = 28\text{A}.$$

$$I_3 = \sqrt{[5^2 + 20^2 + 2 \times 5 \times 20 \cos(60 + 30 - 0)^\circ]} = 20{\cdot}6\text{A}.$$

A vector diagram drawn to scale is shown in Fig. 282.

If the phase rotation is reversed the load and line currents are now

$$I'_{\mathrm{I}} = \frac{100}{10} J^{(15)/90} = 10 J^{15/90}$$

$$I'_{\mathrm{I}} = 10\text{A}.$$

$$I'_{\mathrm{II}} = \frac{100}{5} J^{(120-30)/90} = 20 J^{90/90}$$

$$I'_{\mathrm{II}} = 20\text{ A}.$$

$$I'_{\mathrm{III}} = \frac{100}{20} J^{(240-0)/90} = 5 J^{240/90}$$

$$I'_{\mathrm{III}} = 5\text{A}.$$

$$I'_1 = \sqrt{[10^2 + 5^2 + 2 \times 10 \times 5 \cos(60 - 0 + 15)^\circ]} = 12{\cdot}3$$

$$I'_2 = \sqrt{[20^2 + 10^2 + 2 \times 20 \times 10 \cos(60 - 15 + 30)^\circ]} = 24{\cdot}6$$

$$I'_3 = \sqrt{[5^2 + 20^2 + 2 \times 5 \times 20 \cos(60 - 30 + 0)^\circ]} = 24{\cdot}4$$

Methods of Determining Load Currents for Unbalanced Star-connected Load. Two general methods are available. One method involves the conversion of the unbalanced star-connected load into an equivalent delta-connected load and determining the line currents in the manner described above. The other method involves

the determination of the phase voltages at the load, from which the load currents are readily calculated.

The first method is more general in its application, as it requires no restrictions to be made concerning the symmetry of the system. For this reason we shall consider it before dealing with the apparently simpler method—involving the determination of the phase voltages at the load—which, however, loses its simplicity when the system is unsymmetrical.

Conversion of an Unbalanced Star-connected Load into an Equivalent Delta-connected Load. In order that the delta-connected load may be the equivalent of the star-connected load, the joint impedances and admittances between corresponding terminals must be the same in both cases. Thus, if Z_1, Z_2, Z_3 denote the impedances of the branches of the star-connected load, and Z_a, Z_c denote the corresponding impedances of the branches of the equivalent delta-connected load (see Fig. 284), then we have

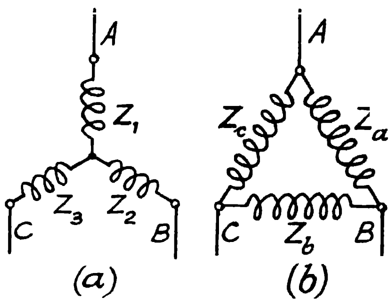


FIG. 284. CIRCUIT DIAGRAMS FOR EQUIVALENT STAR AND DELTA CIRCUITS

Joint impedance between terminals A and B of star-connected load

$$= Z_1 + Z_2$$

Joint impedance between terminals B and C of star-connected load

$$= Z_2 + Z_3$$

Joint impedance between terminals C and A of star-connected load

$$= Z_3 + Z_1$$

Joint impedance between terminals A and B of delta-connected load

$$= \frac{Z_a(Z_b + Z_c)}{Z_a + Z_b + Z_c}$$

Joint impedance between terminals B and C of delta-connected load

$$= \frac{Z_b(Z_a + Z_c)}{Z_a + Z_b + Z_c}$$

Joint impedance between terminals C and A of delta-connected load

$$\frac{Z_c(Z_a + Z_b)}{Z_a + Z_b + Z_c}$$

Hence for the delta-connected load to be the equivalent of the star-connected load, we must have

$$Z_1 + Z_2 = \frac{Z_a(Z_b + Z_c)}{Z_a + Z_b + Z_c} = \frac{Z_b + Z_c}{1 + Z_b/Z_a + Z_c/Z_a}$$

$$Z_2 + Z_3 = \frac{Z_b(Z_a + Z_c)}{Z_a + Z_b + Z_c} = \frac{Z_a + Z_c}{1 + Z_a/Z_b + Z_c/Z_b}$$

$$Z_3 + Z_1 = \frac{Z_c(Z_a + Z_b)}{Z_a + Z_b + Z_c} = \frac{Z_a + Z_b}{1 + Z_a/Z_c + Z_b/Z_c}$$

Whence

$$\left.\begin{aligned} Z_a &= Z_1 + Z_2 + (Z_1Z_2/Z_3) \quad . \quad . \quad . \\ Z_b &= Z_2 + Z_3 + (Z_2Z_3/Z_1) \quad . \quad . \quad . \\ Z_c &= Z_3 + Z_1 + (Z_3Z_1/Z_2) \quad . \quad . \quad . \end{aligned}\right\} \quad (206)$$

The method of obtaining the latter group of equations from the preceding group is as follows—

Dividing each equation by the numerator of its right-hand side, we have

$$\frac{Z_1 + Z_2}{Z_a(Z_b + Z_c)} = \frac{Z_2 + Z_3}{Z_b(Z_a + Z_c)} = \frac{Z_3 + Z_1}{Z_c(Z_a + Z_b)} = \frac{1}{Z_a + Z_b + Z_c}$$

from which we obtain

$$\frac{(Z_1 + Z_2) - (Z_2 + Z_3) + (Z_3 + Z_1)}{Z_a(Z_b + Z_c) - Z_b(Z_a + Z_c) + Z_c(Z_a + Z_b)} = \frac{2Z_1}{2Z_aZ_c} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\frac{(Z_1 + Z_2) + (Z_2 + Z_3) - (Z_3 + Z_1)}{Z_a(Z_b + Z_c) + (Z_b(Z_a + Z_c) - Z_c(Z_a + Z_b} = \frac{2Z_2}{2Z_aZ_b} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\frac{-(Z_1 + Z_2) + (Z_2 + Z_3) + (Z_3 + Z_1)}{-Z_a(Z_b + Z_c) + Z_b(Z_a + Z_c) + Z_c(Z_a + Z_b)} = \frac{2Z_3}{2Z_bZ_c} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\text{Hence} \quad \frac{Z_a}{Z_b} = \frac{Z_1}{Z_3}, \quad \frac{Z_b}{Z_c} = \frac{Z_2}{Z_1}, \quad \frac{Z_c}{Z_a} = \frac{Z_3}{Z_2}$$

Substituting in the original equations we have

$$Z_1 + Z_2 = \frac{Z_c(Z_2/Z_1) + Z_c}{1 + Z_3/Z_1 + Z_3/Z_2} = \frac{Z_cZ_2(Z_1 + Z_2)}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}$$

whence

$$Z_c = Z_3 + Z_1 + (Z_1Z_3/Z_2)$$

Similarly

$$Z_b = Z_2 + Z_3 + (Z_2Z_3/Z_1)$$
$$Z_a = Z_1 + Z_2 + (Z_1Z_2/Z_3)$$

Denoting the *admittances* of the star-connected load by Y_1, Y_2, Y_3, and the corresponding admittances of the delta-connected load by Y_a, Y_b, Y_c, then for the two loads to be equivalent we must have

$$\frac{Y_1Y_2}{Y_1 + Y_2} = Y_a + \frac{Y_bY_c}{Y_b + Y_c} = \frac{Y_aY_b + Y_bY_c + Y_cY_a}{Y_b + Y_c}$$

$$\frac{\dot{Y}_2\dot{Y}_3}{\dot{Y}_2 + \dot{Y}_3} = \dot{Y}_b + \frac{\dot{Y}_c\dot{Y}_a}{\dot{Y}_c + \dot{Y}_a} = \frac{\dot{Y}_a\dot{Y}_b + \dot{Y}_b\dot{Y}_c + \dot{Y}_c\dot{Y}_a}{\dot{Y}_c + \dot{Y}_a}$$

$$\frac{\dot{Y}_3\dot{Y}_1}{\dot{Y}_3 + \dot{Y}_1} = \dot{Y}_c + \frac{\dot{Y}_a\dot{Y}_b}{\dot{Y}_a + \dot{Y}_b} = \frac{\dot{Y}_a\dot{Y}_b + \dot{Y}_b\dot{Y}_c + \dot{Y}_c\dot{Y}_a}{\dot{Y}_a + \dot{Y}_b}$$

Whence

$$\left.\begin{aligned} \dot{Y}_a &= \frac{\dot{Y}_1\dot{Y}_2}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \quad . \quad . \quad . \quad . \\ \dot{Y}_b &= \frac{\dot{Y}_2\dot{Y}_3}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \quad . \quad . \quad . \quad . \\ \dot{Y}_c &= \frac{\dot{Y}_3\dot{Y}_1}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad (207)$$

Graphical Construction for Obtaining the Values of the Equivalent Impedances. As equation (206) shows the equivalent impedances

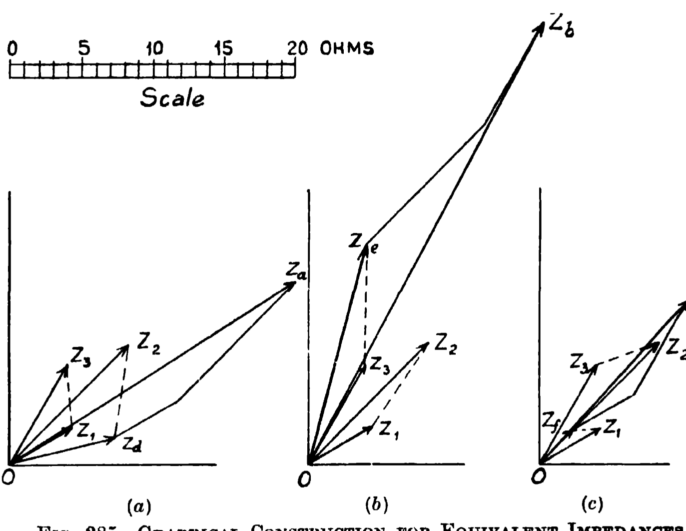


FIG. 285. GRAPHICAL CONSTRUCTION FOR EQUIVALENT IMPEDANCES

$\dot{Z}_a$, $\dot{Z}_b$. $\dot{Z}_c$, to be complex quantities, their calculation involves the application of the symbolic method. If desired, however, the quantities may be readily obtained by a simple graphical construction, which involves the simple geometric processes of addition and proportion.

Thus, let the complex quantities Z_1Z_2/Z_3, Z_2Z_3/Z_1, Z_3Z_1/Z_2, be denoted by Z_d, Z_e, Z_f, respectively. Then we have

$$\frac{Z_1}{Z_3} = \frac{Z_d}{Z_2} \qquad \frac{Z_2}{Z_1} = \frac{Z_e}{Z_3} \qquad \frac{Z_3}{Z_2} = \frac{Z_f}{Z_1}$$

Hence to determine, say, Z_d, vectors OZ_1, OZ_2, OZ_3, representing the star-connected impedances Z_1, Z_2, Z_3, are drawn from a common point, O (Fig. 285*a*), and a triangle OZ_2Z_d, similar to triangle OZ_3Z_1, is constructed upon OZ_2. Then OZ_d represents the quantity $Z_d = Z_1Z_2/Z_3$, since $OZ_d/OZ_2 = OZ_1/OZ_3$.

Therefore Z_a is given by the geometric sum of OZ_1, OZ_2, OZ_d, i.e. by OZ_a (Fig. 285*a*).

The quantities Z_b, Z_c, are determined in a similar manner, as shown in Fig. 285*b*, *c*.

Example. Determine the line currents in an unbalanced star-connected load when supplied from a symmetrical three-phase system at a line pressure of 100 V., the impedances of the branches of the load being $Z_1 = 5\underline{/30°}$, $Z_2 = 12\underline{/45°}$, and $Z_3 = 8\underline{/60°}$ ohms.

The graphic solution for the equivalent impedances of the delta-connected load is given in Fig. 285. The analytical solution is as follows—

The compound terms, $Z_1 Z_2/Z_3$, etc., in the equations (206) for the equivalent impedances Z_a, Z_b, Z_c, are first evaluated, using the polar form of symbolic notation, and the vector addition is then carried out, using the rectangular form of notation. Thus

$$\frac{Z_1Z_2}{Z_3} = \frac{5 \times 12}{8}\, j^{(30+45-60)/90} = 7{\cdot}5\, j^{15/90}$$

$$\frac{Z_2Z_3}{Z_1} = \frac{12 \times 8}{5}\, j^{(45+60-30)/90} = 19{\cdot}2\, j^{75/90}$$

$$\frac{Z_3Z_1}{Z_2} = \frac{8 \times 5}{12}\, (j^{60+30-45})/90 = 3{\cdot}33\, j^{45/90}$$

Hence

$$\begin{aligned} Z_a &= (5\cos 30° + j5\sin 30°) + (12\cos 45° + j12\sin 45°) \\ &\qquad + (7{\cdot}5\cos 15° + j7{\cdot}5\sin 15°) \\ &= (4{\cdot}33 + 8{\cdot}48 + 7{\cdot}24) + j(2{\cdot}5 + 8{\cdot}48 + 1{\cdot}94) \\ &= \boldsymbol{20{\cdot}05 + j12{\cdot}92} \end{aligned}$$

$$\begin{aligned} Z_b &= (12\cos 45° + j12\sin 45°) + (8\cos 60° + j8\sin 60°) \\ &\qquad + (19{\cdot}2\cos 75° + j19{\cdot}2\sin 75°) \\ &= (8{\cdot}48 + 4 + 4{\cdot}97) + j(8{\cdot}48 + 6{\cdot}93 + 18{\cdot}55) \\ &= \boldsymbol{17{\cdot}45 + j33{\cdot}96} \end{aligned}$$

$$\begin{aligned} Z_c &= (8\cos 60° + j8\sin 60°) + (5\cos 30° + j5\sin 30°) \\ &\qquad + (3{\cdot}33\cos 45° + j3{\cdot}33\sin 45°) \\ &= (4 + 4{\cdot}33 + 2{\cdot}36) + j(6{\cdot}93 + 2{\cdot}5 + 2{\cdot}36) \\ &= \boldsymbol{10{\cdot}69 + j11{\cdot}79} \end{aligned}$$

Whence

$$\begin{aligned}
Z_a &= \sqrt{(20{\cdot}05^2 + 12{\cdot}92^2)} &&= 23{\cdot}83\,\Omega \\
\varphi_a &= \tan^{-1} 12{\cdot}92/20{\cdot}05 &&= 32{\cdot}8^\circ \\
Z_b &= \sqrt{(17{\cdot}45^2 + 33{\cdot}96^2)} &&= 38{\cdot}1\,\Omega \\
\varphi_b &= \tan^{-1} 33{\cdot}96/17{\cdot}45 &&= 62{\cdot}8^\circ \\
Z_c &= \sqrt{(10{\cdot}69^2 + 11{\cdot}79^2)} &&= 15{\cdot}9\,\Omega \\
\varphi_c &= \tan^{-1} 11{\cdot}79/10{\cdot}69 &&= 47{\cdot}8^\circ
\end{aligned}$$

Assuming the phase rotation to be clockwise, and denoting the currents in the branches of the equivalent delta-connected load by I_a, I_b, I_c, and the corresponding line currents by I_1, I_2, I_3, we have

$$\begin{aligned}
I_a &= \frac{V_{1-2}}{Z_a} = \frac{100}{23{\cdot}83}\, J^{-32.8/90} \qquad 4{\cdot}2\, J^{-32.8/90} \\
I_b &= \frac{V_{2-3}}{Z_b} = \frac{100}{38{\cdot}1}\, J^{-(120 + 62.8)/90} = 2{\cdot}62\, J^{-182.8/90} \\
I_c &= \frac{V_{3-1}}{Z_c} = \frac{100}{15{\cdot}9}\, J^{-(240 + 47.8)/90} = 6{\cdot}29\, J^{-287.8/90}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 &= \sqrt{[I_a^2 + I_c^2 + 2I_aI_c \cos(60 + \varphi_c - \varphi_a)^\circ]} \\
&= \sqrt{[4{\cdot}2^2 + 6{\cdot}29^2 + 2 \times 4{\cdot}2 \times 6{\cdot}29 \cos(60 + 47{\cdot}8 - 32{\cdot}8)^\circ]} = 8{\cdot}42\text{A}. \\
I_2 &= \sqrt{[I_b^2 + I_a^2 + 2I_bI_a \cos(60 + \varphi_a - \varphi_b)^\circ]} \\
&= \sqrt{[2{\cdot}62^2 + 4{\cdot}2^2 + 2 \times 2{\cdot}62 \times 4{\cdot}2 \cos(60 + 32{\cdot}8 - 62{\cdot}8)^\circ]} = 6{\cdot}6\text{A}. \\
I_3 &= \sqrt{[I_c^2 + I_b^2 + 2I_cI_b \cos(60 + \varphi_b - \varphi_c)^\circ]} \\
&= \sqrt{[6{\cdot}29^2 + 2{\cdot}62^2 + 2 \times 6{\cdot}29 \times 2{\cdot}62 \cos(60 + 62{\cdot}8 - 47{\cdot}8)^\circ]} = 7{\cdot}42\text{A}.
\end{aligned}$$

Determination of the Currents in an Unbalanced Star-connected Non-inductive Load Supplied from a Symmetrical System. *Direct Geometrical Method.* In a symmetrical system the line voltage vectors may be drawn in the form of an equilateral triangle, e.g. ABC (Fig. 286). If the point O in this triangle represents the potential of the neutral point of an unbalanced load connected to this system, then vectors drawn from O to the corners of the triangle will represent the voltages across the branches of the load. These voltages can be expressed in rectangular co-ordinates with respect to rectangular axes drawn through O when the line-voltage triangle ABC is drawn with the side BC horizontal.

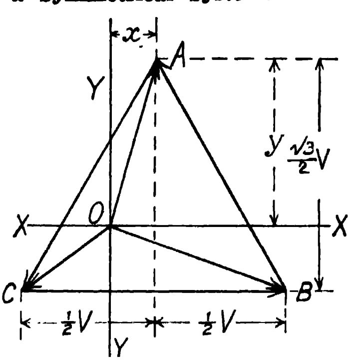


FIG. 286

Thus let x, y be the co-ordinates of A; V denote the numerical value of the line voltage; and E_{I}, E_{II}, E_{III} denote the phase voltages (represented by OA, OB, OC). Then the co-ordinates of B are

$x_B = x + \frac{1}{2}V$, $y_B = -(\sqrt{3}V/2 - y)$; and the co-ordinates of C are $x_C = -(\frac{1}{2}V - x)$, $y_C = -(\sqrt{3}V/2 - y)$.

Hence $E_{\mathrm{I}} = x + jy$; $E_{\mathrm{II}} = (x + \frac{1}{2}V) - j(\sqrt{3}V/2 - y)$; $E_{\mathrm{III}} = -(\frac{1}{2}V - x) - j(\sqrt{3}V/2 - y)$.

Let R_1, R_2, R_3 denote the resistances of the branches of the load; Y_1, Y_2, Y_3, the admittances of these branches, and I_1, I_2, I_3, the currents in them.

Then since $I_1 + I_2 + I_3 = 0$,

$$I_1 = -(I_2 + I_3) = E_{\mathrm{I}}Y_1 = Y_1(x + jy).$$

Also $$I_2 = E_{\mathrm{II}}Y_2, \text{ and } I_3 = E_{\mathrm{III}}Y_3.$$

Hence

$$I_2 + I_3 = Y_2[(x + \tfrac{1}{2}V) - j(\tfrac{1}{2}V\sqrt{3} - y)] + Y_3[-(\tfrac{1}{2}V - x) - j(\tfrac{1}{2}V\sqrt{3} - y)]$$

Adding this to the preceding equation for $-(I_2 + I_3)$, we have

$$0 = [x(Y_1 + Y_2 + Y_3) + \tfrac{1}{2}V_2(Y_2 - Y_3)] + j[y(Y_1 + Y_2 + Y_3) - (Y_2 + Y_3)\tfrac{1}{2}V\sqrt{3}]$$

Therefore $$x(Y_1 + Y_2 + Y_3) + \tfrac{1}{2}V(Y_2 - Y_3) = 0,$$

and $$y(Y_1 + Y_2 + Y_3) - (Y_2 + Y_3)\tfrac{1}{2}V\sqrt{3} = 0.$$

Whence $$x = -\tfrac{1}{2}V(Y_2 - Y_3)/(Y_1 + Y_2 + Y_3)$$

$$y = (\tfrac{1}{2}V\sqrt{3})(Y_2 + Y_3)/(Y_1 + Y_2 + Y_3).$$

Example. Three non-inductive resistances of 5, 10, and 15 ohms are connected in star and supplied from a 230-V. symmetrical three-phase system. Calculate the line currents.

Denoting the voltage across the 5Ω. resistance by E_{I}, and the voltages across the 10Ω. and 15Ω. resistances by E_{II} and E_{III} respectively, the co-ordinates of E_{I} with respect to the neutral point, O (Fig. 286), are given by

$$x = -\tfrac{1}{2} \times 230(\tfrac{1}{10} - \tfrac{1}{15})/(\tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{15})$$
$$= -90{\cdot}5$$
$$y = \tfrac{1}{2} \times 230\sqrt{3}(\tfrac{1}{10} + \tfrac{1}{15})/(\tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{15})$$
$$= 90{\cdot}5$$

Therefore, $$E_{\mathrm{I}} = -10{\cdot}45 + j90{\cdot}5$$
$$E_{\mathrm{II}} = (-10{\cdot}45 + 115) - j(199 - 90{\cdot}5)$$
$$= 104{\cdot}55 - j108{\cdot}5$$
$$E_{\mathrm{III}} = -(115 + 10{\cdot}45) - j(199 - 90{\cdot}5)$$
$$= -125{\cdot}55 - j108{\cdot}5$$

Whence $$I_1 = E_{\mathrm{I}}/5 = -2{\cdot}1 + j18{\cdot}1$$
$$I_2 = E_{\mathrm{II}}/10 = 10{\cdot}5 - j10{\cdot}85$$
$$I_3 = E_{\mathrm{III}}/15 = -8{\cdot}37 - j7{\cdot}23$$

Check— $$I_1 + I_2 + I_3 = 0{\cdot}03 - j0{\cdot}02$$

[NOTE. The calculations are made to slide-rule accuracy only.]

Hence, $$I_1 = \sqrt{(2{\cdot}1^2 + 18{\cdot}1^2)} = 18{\cdot}2 \text{ A.}$$
$$I_2 = \sqrt{(10{\cdot}5^2 + 10{\cdot}85^2)} = 15{\cdot}1 \text{ A.}$$
$$I_3 = \sqrt{(8{\cdot}37^2 + 7{\cdot}23^2)} = 11{\cdot}05 \text{ A.}$$

Direct Determination of Potential of Neutral Point of Load

Determination of the Potential of the Neutral Point and the Phase Voltages for a Balanced Star-connected Load Supplied from a Three-phase System. In this case the potential of the neutral point and the voltage across each branch of the load is easily determined graphically by drawing the line-voltage vectors in the form of a triangle and determining the centre of gravity of this triangular area. The point so obtained represents the potential of the neutral point, and vectors drawn from it to the corners of the triangle

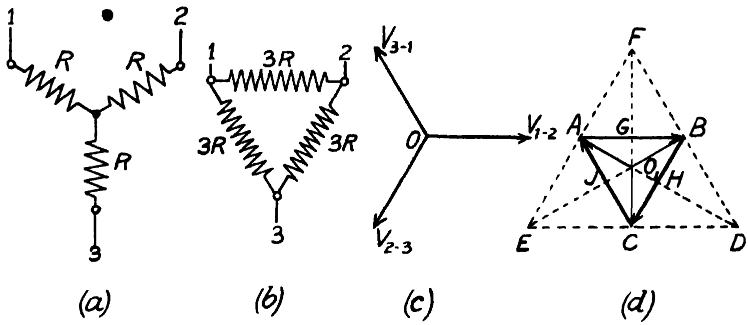


FIG. 287. METHOD OF DETERMINING POTENTIAL OF NEUTRAL POINT OF A BALANCED STAR-CONNECTED NON-INDUCTIVE LOAD

represent the voltages across the branches of the load. The construction is shown in Fig. 287, in which the triangle ABC is drawn, having its sides AB, BC, CA, equal and parallel to the line-voltage vectors OV_{1-2}, OV_{2-3}, OV_{3-1} (Fig. 287*c*) respectively. The mid-points, G, H, J, of these sides are joined to the opposite corners. The common point, O_1, of intersection of these lines is the centre of gravity of the triangular area ABC, and represents the potential of the neutral point of the load. The voltages across the branches of the load are represented by the vectors O_1A, O_1B, O_1C (Fig. 287*d*).

Proof. Consider for simplicity a *non-inductive*, star-connected, balanced load (Fig. 287*a*). Let this be replaced by an equivalent delta-connected load (Fig. 287*b*). Then the currents in the branches of the latter will be proportional to, and in phase with, the line voltages: they may, therefore, be represented by the vectors OV_{1-2}, OV_{2-3}, OV_{3-1} (Fig. 287*c*), and by the triangle ABC (Fig. 287*d*). The current scale and the magnitudes of the currents may be easily calculated. Thus, if R is the resistance of each branch of the star-connected load, the resistance of each branch of the equivalent delta-connected load is equal to $3R$, and the currents in the branches of this load are given by

$$I_{I} = V_{1-2}/3R, \qquad I_{II} = V_{2-3}/3R, \qquad I_{III} = V_{3-1}/3R.$$

Hence the current scale of the vector diagram is equal to $1/3R$ times the voltage scale.

Now the line currents are equal to the vector differences of the currents in adjacent branches of the delta-connected load: they are, therefore, represented in the vector diagram by the diagonals AD, BE, CF, of the parallelograms, $ABDC$, $BCEA$, $CAFB$, respectively, described on the sides of the triangle ABC.

But the voltages across the branches of the star-connected load are proportional to, and in phase with, the line currents, and are, therefore, given by

$$E_1 = R\left\{\frac{1}{3R}(V_{1\cdot 2} - V_{3\cdot 1})\right\} = \frac{1}{3}(V_{1\cdot 2} - V_{3\cdot 1})$$

$$E_2 = R\left\{\frac{1}{3R}(V_{2\cdot 3} - V_{1\cdot 2})\right\} = \frac{1}{3}(V_{2\cdot 3} - V_{1\cdot 2})$$

$$E_3 = R\left\{\frac{1}{3R}(V_{3\cdot 1} - V_{2\cdot 3})\right\} = \frac{1}{3}(V_{3\cdot 1} - V_{2\cdot 3})$$

Hence AD, BE, CF, represent the voltages across the branches of the star-connected load to a scale three times the original voltage scale of the diagram.

Since the diagonals AD, BE, CF, bisect the sides BC, CA, AB, of the triangle ABC, the former intersect one another at a common point, O_1, and

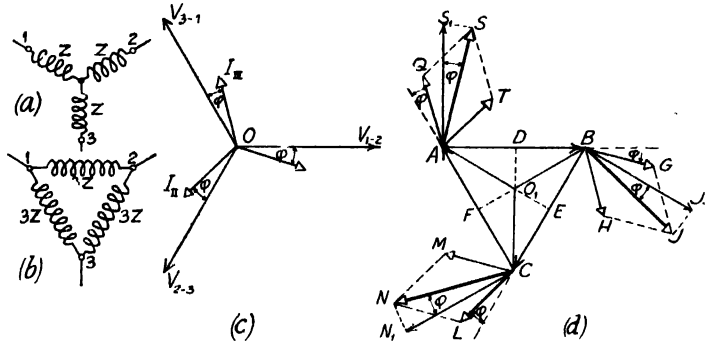


Fig. 288. Method of Determining Potential of Neutral Point of a Balanced Star-connected Inductive Load

(a, b) Equivalent circuit diagrams; (c) Vector diagram for (b); (d) Construction for obtaining potential of neutral point.

the distances O_1G, O_1H, O_1J, are equal to one-third of the distances GC, HA, JB, respectively. Therefore, O_1A, O_1B, O_1C, represent the voltages across the branches of the star-connected load to the original voltage scale, and the point O_1 represents the potential of the neutral point of the load.

The construction for the proof in the case of a balanced *inductive* load differs slightly from that for the case when the load is non-inductive, as the currents in the branches of the load are not in phase with the voltages across those branches. Thus, in Fig. 288*d*, the triangle ABC represents the vector triangle for the line voltages. From the corners of this triangle are drawn the vectors BG, CL, AQ, representing the currents in the branches of the equivalent delta-connected load. If these vectors are made equal to the vectors AB, BC, CA, respectively, the current scale will be equal to $1/3Z$ times the voltage scale of the diagram, where Z denotes the impedance of each branch of the star-connected load.

The line-current vectors, BJ, CN, AS, are then determined by constructing

the parallelograms $BGJH$, $CLNM$, $AQST$; and each parallelogram is rotated so as to bring the current vectors BG, CL, AQ, in line with the corresponding voltage vectors, AB, BC, CA. Then the new positions of the line-current vectors will represent the voltages across the branches of the star-connected load to a scale three times the original voltage scale of the diagram. Hence if from the points A B C the lines AE, BF, CD, are drawn parallel to BJ_1 CN_1, AS_1, respectively, the former will intersect at a common point, O_1 which represents the potential of the neutral point of the load. Moreover, since the points D, E, F, are the mid-points of the sides AB, BC, CA respectively, the point O_1 is the centre of gravity of the triangular area ABC.

Hence, in all cases of balanced star-connected loads, the potential of the neutral point of the load may be obtained by determining the centre of gravity of the triangular area formed by the vector triangle for the line voltages.

Potential Difference Between Neutral Points of Generator and Balanced Star-connected Load. When the supply system is symmetrical the neutral point of a balanced star-connected load is at the same potential as that of the generator, since the line-voltage vectors then form either an equilateral triangle (when the number of phases is equal to three) or a regular polygon (when the number of phases is greater than three), and the potential of the neutral point of the system is represented by the centre of gravity of this triangle or polygon, which also represents the potential of the neutral point of the load. Hence the voltages across each branch of the load are equal to, and in phase with, the voltage across the corresponding phase of the generator, assuming the voltage drop in the connecting wires to be negligible.

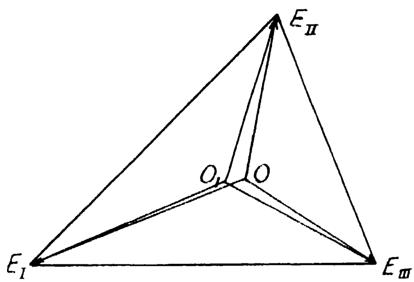


FIG. 289. SHOWING DIFFERENCE OF POTENTIAL BETWEEN NEUTRAL POINTS OF AN UNSYMMETRICAL SYSTEM AND OF A BALANCED STAR-CONNECTED LOAD

But when the system is unsymmetrical, the potential of the neutral point of the system is not generally represented by the centre of gravity of the line-voltage vector triangle or polygon. In this case the voltages across the branches of the load are neither equal to, nor in phase with, the corresponding phase voltages of the generator. For example, consider an unsymmetrical three-phase system in which the phase voltages are represented by the vectors OE_{I}, OE_{II}, OE_{III}, Fig. 289. The line voltages are then represented by the sides of the triangle, E_I, E_{II}, E_{III}, formed by joining the extremities of these vectors, and the potential of the neutral point is represented by O. The centre of gravity of this triangle is at O_1,

which therefore represents the potential of the neutral point of a balanced star-connected load supplied from the system. The voltages across the branches of the load are represented by vectors drawn from O_1 to the points E_{I}, E_{II}, E_{III}.

The potential difference between the two neutral points is represented, to the voltage scale of the diagram, by the distance OO_1.

Hence, when a non-inductive star-connected balanced potential circuit is employed in connection with the three- and four-wattmeter methods of measuring power in three- and four-phase unbalanced systems (p. 195), the voltages across the branches of the potential circuit are not necessarily equal to, nor in phase with, the phase voltages of the system. Under these conditions, the readings of the separate wattmeters do not represent the power in the phases of the system, although the sum of the readings is equal to the total power.

For example, in a three-phase unsymmetrical system the voltages across the branches of the potential circuits may be represented by O_1E_{I}, O_1E_{II}, O_1E_{III}, Fig. 289, and the phase voltages of the system by OE_{I}, OE_{II}, OE_{III}. Now O_1E_{I} is equal to the vector difference of OE_{I} and OO_1; O_1E_{II} is equal to the vector difference of OE_{II} and OO_1; O_1E_{III} is equal to the vector difference of OE_{III} and OO_1. Taking instantaneous values, and denoting the potential difference, OO_1, between the neutral points by v_o, the line currents by i_1, i_2, i_3, and the phase voltages of the system by e_{I}, e_{II}, e_{III}, the power measured by the separate wattmeters is given by

$$p_1 = i_1(e_{\text{I}} - v_o) = i_1e_{\text{I}} - i_1v_o$$

$$p_2 = i_2(e_{\text{II}} - v_o) = i_2e_{\text{II}} - i_2v_o$$

$$p_3 = i_3(e_{\text{III}} - v_o) = i_3e_{\text{III}} - i_3v_o$$

Whence $$p_1 + p_2 + p_3 = i_1e_{\text{I}} + i_2e_{\text{II}} + i_3e_{\text{III}} - v_o(i_1 + i_2 + i_3)$$
$$= i_1e_{\text{I}} + i_2e_{\text{II}} + i_3e_{\text{III}}$$

since $i_1 + i_2 + i_3 = 0$.

Variation of the Neutral Point Potential of a Star-connected, Non-inductive Circuit when the Resistance of One Branch is Varied. Consider a star-connected, non-inductive circuit, of which one branch is variable, as in Fig. 290, to be supplied from a symmetrical three-phase system at constant voltage. The variation of the resistance of one branch will then cause a variation in magnitude and phase of the currents in all the branches, as well as a variation

of the potential of the neutral point. If the neutral point of the system is assumed to be at zero potential. the potential of the neutral point of the load may, according to the relative values of the resistances of the variable and fixed branches, have values between zero and the phase voltage of the system. For example, when the resistance of the variable branch is equal to that of each of the fixed branches, the potential of the neutral point is zero and when the resistance of the variable branch is zero, the potential

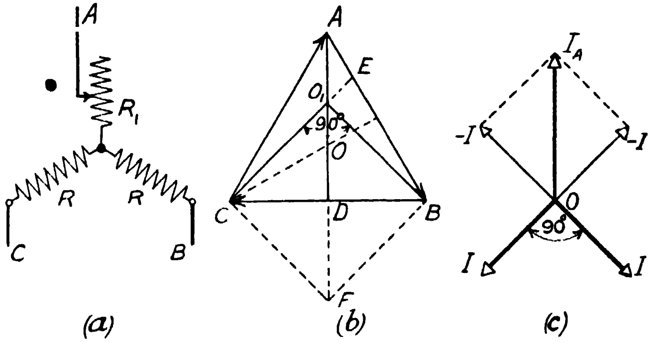


FIG. 290. CIRCUIT AND VECTOR DIAGRAMS FOR STAR-CONNECTED NON-INDUCTIVE LOAD WITH ONE BRANCH VARIABLE

of the neutral point is equal to the phase voltage of the system. Again, when the resistance of the variable branch is infinite, the potential of the neutral point is now reversed and is equal to one-half of the phase voltage of the system.

The variation of the potential of the neutral point and the phase difference between the currents in the branches of fixed resistance for various ratios of variable resistance/fixed resistance is shown in the curves of Fig. 291.

Since the supply system is symmetrical, the currents in the branches of fixed resistance, and the voltages across these branches, must be always equal to each other, and the vector difference of these voltages must be equal to the (constant) voltage between the line wires to which the fixed branches are connected. Hence, if the line-voltage vectors are drawn in the form of an equilateral triangle, *ABC* (Fig. 290*b*), and the side *BC* represents the voltage between the line wires to which the fixed branches are connected, the potential of the neutral point of the load is represented by a point in the line *AD*, where *D* is the mid-point of the side *BC*.

·When the resistance of the variable branch is equal to that of each of the fixed branches, the potential of the neutral point of

the load is represented by the point O (the distance OD being one-third of AD), and is zero, assuming the neutral point of the system to be at zero potential. When the resistance of the variable branch is decreased, the potential of the neutral point of the load increases, and is represented by a point, such as O_1 in OA. In the extreme case, when the resistance of this branch is zero, the potential of the neutral point of the load is represented by A, and the difference of potential between the two neutral points is equal to the phase

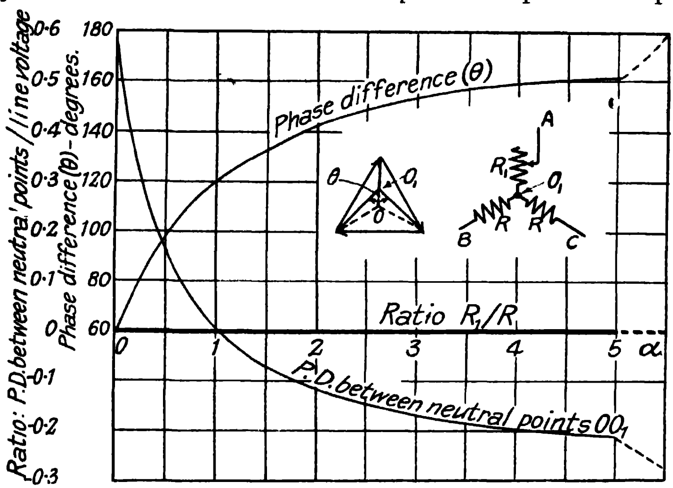


FIG. 291. CURVES RELATING TO FIG. 290

voltage of the system, OA. In the other extreme case, when the resistance of the variable branch is infinite, the potential of the load neutral point is represented by D, and since OD is equal to $\frac{1}{2}OA$, the potential difference between the two neutral points is now equal to one half of the phase voltage of the system, and is reversed in direction.

The currents in the branches of fixed resistance, and the voltages across these branches, are represented, to different scales, by O_1C and O_1B, and the phase difference between the currents, or voltages, is represented by the angle BO_1C.

The current in the branch of variable resistance is given by the reversed vector sum of the currents in the branches of fixed resistance, and is represented by OI_A (Fig. 290*c*). This current is in phase with the voltage across the variable branch, which is represented by O_1A (Fig. 290*b*). Thus, from the vector diagram, or the curves of Fig. 291, the value of the resistance of the variable branch,

expressed in terms of the resistance of the fixed branches, necessary to obtain a given phase difference between the currents in the fixed branches may readily be obtained.

For example, suppose the currents in the fixed branches are to have a phase difference of 90°, the point O_1 (Fig. 290*b*) is determined such that the angle BO_1C is 90°. The value of the resistance of the variable branch may then be obtained either by calculation or from measurements on the vector diagram. Thus, if V denotes the line voltage, the voltages across the branches of the load are, from the geometry of Fig. 290*b*, equal to $V\sqrt{2}/2$; $V\sqrt{2}/2$; $V(\sqrt{3}-1)/2$. Hence if R denotes the resistance of each of the fixed branches, the current, I, in these branches is equal to $V\sqrt{2}/2R$, and the current, I_A, in the variable branch is numerically equal to the vector sum of the currents in the fixed branches, i.e. $I_A = \sqrt{2} . I = V/R$. Hence the resistance of the variable branch is given by $R_1 = [V(\sqrt{3}-1)/2]/(V/R) = R(\sqrt{3}-1)/2 = 0{\cdot}366R$.

To obtain the value of R_1 from measurements on the vector diagram, produce CO_1 (Fig. 290*b*) to cut the side AB at E, and measure AE and BE. Then $AE/BE = R_1/R$.

Conversely, if the side AB be divided at E such that $AE/EB = R_1/R$ and the point E is joined to the opposite corner C, then the point, O_1, of intersection of CE and AD gives the potential of the neutral point of the load.

Proof. From B and C (Fig. 290*b*) draw BF and CF parallel to O_1C and O_1B respectively, and produce AD to the point of intersection, F, of BF and CF. Then triangles AO_1E, AFB, are similar, and therefore

$$AO_1/AF = AE/AB.$$

Whence $$AO_1/O_1F = AE/EB.$$

Now OA_1 represents, to the voltage scale of the diagram, the voltage across the variable branch of the load, and O_1F represents, to the current scale, the current in this branch, since, by construction, O_1F is the vector sum of O_1B and O_1C, and the latter represent the currents in the fixed branches of the load. Hence, if the diagram is drawn for a voltage scale of 1 cm. = q volts, the current scale will be 1 cm. = q/R amp.

Whence $$R_1 = \frac{q \,.\, AO_1}{(q/R)O_1F} = R\,\frac{AO_1}{O_1F} = R\,\frac{AE}{EB}$$

Now, when $\angle BO_1C = 90°$, $AO_1 = AB(\sqrt{3}-1)/2$,

$$O_1F = O_1B \,.\, \sqrt{2} = \sqrt{2}(\sqrt{2} \,.\, AB/2) = AB.$$

Therefore $$\frac{AO_1}{O_1F} = \frac{AB[(\sqrt{3}-1)/2]}{AB} = \frac{\sqrt{3}-1}{2} = 0{\cdot}366,$$

i.e. $$R_1 = 0{\cdot}366\,R,$$

and generally $$R_1/R = AE/EB.$$

This graphical method of determining the potential of the neutral point may be extended to unsymmetrical systems, and to the more

general cases, when all branches of the load are variable and the loads are inductive, *provided that the power factor of each branch of the load has the same value.*

In the general case, where the branches of the load have the impedances Z_1, Z_2, Z_3, and the line voltages of the system are represented by the vector triangle ABC (Fig. 292*b*)—in which the

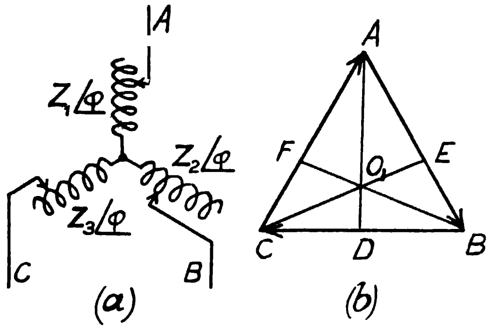


FIG. 292. CIRCUIT AND VECTOR DIAGRAMS FOR STAR-CONNECTED VARIABLE LOAD OF CONSTANT POWER FACTOR

side AB represents the voltage between the lines to which the impedances Z_1, Z_2, are connected; the side BC represents the voltage between the lines to which the impedances Z_2, Z_3, are connected, and so on—the side AB is divided at E such that $AE/EB = Z_1/Z_2$; BC is divided at D such that $BD/DC = Z_2/Z_3$, CA is divided at F such that $CF/FA = Z_3/Z_1$. Then the lines joining the points A, D; B, F; C, E will intersect at a common point, O_1, which represents the potential of the neutral point of the load.

"Floating" Neutral Point. When the neutral point of the load is isolated from the neutral point of the generator, the potential of the former is subject to variations according to the unbalance of the load, and under certain conditions of loading a considerable difference of potential may exist between the two neutral points. Such an isolated neutral point is called a "floating" neutral point.

All star-connected loads supplied from polyphase systems without neutral wires have floating neutral points, and any unbalancing of the load causes variations not only of the potential of the neutral point but also of the voltages across the several branches of the load. Hence, when single-phase electric lighting loads are to be supplied from three-phase three-wire systems, the former must be delta-connected in order that the voltages across the branches of the load may not be appreciably affected by a slight unbalancing

of the loads. If a star connection of the load is desired, then the four-wire system must be employed, and the neutral points of load and generator must be connected together.

Determination of the Potential of the Neutral Point and the Phase Voltages for an Unbalanced, Star-connected, Inductive Load Supplied from a Three-phase System. The solution will be obtained for the general case where the power-factors of the branches of the load may all be unequal. The line voltages of the system supplying the load are assumed to be known, as will generally be the case in practice. The simplest solution is a graphical one, the construction being shown in Fig. 293. Before the construction is commenced, however, the impedances $Z_a\underline{/\varphi_a}$, $Z_b\underline{/\varphi_b}$, $Z_c\underline{/\varphi_c}$, of the equivalent delta-connected load must be determined, either by calculation or graphically. Having obtained these quantities, the vector triangle *ABC* (Fig. 293) is drawn to represent the line voltages of the supply system. From the corners *B*, *C*, *A* of this triangle the vectors *BD*, *CG*, *AK*, are drawn to represent the currents in the branches of the equivalent delta-connected load; *BD* representing the current in the branch supplied at the voltage represented by *AB*; *CG* representing the current in the branch supplied at the voltage represented by *BC*; and so on. The line currents are then determined by constructing the parallelograms *BDEF*, *CGHJ*, *AKLM*, and are represented by the vectors *BF*, *CJ*, *AM*. If each of these vectors is multiplied by the impedance of the branch of the star-connected load through

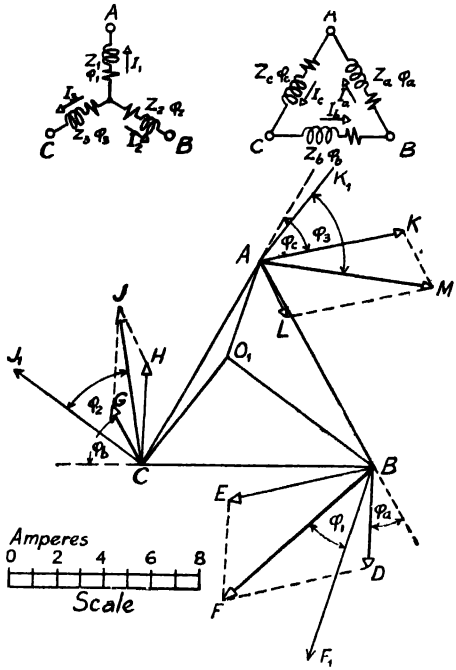


FIG. 293. GRAPHICAL SOLUTION FOR DETERMINING POTENTIAL OF NEUTRAL POINT OF UNBALANCED STAR-CONNECTED LOAD (ANY POWER FACTOR)

which the current (which is represented by the vector under consideration) passes, the resulting vectors will represent the voltages across the branches of the original star-connected load. This multiplication is carried out by rotating the vectors through the appropriate phase angles, φ_1, φ_2, φ_3, of the branches of the star-connected load, and at the same time changing the scale. The calculation of the new scale for the voltage may be avoided by employing the direct construction shown in Fig. 293, by means of which the load-voltage vectors are determined to the same scale as the line-voltage vectors. Thus the line-current vectors are rotated through the appropriate angles, φ_1, φ_2, φ_3, to the positions BF_1, CJ_1, AM_1, and parallels AO_1, BO_1, CO_1, are drawn from the corners A, B, C of the line-voltage vector triangle. These lines meet at a common point, O_1, which represents the potential of the neutral point of the load. Hence the vectors, AO_1, BO_1, CO_1, represent the voltages across the branches of the load to the same scale as the vectors AB, BC, CA, represent the line voltages.

Proof. Let the impedances of the star-connected load be denoted by $Z_1\underline{/\varphi_1}$, $Z_2\underline{/\varphi_2}$, $Z_3\underline{/\varphi_3}$, and those of the equivalent delta-connected load by $Z_a\underline{/\varphi_a}$, $Z_b\underline{/\varphi_b}$, $Z_c\underline{/\varphi_c}$ Then if the line voltages are denoted by V_{1-2}, V_{2-3}, V_{3-1}, the currents in the branches of the delta-connected load are given by

$$I_a = V_{1-2}/Z_a, \qquad I_b = V_{2-3}/Z_b, \qquad I_c = V_{3-1}/Z_c$$

Hence the line currents are given by

$$I_1 = I_a - I_c, \qquad I_2 = I_b - I_a, \qquad I_3 = I_c - I_b$$

Therefore the voltages across the branches of the load are given by

$$V_{1-0} = I_1 Z_1 = Z_1(V_{1-2}/Z_a - V_{3-1}/Z_c)$$
$$V_{2-0} = I_2 Z_2 = Z_2(V_{2-3}/Z_b - V_{1-2}/Z_a)$$
$$V_{3-0} = I_3 Z_3 = Z_3(V_{3-1}/Z_c - V_{2-3}/Z_b)$$

Whence

$$V_{1-0} - V_{2-0} = V_{1-2}\left(\frac{Z_1 + Z_2}{Z_a}\right) - V_{2-3}\frac{Z_2}{Z_b} - V_{3-1}\frac{Z_1}{Z_c}$$

$$= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) - V_{2-3}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) - V_{3-1}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right)$$

$$= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) - (V_{2-3} + V_{3-1})\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right)$$

$$= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) + V_{1-2}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right)$$

$$= V_{1-2}$$

Similarly, $V_{2-0} - V_{3-0} = V_{2-3}$,

and $V_{3-0} - V_{1-0} = V_{3-1}$.

Thus the voltages across the branches of an unbalanced star-connected load supplied from a three-phase system may be represented by vectors drawn from a particular point, inside the vector triangle for the line voltages, to the corners of this triangle, and both quantities are represented to the same scale. Hence, in the construction of Fig. 293, since the lines AO_1, BO_1, CO_1, were drawn parallel to the vectors representing the quantities I_1Z_1, I_2Z_2, I_3Z_3, their common point of intersection represents the potential of the neutral point of the load, and their lengths represent the magnitudes of the voltages across the branches of the load.

Calculation of Generator and Load Voltages

Determination of the Generator Line Voltage for Three-phase, Three-wire, Systems, the Voltage at the Load Being Known. In many cases of the supply of electrical energy for power purposes the position of the "load," or place where the energy is utilized, is at a considerable distance from the generator, and therefore the impedance of the line wires will affect the voltages at the generator and load. As the load must usually be supplied at a definite voltage we must show how the voltage at the generator may be determined. We shall assume the load to be concentrated at a single point and to be supplied from the generator through a single-circuit transmission line, as these conditions are representative of the practical case of a sub-station, or a distributing station, being supplied from a central generating station.

Case I. Star-connected Load The line currents and the potential of the neutral point of the load are first determined. The pressure drop in each line wire is then calculated and is added vectorially to the voltage across the corresponding branch of the load. The quantities so obtained represent the voltages between each terminal of the generator and the neutral point of the load. The voltages between the terminals of the generator are therefore determined.

The vector diagram is shown in Fig. 294, in which ABC represents the vector triangle for the known line voltages at the load; O_1 represents the potential of the neutral point of the load; and O_1A, O_1B, O_1C, represent the voltages across the branches of the load. The line currents are represented by the vectors O_1I_1, O_1I_2, O_1I_3. The pressure drops in the line wires are represented by the vectors AD, BE, CF, these vectors being drawn in the positions shown for convenience of carrying out the vector addition. [*Note.* AD represents the pressure drop in line 1, CF the pressure drop in line 2, and BE the pressure drop in line 3.] By adding, vectorially, the pressure drop in any line wire to the voltage across the corresponding branch of the load we obtain the voltage between the neutral point of the load and the terminal of the generator. These voltages are represented by the vectors O_1D, O_1E, O_1F. Therefore

the triangle DEF is the vector triangle for the line voltages at the terminals of the generator.

Observe that when the loads are balanced and the line wires have equal impedances, the vector triangle for the generator voltages is of similar shape to that for the load voltages (Fig. 294*a*), but that when the loads are unbalanced, one of the voltage triangles is distorted relatively to the other (Fig. 294*b*).

Case II. Delta-connected Load. The vector diagram for this case is shown in Fig. 295. The known (line) voltages across the branches

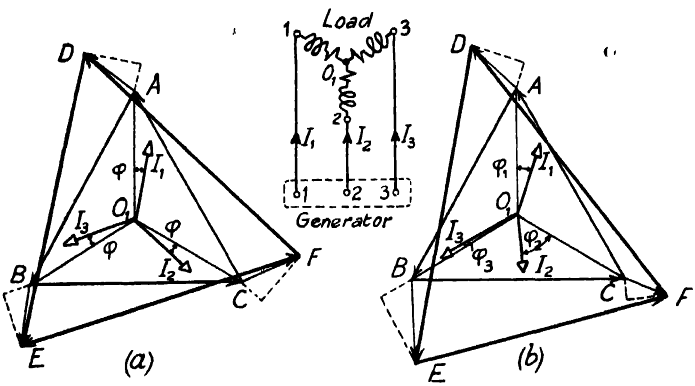


FIG. 294. GRAPHICAL METHOD OF DETERMINING GENERATOR VOLTAGE FOR STAR-CONNECTED SYSTEM
(*a*) Balanced load; (*b*) Unbalanced load.

of the load are represented by the sides AB, BC, CA, of the vector triangle ABC, and the currents in the line wires are represented by the vectors AD, BE, CF, these vectors being obtained from the vectors AI_{I}, BI_{II}, CI_{III}, representing the currents in the branches of the load. Observe that the vectors AD, BE, CF are so drawn in relation to the vectors for the load voltages that if BA represents the voltage between lines 1 and 2, AD will represent the current in line 1; if AC represents the voltage between lines 3 and 1, CF will represent the current in line 3; and so on.

The vectors BG, AH, CJ, representing the pressure drop in each of the line wires, are now added at the appropriate corners of the voltage triangle ABC (BG representing the pressure drop in line 2, CJ the pressure drop in line 3, and AH the pressure drop in line 1), and by joining the free ends, G, H, J, we obtain the vector triangle, GHJ, for the generator voltages.

The above observations regarding the nature of the load and the

shapes of the vector triangles for the generator and load voltages apply equally well to the present case.

Determination of Load Currents and Voltages for Three-phase, Three-wire, Systems, the Generator E.M.F. being Known. In the preceding section we determined the voltage necessary at the terminals of the generator in order to give a definite voltage at the load. When the load voltages are symmetrical and the load is balanced, the voltages at the generator will also be symmetrical. But if the load voltages are to be symmetrical and equal when the

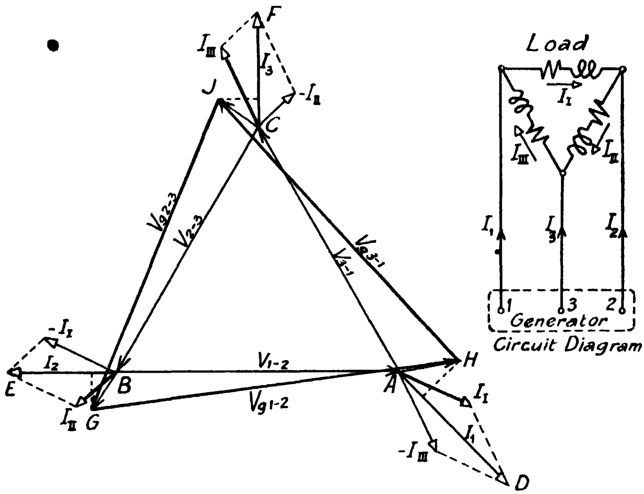


FIG. 295. GRAPHICAL METHOD OF DETERMINING GENERATOR VOLTAGE FOR DELTA-CONNECTED LOAD

load is unbalanced, the voltages between the terminals of the generator must be unsymmetrical and unequal, due to the pressure drop in the line wires. Under these conditions there may be difficulty in obtaining the required voltages. It will, therefore, be of interest to consider the converse of the above case, viz. the determination of the voltages at the load when the internal E.M.Fs. of the generator are known.

When a three-phase generator is unsymmetrically loaded, the voltages at its terminals will, in general, be unsymmetrical, although the E.M.Fs. generated in the phases may be symmetrical. Since the generated E.M.F. may be considered as equivalent to the no-load E.M.F., we may determine the terminal voltages when the generator is loaded by compounding the no-load E.M.Fs. of the

several phases with the pressure drop in these phases due to impedance and armature reaction. The currents in the load may therefore be determined from the no-load generator E.M.Fs. and the impedances of the load, line wires and generator.

Case I. Generator and Load Star-connected. The currents may be determined by two methods, one involving the calculation of the equivalent delta-connected circuit, and the other involving the determination of the difference of potential between the neutral points of the generator and the load. In the first method the total

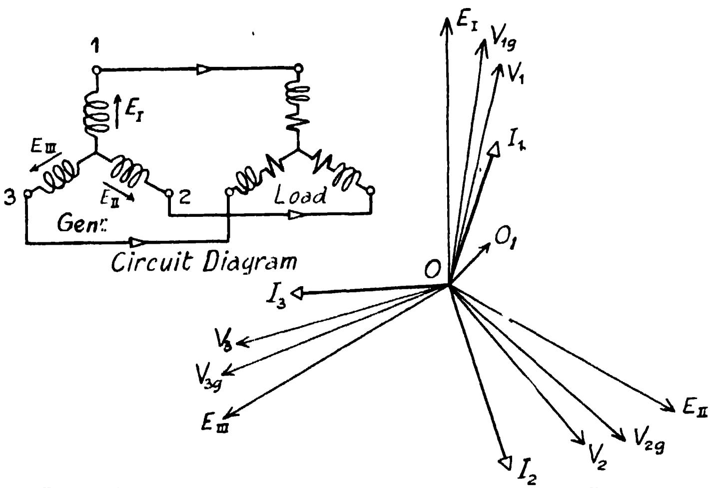


FIG. 296. GRAPHICAL METHOD OF DETERMINING LOAD VOLTAGE FOR STAR-CONNECTED SYSTEM

impedance of each phase of the star-connected circuit (i.e. the sum of the impedances of each phase of the generator, load, and connecting wire) is calculated, and the impedance of the equivalent delta-connected circuit is determined. This delta-circuit is then assumed to be supplied at the no-load generator line voltage, and the branch circuit and line currents are calculated. The line currents will be the same as those in the original star-connected circuit.

If the potential difference between the neutral points of the generator and load is known, the line currents may be obtained by dividing the total impedance of each phase into the difference between the no-load E.M.F. of the appropriate phase of the generator and the potential difference between the two neutral points. Thus if the no-load phase E.M.Fs. of the generator are represented in Fig. 296 by the vectors OE_{I}, OE_{II}, OE_{III}, and O_1 represents the

potential of the neutral point of the load, the current in phase I of the system is given by O_1E_I/(sum of impedances of phase I of generator, load, and connecting line wire). The currents in phases II and III of the system are obtained by dividing the E.M.Fs. represented by O_1E_{II}, O_1E_{III}, by the appropriate impedances. The currents are represented by the vectors OI_1, OI_2, OI_3.

The voltages across the branches of the load are obtained by multiplying the line currents by the appropriate load impedances. These voltages are represented in the vector diagram of Fig. 296 by the vectors OV_1, OV_2, OV_3.

The voltages at the terminals of the generator may be obtained by compounding the voltages across the branches of the load with the pressure drops in the line wires. The latter are represented by the vectors V_1V_{1g}, V_2V_{2g}, V_3V_{3g}. Thus the phase voltages of the generator when loaded are represented by the vectors OV_{1g}, OV_{2g}, OV_{3g}, and the pressure drops in the phases of the generator are represented by the vectors E_IV_{1g}, $E_{II}V_{2g}$, $E_{III}V_{3g}$.

Case II. Generator and Load Delta-connected. In this case we replace the delta connections of the generator and load by equivalent star-connected circuits. The solution is then obtained in the same manner as for the preceding case.

Determination of Load Currents and Voltages for Three-phase, Four-wire Systems, i.e. Three-phase Systems with Star-connected Loads and Neutral Wire. *General Case— Voltage Drop in all Parts of Circuit Considered.* The direct analytical solution to this case involves the application of Kirchhoff's laws and is given on p. 491.

An alternative analytical solution may be obtained very simply by employing the principle of super-position of electric currents. Thus, since the resultant E.M.F. in each phase, taken from the neutral point of the generator to the neutral point of the load, is equal to the vector difference of the no-load phase E.M.F. of the generator and the potential difference between the two neutral points, the actual current in each circuit may be obtained by super-imposing the fictitious currents due to (1) the no-load generator E.M.F. acting alone, (2) the potential difference between the neutral points acting alone.

Let the no-load phase E.M.Fs. of the generator be denoted by E_I, E_{II}, E_{III}; the impedances of each phase of the system (including the impedances of the generator, connecting line wire and load, but not the impedance of the neutral wire) by Z_{1t}, Z_{2t}, Z_{3t}; the impedance of the neutral wire by Z_{0l}; the actual currents in the neutral wire and the branches of the load by I_0, I_1, I_2, I_3, respectively,

and the potential difference between the two neutral points by V_0. Then

$$I_0 = -(I_1 + I_2 + I_3), \qquad V_0 = -I_0 Z_{0l}$$

$$I_1 = \frac{E_{\text{I}} - V_0}{Z_{1t}} = \frac{E_{\text{I}}}{Z_{1t}} - \frac{V_0}{Z_{1t}} = I_1' - I_1'' \quad . \quad . \quad . \quad (208)$$

$$I_2 = \frac{E_{\text{II}} - V_0}{Z_{2t}} = \frac{E_{\text{II}}}{Z_{2t}} - \frac{V_0}{Z_{2t}} = I_2' - I_2'' \quad . \quad . \quad . \quad (209)$$

$$I_3 = \frac{E_{\text{III}} - V_0}{Z_{3t}} = \frac{E_{\text{III}}}{Z_{3t}} - \frac{V_0}{Z_{3t}} = I_3' - I_3'' \quad . \quad . \quad . \quad (210)$$

where I_1', I_2', I_3', denote the phase currents which would be obtained if the two neutral points were at the same potential, and I_1'', I_2'', I_3'', denote the phase currents which would be obtained if the generator E.M.Fs. were zero and the potential V_0 existed between the neutral points.

Hence $I_0 = -[(I_1' - I_1'') + (I_2' - I_2'') + (I_3' - I_3'')]$

$$= -(I_1' + I_2' + I_3') + (I_1'' + I_2'' + I_3'')$$

$$= -I'_0 + (I_1'' + I_2'' + I_3'')$$

or

$$I_0' = -I_0 + I_1'' + I_2'' + I_3''$$

$$= V_0/Z_{0l} + V_0/Z_{1t} + V_0/Z_{2t} + V_0/Z_{3t}$$

$$= V_0(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$$

where I_0' denotes the current which would be obtained in the neutral wire if the impedance of this wire were zero, and Y_{0l}, Y_{1t}, Y_{2t}, Y_{3t}, denote the admittances of the neutral wire and the several phases, respectively.

Whence $$V_0 = I_0'/(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t}) \quad . \quad . \quad (211)$$

The load currents I_1, I_2, I_3, may be determined either directly from the E.M.Fs. and impedances or, indirectly, by the superposition of the fictitious currents I_1', I_1''; I_2', I_2''; I_3', I_3''. In the latter case we have

$$I_0' = V_0(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$$

$$I_1'' = V_0 Y_{1t}; \quad I_2'' = V_0 Y_{2t}; \quad I_3'' = V_0 Y_{3t}$$

Whence $I_1'' : I_0' :: Y_{1t} : (Y_{0t} + Y_{1t} + Y_{2t} + Y_{3t})$

$$I_2'' : I_0' :: Y_{2t} : (Y_{0t} + Y_{1t} + Y_{2t} + Y_{3t})$$
$$I_3'' : I_0' :: Y_{3t} : (Y_{0t} + Y_{1t} + Y_{2t} + Y_{3t})$$

The geometrical construction for determining the currents I_1'', I_2'', I_3'', is as follows—

A vector polygon is constructed to represent the admittances

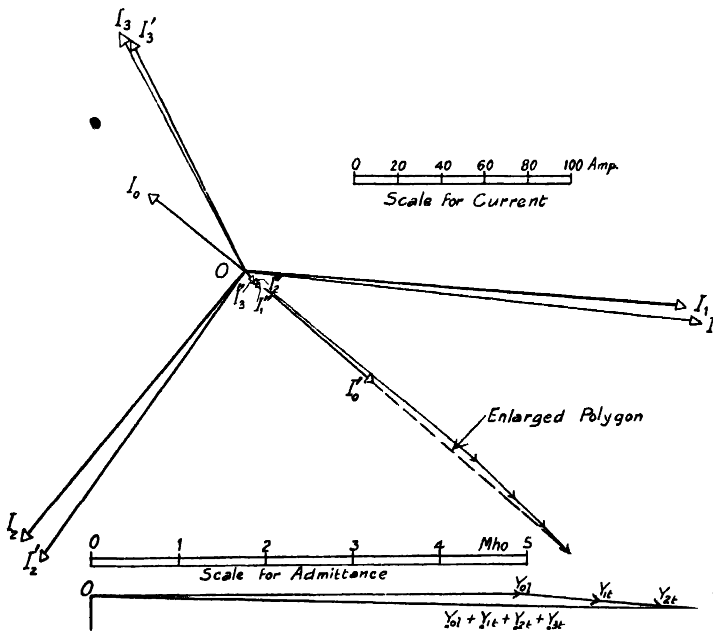


FIG. 297. GRAPHICAL METHOD OF DETERMINING THE FICTITIOUS CURRENTS I_1'', I_2'', I_3'' FOR A THREE-PHASE FOUR-WIRE SYSTEM

Y_{1t}, Y_{2t}, Y_{3t}, Y_{0t}, $(Y_{0t} + Y_{1t} + Y_{2t} + Y_{3t})$; the currents I_1', I_2', I_3', are calculated; their vector sum (I_0') is determined graphically, and upon the vector representing this quantity is constructed a polygon similar to the admittance vector polygon, the vector I_0' corresponding to the side of the latter which represents the quantity $(Y_{0t} + Y_{1t} + Y_{2t} + Y_{3t})$. The currents I_1'', I_2'', I_3'', in the admittances Y_{1t}, Y_{2t}, Y_{3t}, are then represented by the sides of the current polygon which are similar to the sides representing the latter quantities in the admittance polygon.

The geometrical construction in Fig. 297 refers to the worked example which follows.

Example. An unbalanced star-connected load of unity power factor—the branches of which have resistances of 1·0, 1·3, 1·85 ohms—is supplied from a symmetrical three-phase star-connected generator through a four-core cable 1000 yd. long. The cross sections of the cores are 0·25, 0·25, 0·25, 0·125 sq. in., and the core of smallest cross section forms the neutral conductor. The impedance of each phase of the generator is $0{\cdot}13\underline{/76^\circ}$ ohm. The no-load E.M.F. of the generator per phase is 240 V. and the phase rotation is counter-clockwise. Determine the currents and voltages at the load.

The resistance of the cable connecting the load and generator is 0·1 Ω. for the principal conductors and 0·2 Ω. for the neutral, the inductance being negligible in each case.

The impedance per phase of the generator is

$$\begin{aligned} Z_g &= 0{\cdot}13 \cos 76^\circ + j\, 0{\cdot}13 \sin 76^\circ \\ &= 0{\cdot}0314 + j0{\cdot}1262 \end{aligned}$$

The total impedances per phase are therefore

$$\begin{aligned} Z_{1t} &= 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}0 = 1{\cdot}1314 + j0{\cdot}1262 \\ Z_{2t} &= 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}3 = 1{\cdot}4314 + j0{\cdot}1262 \\ Z_{3t} &= 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}85 = 1{\cdot}9814 + j0{\cdot}1262 \end{aligned}$$

and the impedance of the neutral wire is

$$Z_{0l} = 0{\cdot}2 + j0$$

Whence the admittances per phase are

$$\begin{aligned} Y_{1t} &= \frac{1{\cdot}1314}{1{\cdot}1314^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}1314^2 + 0{\cdot}1262^2} = 0{\cdot}872 - j0{\cdot}0973 \\ Y_{2t} &= \frac{1{\cdot}4314}{1{\cdot}4314^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}4314^2 + 0{\cdot}1262^2} = 0{\cdot}693 - j0{\cdot}0611 \\ Y_{3t} &= \frac{1{\cdot}9814}{1{\cdot}9814^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}9814^2 + 0{\cdot}1262^2} = 0{\cdot}502 - j0{\cdot}032 \\ Y_{0l} &= 5{\cdot}0 + j0 \end{aligned}$$

$$\therefore\ Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l} = 7{\cdot}067 - j0{\cdot}1904$$

$$\begin{aligned} \frac{1}{Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}} &= \frac{7{\cdot}067}{7{\cdot}067^2 + 0{\cdot}1904^2} + j\frac{0{\cdot}1904}{7{\cdot}067^2 + 0{\cdot}1904^2} \\ &= 0{\cdot}1412 + j0{\cdot}0038 \end{aligned}$$

Since the no-load phase E.M.Fs. are given by the expressions

$$E_{\mathrm{I}} = 240(1 + j0), \qquad E_{\mathrm{II}} = 240(-0{\cdot}5 - j0{\cdot}866) = -120 - j208,$$
$$E_{\mathrm{III}} = 240(-0{\cdot}5 + j0{\cdot}866) = -120 + 208,$$

the fictitious currents I_1', I_2', I_3', are

$$\begin{aligned} I_1' &= E_{\mathrm{I}} Y_{1t} = 240(0{\cdot}872 - j0{\cdot}0973) = 209{\cdot}3 - j23{\cdot}4 \\ I_2' &= E_{\mathrm{II}} Y_{2t} = (-120 - j208)(0{\cdot}693 - j0{\cdot}061) = -95{\cdot}9 - j136{\cdot}9 \\ I_3' &= E_{\mathrm{III}} Y_{3t} = (-120 + j208)(0{\cdot}502 - j0{\cdot}032) = -53{\cdot}5 + j108{\cdot}3 \end{aligned}$$

Whence $I_0' = I_1' + I_2' + I_3' = 59{\cdot}9 - j52$

The fictitious currents I_1'', I_2'', I_3'', are now obtained thus

$$\begin{aligned} I_1'' &= I_0' Y_{1t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}) \\ &= (59{\cdot}9 - j52)(0{\cdot}1412 + j0{\cdot}0038)(0{\cdot}872 - j0{\cdot}0973) \\ &= 6{\cdot}867 - j7{\cdot}053 \\ I_2'' &= I_0' Y_{2t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}) \\ &= (59{\cdot}9 - j52)(0{\cdot}1412 + j0{\cdot}0038)(0{\cdot}693 - j0{\cdot}061) \\ &= 5{\cdot}565 - j5{\cdot}458 \\ I_3'' &= I_0' Y_{3t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}) \\ &= (59{\cdot}9 - j52)(0{\cdot}1412 + j0{\cdot}0038)(0{\cdot}502 - j0{\cdot}032) \end{aligned}$$

Hence $\dot{I}_1 = \dot{I}_1' - \dot{I}_1'' = \mathbf{209{\cdot}3 - j23{\cdot}4 - (6{\cdot}87 - j7{\cdot}05) = 202{\cdot}4 - j16{\cdot}35}$

$$I_1 = \sqrt{(202{\cdot}4^2 + 16{\cdot}35^2)} = 203 \text{ A.}$$

$$\dot{I}_2 = \dot{I}_2' - \dot{I}_2'' = \mathbf{-95{\cdot}9 - j136{\cdot}9 - (5{\cdot}56 - j5{\cdot}46) = -101{\cdot}5 - j131{\cdot}4}$$

$$I_2 = \sqrt{(101{\cdot}5^2 + 131{\cdot}4^2)} = 166 \text{ A.}$$

$$\dot{I}_3 = \dot{I}_3' - \dot{I}_3'' = \mathbf{-53{\cdot}5 + j108{\cdot}3 - (4{\cdot}11 - j3{\cdot}85) = -57{\cdot}6 + j112{\cdot}1}$$

$$I_3 = \sqrt{(57{\cdot}6^2 + 112{\cdot}1^2)} = 126 \text{ A.}$$

$$\dot{I}_0 = -(\dot{I}_1 + \dot{I}_2 + \dot{I}_3) = -\ [(\mathbf{202{\cdot}4 - j16{\cdot}35}) + (\mathbf{-101{\cdot}5 - j131{\cdot}4}) + (\mathbf{-57{\cdot}6 + j112{\cdot}1})] = \mathbf{-43{\cdot}5 + j35{\cdot}6}$$

$$I_0 = \sqrt{(43{\cdot}5^2 + 35{\cdot}6^2)} = 56{\cdot}2 \text{ A.}$$

Phase difference between I_1 and $E_1 = \tan^{-1} - 16{\cdot}35/202{\cdot}4 = -5°$

„ „ „ I_2 „ $E_1 = \tan^{-1} - 131{\cdot}4/- 101{\cdot}5 = -(180 - 52{\cdot}3)° = -127{\cdot}7°$

„ „ „ I_3 „ $E_1 = \tan^{-1} 112{\cdot}1/- 57{\cdot}6 = -(180 + 62{\cdot}8)° = -242{\cdot}8°$

„ „ „ I_0 „ $E_1 = \tan^{-1} 35{\cdot}6/- 43{\cdot}5 = -(180 + 39{\cdot}3)° = -219{\cdot}3°$

The pressures across the branches of the load are in phase with the respective currents, and their magnitudes are

$$V_1 = 1{\cdot}0 \times I_1 = 203 \text{ V.}$$
$$V_2 = 1{\cdot}3 \times I_2 = 215{\cdot}8 \text{ V.}$$
$$V_3 = 1{\cdot}85 \times I_3 = 233 \text{ V.}$$

The pressure drop in the neutral wire is

$$V_0 = 0{\cdot}2 \times I_0 = 11{\cdot}2 \text{ V.}$$

As a check on the above calculations we may calculate the potential difference between the neutral points by means of equation (211), thus

$$\dot{V}_0 = \dot{I}_0'/(\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t} + \dot{Y}_{0l}) = (\mathbf{59{\cdot}9 - j52})\ (\mathbf{0{\cdot}1412 + j0{\cdot}0038}) = \mathbf{8{\cdot}58 - j7{\cdot}06}$$

$$V_0 = \sqrt{(8{\cdot}58^2 + 7{\cdot}06^2)} = 11{\cdot}12 \text{ V}$$

Phase difference between E_1 and $V_0 = \tan^{-1} - 7{\cdot}06/8{\cdot}58 = -39{\cdot}5°$

These values are sufficiently close to the previous values for all practical purposes.

It will be of interest to calculate the pressure drop in the principal line wires and the terminal pressure at the generator.

Pressure drop in the "outer" line wires

$$V_{1l} = 0{\cdot}1 \times I_1 = 20{\cdot}3 \text{ V.}$$
$$V_{2l} = 0{\cdot}1 \times I_2 = 16{\cdot}6 \text{ V.}$$
$$V_{3l} = 0{\cdot}1 \times I_3 = 12{\cdot}6 \text{ V.}$$

The pressures between the terminals of the generator and the neutral point of the load are

$$V_{1g-0} = 203 + 20{\cdot}3 = 223{\cdot}3 \text{ V.}$$
$$V_{2g-0} = 215{\cdot}8 + 16{\cdot}6 = 232{\cdot}4 \text{ V}$$
$$V_{3g-0} = 233 + 12{\cdot}6 = 245{\cdot}6 \text{ V.}$$

The generator terminal pressures per phase (i.e. the pressure between the terminals of the generator and the neutral point of the generator) are

$$\dot{V}_{\mathrm{I}} = \dot{E} - \dot{I}_1\dot{Z}_g = (240 + j0) - (202{\cdot}4 - j16{\cdot}35)\ (0{\cdot}0314 + j0{\cdot}1262)$$
$$= 231{\cdot}6 - j25{\cdot}4$$
$$V_{\mathrm{I}} = \sqrt{(231{\cdot}6^2 + 25{\cdot}4^2)} = 232{\cdot}9 \text{ V.}$$

$$\dot{V}_{\mathrm{II}} = \dot{E}_{\mathrm{II}} - \dot{I}_2\dot{Z}_g = (-120 - j208) - (-101{\cdot}5 - j131{\cdot}4)(0{\cdot}0314 + j0{\cdot}1262)$$
$$= -133{\cdot}4 - j191$$
$$V_{\mathrm{II}} = \sqrt{(133{\cdot}4^2 + 191^2)} = 233 \text{ V.}$$

$$\dot{V}_{\mathrm{III}} = \dot{E}_{\mathrm{III}} - \dot{I}_3\dot{Z}_g = (-120 + j208) - (-57{\cdot}6 + j112{\cdot}1)(0{\cdot}0314 + j0{\cdot}1262)$$
$$= -104 + j211{\cdot}7$$
$$V_{\mathrm{III}} = \sqrt{(104^2 + 211{\cdot}7^2)} = 235{\cdot}9 \text{ V.}$$

The line voltages at the terminals of the generator are

$$\dot{V}_{g1\text{-}2} = \dot{V}_{\mathrm{I}} - \dot{V}_{\mathrm{II}} = 231{\cdot}6 - j25{\cdot}4 - (-133{\cdot}4 - j191) = 365 + j165{\cdot}6$$
$$V_{g1\text{-}2} = \sqrt{(365^2 + 165{\cdot}6^2)} = 401 \text{ V.}$$

$$\dot{V}_{g2\text{-}3} = \dot{V}_{\mathrm{II}} - \dot{V}_{\mathrm{III}} = -133{\cdot}4 - j191) - (-104 + j211{\cdot}7) = -29{\cdot}4 - j402{\cdot}7$$
$$V_{g2\text{-}3} = \sqrt{(29{\cdot}4^2 + 402{\cdot}7^2)} = 403{\cdot}5 \text{ V.}$$

$$\dot{V}_{g3\text{-}1} = \dot{V}_{\mathrm{III}} - \dot{V}_{\mathrm{I}} = -104 + j211{\cdot}7 - (231{\cdot}6 - j25{\cdot}4 - = -335{\cdot}6 + j237$$
$$V_{g3\text{-}1} = \sqrt{(335{\cdot}6^2 + 237^2)} = 411 \text{ V.}$$

and the line voltages at the load are

$$\dot{V}_{1\text{-}2} = \dot{V}_1 - \dot{V}_2 = 202{\cdot}4 - j16{\cdot}4 - (-132 - j170{\cdot}8) = 334{\cdot}4 + j154{\cdot}4$$
$$V_{1\text{-}2} = \sqrt{(334{\cdot}4^2 + 154{\cdot}4^2)} = 368{\cdot}5 \text{ V.}$$
$$\dot{V}_{2\text{-}3} = \dot{V}_2 - \dot{V}_3 = -132 - j170{\cdot}8 - (-106{\cdot}5 + j207{\cdot}3) = -25{\cdot}5 - j378$$
$$V_{2\text{-}3} = \sqrt{(25{\cdot}5^2 + 378^2)} = 379 \text{ V.}$$
$$\dot{V}_{3\text{-}1} = \dot{V}_3 - \dot{V}_1 = -106{\cdot}5 + j207{\cdot}3 - (202{\cdot}4 - j16{\cdot}4) = -308{\cdot}9 + j223{\cdot}7$$
$$V_{3\text{-}1} = \sqrt{(308{\cdot}9^2 + 223{\cdot}7^2)} = 381{\cdot}5 \text{ V.}$$

[Note. In this example the resistances of the line wires have been chosen much higher than the values which would be adopted in practice.]

CHAPTER XX

APPLICATION OF KIRCHHOFF'S LAWS TO THREE-PHASE CIRCUITS

KIRCHHOFF's laws enable analytical solutions to be obtained for any electric network, whether it is supplied from a direct-current system or an alternating-current (single-phase or polyphase) system. The laws may be stated thus—

1. At every junction of two or more branches of an electric circuit the algebraic sum of all currents is zero, i.e. the sum of the currents flowing towards the junction must equal the sum of the currents flowing away from the junction.

For example, if at a junction of five conductors in a direct-current network, the currents I_1, I_2, I_3, in three of the conductors flow towards the junction and the currents I_4, I_5, in the other two conductors flow away from the junction, then

$$I_1 + I_2 + I_3 - I_4 - I_5 = 0,$$
$$\text{or } I_1 + I_2 + I_3 = I_4 + I_5.$$

2. In every closed electric circuit carrying a current the algebraic sum of all E.M.Fs. taken in order round the circuit is zero, i.e. the E.M.Fs. due to the current—viz. the induced E.M.Fs. due to self, or mutual, inductance and the potential drop due to resistance, etc.—must balance the impressed E.M.F. (or the E.M.F. generated, or induced, in the circuit by external means).

For example, in a direct-current circuit of resistance, R,

$$\Sigma (V - E - IR) = 0, \text{ or } V = \Sigma (E + IR),$$

where V denotes the impressed E.M.F.; E, the internal generated or induced E.M.F.; and I, the current.

These laws may be called the generalized laws of electric circuits, as by their application the currents and voltages in any network can be determined. But with complicated networks the equations connecting the currents and voltages may become so numerous that much tedious algebraic work is involved in their solution.

Conditions to be Observed in the Application of Kirchhoff's Laws to A.C. Circuits. In applying these laws to A.C. circuits we may consider either the instantaneous values of the E.M.Fs. and currents or the R.M.S. values of these quantities. When dealing with instantaneous values only the magnitudes and directions of the E.M.Fs. and currents need be considered. The calculations, therefore, are carried out by ordinary algebraic methods.

But when R.M.S. values are employed the relative phase differences of the currents and E.M.Fs. must be considered as well as their magnitudes. Hence, in this case, the calculations must be carried out by the symbolic method, as we are dealing with complex quantities.

The *method of calculation* in the case of complex circuits or networks is as follows—

The complex circuit is reduced to a number of closed circuits, or meshes, in each of which a current is assumed to circulate in a definite direction independently of the currents in the adjacent meshes. The E.M.Fs. in each mesh are then calculated and are equated to zero in accordance with Kirchhoff's second law, the number of equations so obtained being equal to the number of meshes. The values of the fictitious circulating currents are then obtained by solving these equations, and when these currents are known, the currents in, and the potential difference across, all parts of the network may easily be obtained.

The process of obtaining the general equations for the meshes is quite simple, but the reduction of these equations to obtain the equations for the several currents is usually complicated when the number of separate currents exceeds three.

In order to illustrate the method of procedure, we shall consider a number of cases.

Calculation of Currents in a Three-phase, Three-wire System with Star-connected Generator and Load. Let the no-load phase E.M.Fs.

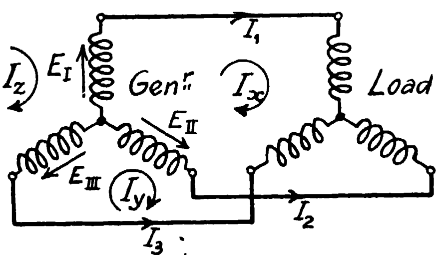


FIG. 298. CIRCUIT DIAGRAM FOR STAR/STAR THREE-PHASE SYSTEM

of the generator be denoted by E_{I}, E_{II}, E_{III}; the line currents by I_1, I_2, I_3; and the total impedances per phase (i.e. the sum of the impedances per phase of the generator, load, and connecting line wire) by Z_{1t}, Z_{2t}, Z_{3t}. Then the system is equivalent to three meshes (Fig. 298), in which the externally-produced E.M.Fs. are $\dot{E}_{I} - \dot{E}_{II}$, $\dot{E}_{II} - \dot{E}_{III}$, $\dot{E}_{III} - \dot{E}_{I}$; and the fictitious circulating currents are I_x, I_y, I_z.

Hence the general E.M.F. equations for the meshes are—

$$E_{\mathrm{I}} - (I_x - I_z)Z_{1t} - (I_x - I_y)Z_{2t} - E_{\mathrm{II}} = 0$$
$$E_{\mathrm{II}} - (I_y - I_x)Z_{2t} - (I_y - I_z)Z_{3t} - E_{\mathrm{III}} = 0$$
$$E_{\mathrm{III}} - (I_z - I_y)Z_{3t} - (I_z - I_x)Z_{1t} - E_{\mathrm{I}} = 0$$

Since $I_x - I_z = I_1$, $I_y - I_x = I_2$, $I_z - I_y = I_3$, the preceding equations may be written in the form

$$I_1 Z_{1t} - I_2 Z_{2t} = E_{\mathrm{I}} - E_{\mathrm{II}} \quad . \quad . \quad . \quad (\alpha)$$
$$I_2 Z_{2t} - I_3 Z_{3t} = E_{\mathrm{II}} - E_{\mathrm{III}} \quad . \quad . \quad . \quad (\beta)$$
$$-I_1 Z_{1t} + I_3 Z_{3t} = E_{\mathrm{III}} - E_{\mathrm{I}} \quad . \quad . \quad . \quad (\gamma)$$

Applying Kirchhoff's first law to the neutral point of the generator we have

$$I_1 + I_2 + I_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

The expressions for the line currents may be readily obtained from these four equations. For example, to obtain I_1, eliminate I_2 and I_3 from equations (α), (γ), (δ), thus

$$I_1 Z_{1t}/Z_{2t} - I_2 = (E_{\mathrm{I}} - E_{\mathrm{II}})/Z_{2t} \quad . \quad . \quad . \quad (\alpha')$$
$$I_1 Z_{1t}/Z_{3t} - I_3 = (E_{\mathrm{I}} - E_{\mathrm{III}})/Z_{3t} \quad . \quad . \quad . \quad (\gamma')$$
$$I_1 + I_2 + I_3 = 0 \quad . \quad . \quad . \quad . \quad (\delta')$$

Adding, we have

$$I_1\left(1 + \frac{Z_{1t}}{Z_{2t}} + \frac{Z_{1t}}{Z_{3t}}\right) = \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{2t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t}}$$

or

$$I_1 Z_{1t}\left(\frac{1}{Z_{1t}} + \frac{1}{Z_{2t}} + \frac{1}{Z_{3t}}\right) = \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{2t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t}}$$

Whence

$$I_1 = \frac{Y_{1t}Y_{2t}(E_{\mathrm{I}} - E_{\mathrm{II}})}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{1t}Y_{3t}(E_{\mathrm{I}} - E_{\mathrm{III}})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (212)$$

$$= \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{1t} + Z_{2t} + Z_{1t}Z_{2t}/Z_{3t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t} + Z_{1t} + Z_{3t}Z_{1t}/Z_{2t}} \quad . \, (212a)$$

Similarly,

$$I_2 = \frac{Y_{2t}Y_{3t}(E_{\mathrm{II}} - E_{\mathrm{III}})}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{2t}Y_{1t}(E_{\mathrm{II}} - E_{\mathrm{I}})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (213)$$

$$= \frac{E_{\mathrm{II}} - E_{\mathrm{III}}}{Z_{2t} + Z_{3t} + Z_{2t}Z_{3t}/Z_{1t}} + \frac{E_{\mathrm{II}} - E_{\mathrm{I}}}{Z_{1t} + Z_{2t} + Z_{1t}Z_{2t}/Z_{3t}} \quad . \, (213a)$$

and

$$I_3 = \frac{Y_{3t}Y_{1t}(E_{III} - E_I)}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{3t}Y_{2t}(E_{III} - E_{II})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (214)$$

$$= \frac{E_{III} - E_I}{Z_{3t} + Z_{1t} + Z_{3t}Z_{1t}/Z_{2t}} + \frac{E_{III} - E_{II}}{Z_{2t} + Z_{3t} + Z_{2t}Z_{3t}/Z_{1t}} \quad . \ (214a)$$

These are the general equations for the line currents.

With symmetrical systems, however, the equations can be expressed in simpler form. Thus, if the phase rotation is counterclockwise and the magnitude of the no-load E.M.F. of each phase is denoted by E, then

$$E_I = EJ^0, \quad E_{II} = EJ^{-120/90}, \qquad E_{III} = EJ^{-240/90}$$

$$E_I - E_{II} = E(J^0 - J^{-120/90}) = E(1 - J^{-120/90})$$

$$E_I - E_{III} = E(J^0 - J^{-240/90}) = E(1 - J^{-240/90})$$

$$E_{II} - E_{III} = E(J^{-120/90} - J^{-240/90})$$

When these expressions are substituted in equation (212) we have, upon re-arrangement,

$$I_1 = \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{Y_{2t}(1 - J^{-120/90}) + Y_{3t}(1 - J^{-240/90})\}$$

$$= \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E(Y_{2t} + Y_{3t} - Y_{2t}J^{-120/90} - Y_{3t}J^{-240/90})$$

Introducing the quantity $Y_{1t}J^0$ into the right-hand side, and noting that $J^0 \equiv 1$, we have

$$I_1 = \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E(Y_{1t} + Y_{2t} + Y_{3t} - (Y_{1t}J^0 + Y_{2t}J^{-120/90}) + Y_{3t}J^{-240/90})$$

$$= EY_{1t}\left(1 - \frac{Y_{1t}J^0 + Y_{2t}J^{-120/90} + Y_{3t}J^{-240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad . \ (212b)$$

Similarly when substitutions are made in equation (213) for E_I, E_{II}, and E_{III}, we have, upon re-arrangement,

$$I_2 = \frac{Y_{2t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{Y_{3t}(J^{-120/90} - J^{-240/90}) + Y_{1t}(J^{\,120/90} - J^0)\}$$

Introducing the quantity $Y_{2t}J^{\cdot 120/90}$,

$$I_2 = \frac{Y_{2t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{Y_{3t}(J^{\cdot 120/90} - J^{\cdot 240/90}) + Y_{2t}(J^{\cdot 120/90} - J^{\cdot 120/90}) + Y_{1t}(J^{\cdot 120/90} - J^0)\}$$

$$= \frac{Y_{2t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{1J^{\cdot 120/90}(Y_{1t} + Y_{2t} + Y_{3t}) - (Y_{1t}J^0 + Y_{2t}J^{\cdot 120/90} + Y_{3t}J^{\cdot 240/90})\}$$

$$= EY_{2t}\left(1J^{\cdot 120/90} - \frac{Y_{1t}J^0 + Y_{2t}J^{\cdot 120/90} + Y_{3t}J^{\cdot 240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad (213b)$$

Similarly, equation (89) finally reduces to

$$I_3 = EY_{3t}\left(1J^{\cdot 240/90} - \frac{Y_{1t}J^0 + Y_{2t}J^{\cdot 120/90} + Y_{3t}J^{\cdot 240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) . \quad (214b)$$

These equations may be readily evaluated when Y_{1t}, Y_{2t}, Y_{3t} are known, since for counter-clockwise phase rotation $J^0 = \mathbf{1} + j\mathbf{0} = 1$; $J^{\cdot 120/90} = \cos - 120° - j \sin - 120° = -\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}$; $J^{\cdot 240/90} = \cos - 240° - j \sin 240° = -\mathbf{0{\cdot}5} + j\,\mathbf{0{\cdot}866}$.

Whence

$$I_1 = EY_{1t}\left(1 \quad \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad . \quad (212c)$$

$$I_2 = EY_{2t}\left(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866} - \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) . \quad (213c)$$

$$I_3 = EY_{3t}\left(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866} - \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) . \quad (214c)$$

The difference of potential (V_0) between the neutral points of generator and load is given by

$$V_0 = E_{\mathrm{I}} - I_1 Z_{1t} = E_{\mathrm{II}} - I_2 Z_{2t} = E_{\mathrm{III}} - I_3 Z_{3t}$$

Hence,

$$V_0 = EJ^0 - E\left(1J^0 - \frac{Y_{1t}J^0 + Y_{2t}J^{\cdot 120/90} + Y_{3t}J^{\cdot 240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right)$$

$$= E\left(\frac{Y_{1t}J^0 + Y_{2t}J^{\cdot 120/90} + Y_{3t}J^{\cdot 240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right). \quad . \quad . \quad (215)$$

$$= E\left(\frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) (215a)$$

With symmetrical systems and clockwise phase rotation, we have

$$E_{I} = EJ^{0}, \qquad E_{II} = EJ^{120/90}, \qquad E_{III} = EJ^{240/90}$$
$$E_{I} - E_{II} = E(J^{0} - J^{120/90}) = E(1 - J^{120/90})$$
$$E_{I} - E_{III} = E(J^{0} - J^{240/90}) = E(1 - J^{240/90})$$
$$E_{II} - E_{III} = E(J^{120/90} - J^{240/90})$$

When these expressions are substituted in equations (212), (213), (214), we have, upon final re-arrangement and reduction,

$$I_1 = EY_{1t}\left(1 - \frac{Y_{1t}J^0 + Y_{2t}J^{120/90} + Y_{3t}J^{240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad . \quad . \quad (212d)$$

$$I_2 = EY_{2t}\left(1J^{120/90} - \frac{Y_{1t}J^0 + Y_{2t}J^{120/90} + Y_{3t}J^{240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad (213d)$$

$$I_3 = EY_{3t}\left(1J^{240/90} - \frac{Y_{1t}J^0 + Y_{2t}J^{120/90} + Y_{3t}J^{240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad (214d)$$

In the present case, $J^{120/90} = \cos 120° + j \sin 120° = -\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}$; $J^{240/90} = \cos 240° + j \sin 240° = -\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}$.
Whence

$$I_1 = EY_{1t}\left(1 - \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad . \quad (212e)$$

$$I_2 = EY_{2t}\left(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866} - \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad (213e)$$

$$I_3 = EY_{3t}\left(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866} - \frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad (214e)$$

and the potential difference between the neutral points of generator and load is given by

$$V_0 = E\left(\frac{Y_{1t} + Y_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + Y_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad (215b)$$

Calculation of Currents in a Star-connected Load Supplied from a Large Three-phase System. The system to which the load is connected is assumed to be so large that the line voltages are unaffected by any unbalancing of the loads.

The equations (212), (213), (214) deduced for the preceding case are therefore applicable to the present case if the line voltages are substituted for the differences between the phase E.M.Fs.

Thus

$$I_1 = \frac{Y_1}{Y_1 + Y_2 + Y_3}(Y_2V_{1-2} - Y_3V_{3-1}) \quad . \quad . \quad . \quad (216)$$

$$I_2 = \frac{Y_2}{Y_1 + Y_2 + Y_3}(Y_3V_{2-3} - Y_1V_{1-2}) \quad . \quad . \quad . \quad (217)$$

$$I_3 = \frac{Y_3}{Y_1 + Y_2 + Y_3}(Y_1V_{3-1} \quad Y_2V_{2-3}) \quad . \quad . \quad . \quad (218)$$

If we wish to calculate with impedances instead of admittances, then

$$I_1 = \frac{V_{1-2}}{Z_1 + Z_2 + Z_1Z_2/Z_3} - \frac{V_{3-1}}{Z_3 + Z_1 + Z_3Z_1/Z_2} \quad . \quad (216a)$$

$$I_2 = \frac{V_{2-3}}{Z_2 + Z_3 + Z_2Z_3/Z_1} - \frac{V_{1-2}}{Z_1 + Z_2 + Z_1Z_2/Z_3} \quad . \quad (217a)$$

$$I_3 = \frac{V_{3-1}}{Z_3 + Z_1 + Z_3Z_1/Z_2} - \frac{V_{2-3}}{Z_2 + Z_3 + Z_2Z_3/Z_1} \quad . \quad (218a)$$

With a symmetrical system and counter-clockwise phase rotation

$$V_{1-2} = VJ^0; \qquad V_{2-3} = VJ^{-120/90}; \qquad V_{3-1} = VJ^{-240/90}$$

Under these conditions, we have

$$I_1 = V\left\{\frac{Y_1}{Y_1 + Y_2 + Y_3}(Y_2J^0 - Y_3J^{-240/90})\right\} \quad . \quad . \quad (216b)$$

$$= V\left(\frac{Y_1}{Y_1 + Y_2 + Y_3}\{Y_2(\boldsymbol{1} + j\boldsymbol{0}) - Y_3(-\boldsymbol{0{\cdot}5} + j\boldsymbol{0{\cdot}866})\}\right)$$

$$. \quad . \quad (216c)$$

$$I_2 = V\left\{\frac{Y_2}{Y_1 + Y_2 + Y_3}(Y_3J^{120/90} - Y_1J^0)\right\} \quad . \quad . \quad (217b)$$

$$= V\left(\frac{Y_2}{Y_1 + Y_2 + Y_3}\{Y_3(-\boldsymbol{0{\cdot}5} - j\boldsymbol{0{\cdot}866}) - Y_1(\boldsymbol{1} + j\boldsymbol{0})\}\right)$$

$$. \quad . \quad (217c)$$

$$I_3 = V\left\{\frac{Y_3}{Y_1 + Y_2 + Y_3}(Y_1J^{-240/90} - Y_2J^{-120/90})\right\} \quad . \quad (218b)$$

$$= V\left(\frac{Y_3}{Y_1 + Y_2 + Y_3}\{Y_1(-\boldsymbol{0{\cdot}5} + j\boldsymbol{0{\cdot}866}) - Y_2(-\boldsymbol{0{\cdot}5} - j\boldsymbol{0{\cdot}866})\}\right) \quad . \quad (218c)$$

or, alternatively,

$$I_1 = V\left(\frac{\mathbf{1 + j0}}{Z_1 + Z_2 + Z_1Z_2/Z_3} - \frac{\mathbf{-0{\cdot}5 + j0{\cdot}866}}{Z_3 + Z_1 + Z_3Z_1/Z_2}\right). \quad (216d)$$

$$I_2 = V\left(\frac{\mathbf{-0{\cdot}5 - j0{\cdot}866}}{Z_2 + Z_3 + Z_2Z_3/Z_1} - \frac{\mathbf{1 + j0}}{Z_1 + Z_2 + Z_1Z_2/Z_3}\right). \quad (217d)$$

$$I_3 = V\left(\frac{\mathbf{-0{\cdot}5 + j0{\cdot}866}}{Z_3 + Z_1 + Z_3Z_1/Z_2} - \frac{\mathbf{-0{\cdot}5 - j0{\cdot}866}}{Z_2 + Z_3 + Z_2Z_3/Z_1}\right). \quad (218d)$$

Example. The diagram (Fig. 299*a*) represents a star-connected unbalanced load which is connected to a three-phase system having sinusoidal E.M.Fs.

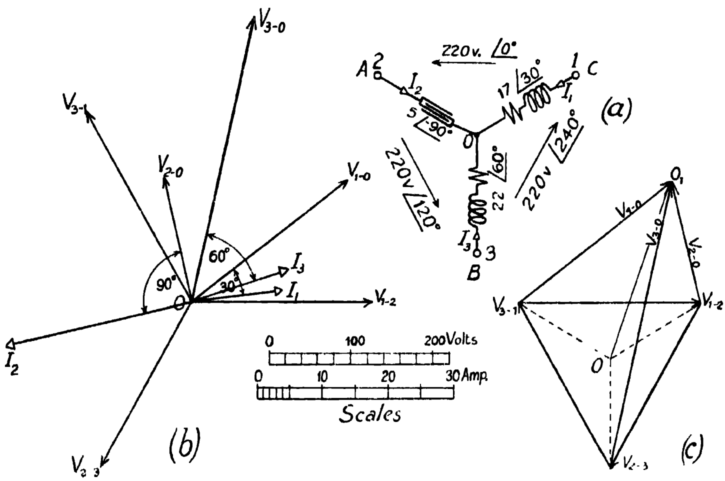


FIG. 299. VECTOR DIAGRAMS FOR WORKED EXAMPLE

the phase relations between the E.M.Fs. being indicated in the diagram. Determine the current in the condenser branch OA. (L. U.)

[NOTE. The reactance of the condenser branch OA is $5/-90^\circ$ ohms, and the impedances of the other branches, OB, OC, are $22/60^\circ$ ohms, and $17/30^\circ$ ohms respectively.]

Denoting the terminals C, A, B, of the load by the numerals 1, 2, 3, respectively, we have, for the impedances of the branches,

$$Z_1 = 17/30^\circ = 17(\cos 30^\circ + j\sin 30^\circ) = \mathbf{14{\cdot}74 + j8{\cdot}5}$$

$$Z_2 = 5/-90^\circ = 5(\cos -90^\circ + j\sin -90^\circ) = \mathbf{0 - j5}$$

$$Z_3 = 22/60^\circ = 22(\cos 60^\circ + j\sin 60^\circ) = \mathbf{11 + j19{\cdot}06}$$

Whence $Z_1Z_2/Z_3 = (17 \times 5/22)J^{(30-90-60)/90} = 3{\cdot}86J^{-120/90}$

$$= 3{\cdot}86(-\mathbf{0{\cdot}5} - \mathbf{0{\cdot}866})$$

$$= \mathbf{-1{\cdot}93 - j3{\cdot}35}$$

$$\dot{Z}_2\dot{Z}_3/\dot{Z}_1 = (5 \times 22/17)J^{(-90+60-30)/90} = 6{\cdot}47J^{-60/90}$$
$$= 6{\cdot}47\,(\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})$$
$$= \mathbf{3{\cdot}235} - j\mathbf{5{\cdot}6}$$

$$\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_1\dot{Z}_2/\dot{Z}_3 = \mathbf{12{\cdot}81} + j\mathbf{0{\cdot}15}$$
$$\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_2\dot{Z}_3/\dot{Z}_1 = \mathbf{14{\cdot}23} + j\mathbf{8{\cdot}46}$$

Hence, from equation (217*d*), we have

$$\dot{I}_2 = V\left(\frac{-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}}{\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_2\dot{Z}_3/\dot{Z}_1} - \frac{\mathbf{1} + j\mathbf{0}}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_1\dot{Z}_2/\dot{Z}_3}\right)$$
$$= 220\left(\frac{(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})\ (\mathbf{14{\cdot}23} - j\mathbf{8{\cdot}46})}{14{\cdot}23^2 + 8{\cdot}46^2} - \frac{\mathbf{12{\cdot}81} - j\mathbf{0{\cdot}15}}{12{\cdot}81^2 + 0{\cdot}15^2}\right)$$
$$= -\mathbf{28{\cdot}78} - j\mathbf{6{\cdot}3}$$

Whence $I_2 = \sqrt{(28{\cdot}78^2 + 6{\cdot}3^2)} = 29{\cdot}46$ A.

Phase difference between I_2 and potential difference (V_{1-2}) between lines 1 and 2 $= \tan^{-1} - 6{\cdot}3/-28{\cdot}78 = -(180 - 12{\cdot}4)^\circ = -167{\cdot}6^\circ$.

As an extension of the problem, it will be of interest to calculate the currents in the other branches and the potential difference across each branch, and to draw a vector diagram for the circuit.

First, the quantity $\dot{Z}_3 + \dot{Z}_1 + \dot{Z}_3\dot{Z}_1/\dot{Z}_2$ is evaluated, thus

$$\dot{Z}_3\dot{Z}_1/\dot{Z}_2 = (17 \times 22/5)\,J^{(30+60+90)/90} = 74{\cdot}8\,J^{180/90}$$
$$= 74{\cdot}8(-\mathbf{1} + j\mathbf{0})$$
$$\dot{Z}_3 + \dot{Z}_1 + \dot{Z}_3\dot{Z}_1/\dot{Z}_2 = -\mathbf{49} + j\mathbf{27{\cdot}55}$$

From equation (216*d*) we have, for the current in the inductive branch *OC*,

$$\dot{I}_1 = V\left(\frac{\mathbf{1} + j\mathbf{0}}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_1\dot{Z}_2/\dot{Z}_3} - \frac{-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}}{\dot{Z}_3 + \dot{Z}_1 + \dot{Z}_3\dot{Z}_1/\dot{Z}_2}\right)$$
$$= 220\left(\frac{\mathbf{12{\cdot}81} - j\mathbf{0{\cdot}15}}{12{\cdot}81^2 + 0{\cdot}15^2} - \frac{(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})\ (-\mathbf{49} + j\mathbf{27{\cdot}55})}{49^2 + 27{\cdot}55^2}\right)$$
$$= \mathbf{13{\cdot}85} + j\mathbf{1{\cdot}794}$$

Whence

$$I_1 = \sqrt{(13{\cdot}85^2 + 1{\cdot}794^2)} = 13{\cdot}96 \text{ A.}$$

Phase difference between I_1 and $V_{1\text{-}2}$ is $\tan^{-1} 1{\cdot}794/13{\cdot}85 = 7{\cdot}4^\circ$

Also from equation (218*d*) the current in the inductive branch *OB* is given by

$$\dot{I}_3 = V\left(\frac{-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}}{\dot{Z}_3 + \dot{Z}_1 + \dot{Z}_3\dot{Z}_1/\dot{Z}_2} - \frac{-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}}{\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_2\dot{Z}_3/\dot{Z}_1}\right)$$
$$= 220\left(\frac{(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866})\ (-\mathbf{49} - j\mathbf{27{\cdot}55})}{49^2 + 27{\cdot}55^2} - \frac{(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})\ (\mathbf{14{\cdot}23} - j\mathbf{8{\cdot}46})}{14{\cdot}23^2 + 8{\cdot}46^2}\right)$$
$$= \mathbf{14{\cdot}94} + j\mathbf{4{\cdot}5}$$

Whence

$I_3 = \sqrt{(14{\cdot}94^2 + 4{\cdot}5^2)} = 15{\cdot}6$ A.

Phase difference between I_3 and $V_{1\text{-}2}$ is $\tan^{-1} 4{\cdot}5/14{\cdot}94 = 16{\cdot}8°$.

The potential difference across the branch OA is given by

$\dot{V}_{2\text{-}0} = \dot{I}_2\dot{Z}_2 = (-\mathbf{28{\cdot}78} - j\mathbf{6{\cdot}3})\ (\mathbf{0} - j\mathbf{5}) = -\mathbf{31{\cdot}5} + j\mathbf{144}$

Whence

$V_{2\text{-}0} = \sqrt{(31{\cdot}5^2 + 144)} = 147{\cdot}6$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{2\text{-}0}$ is

$$\tan^{-1} 144/-31{\cdot}5 = (180 - 77{\cdot}6)° = 102{\cdot}4°$$

The potential difference across the branch OC is given by

$\dot{V}_{1\text{-}0} = \dot{I}_1\dot{Z}_1 = (\mathbf{13{\cdot}85} + j\mathbf{1{\cdot}794})\ (\mathbf{14{\cdot}74} + j\mathbf{8{\cdot}5}) = \mathbf{188{\cdot}8} + j\mathbf{144}$

Whence

$V_{1\text{-}0} = \sqrt{(188{\cdot}8^2 + 144^2)} = 237$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{1\text{-}0}$ is $\tan^{-1} 144/188{\cdot}8 = 37{\cdot}4°$.

The potential difference across the branch OB is given by

$\dot{V}_{3\text{-}0} = \dot{I}_3\dot{Z}_3 = (\mathbf{14{\cdot}94} + j\mathbf{4{\cdot}5})\ (\mathbf{11} + j\mathbf{19{\cdot}06}) = \mathbf{78{\cdot}7} + j\mathbf{334{\cdot}5}$

Whence

$V_{3\text{-}0} = \sqrt{(78{\cdot}7^2 + 334{\cdot}5^2)} = 344$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{3\text{-}0}$ is $\tan^{-1} 334{\cdot}5/78{\cdot}7 = 76{\cdot}8°$.

Check calculations for these potential differences are as follow—

$V_{2\text{-}0} = I_2Z_2 = 29{\cdot}46 \times 5 = 147{\cdot}3$ V.

$V_{1\text{-}0} = I_1Z_1 = 13{\cdot}96 \times 17 = 237$ V.

$V_{3\text{-}0} = I_3Z_3 = 15{\cdot}6 \times 22 = 344$ V.

$$\dot{V}_{1\text{-}0} - \dot{V}_{2\text{-}0}(= \dot{V}_{1\text{-}2}) = (\mathbf{188{\cdot}8} + j\mathbf{144}) - (-\mathbf{31{\cdot}5} + j\mathbf{144})$$
$$= \mathbf{220{\cdot}3} + j\mathbf{0}$$

$$\dot{V}_{2\text{-}0} - \dot{V}_{3\text{-}0}(= \dot{V}_{2\text{-}3}) = (-\mathbf{31{\cdot}5} + j\mathbf{144}) - (\mathbf{78{\cdot}7} + j\mathbf{334{\cdot}5})$$
$$= -\mathbf{110{\cdot}2} - j\mathbf{190{\cdot}5}$$

$$\dot{V}_{3\text{-}0} - \dot{V}_{1\text{-}0}(= \dot{V}_{3\text{-}1}) = (\mathbf{78{\cdot}7} + \mathbf{334{\cdot}5}) - (\mathbf{188{\cdot}8} + j\,\mathbf{144})$$
$$= -\mathbf{110{\cdot}1} + j\mathbf{190{\cdot}5}$$

The vector diagram for the load circuit is given in Fig. 299*b*, in which the line voltages are represented by the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\text{-}1}$, the voltages across the branches of the load by $OV_{1\text{-}0}$, $OV_{2\text{-}0}$, $OV_{3\text{-}0}$, and the line currents by OI_1, OI_2, OI_3.

If the line-voltage vectors are drawn in the form of a triangle, as in Fig. 299*c*, and the vectors representing the voltages across

the branches of the load are so drawn from the corners of this triangle as to meet in a common point, O_1, then this point represents the potential of the neutral point of the load. Moreover, since the point, O (which is the centre of gravity of the line-voltage vector triangle) represents the potential of the neutral point of the supply system (which, according to the circuit diagram (Fig. 299*a*), is symmetrical), the potential difference between the two neutral points is given by

$$\dot{V}_0 = \dot{V}_{1\cdot0} - \dot{V}'_{1\cdot0} = (188\cdot8 + j144) - (0\cdot866 - j0\cdot5)220/\sqrt{3}$$
$$= 78\cdot7 + j207\cdot5$$

where $\dot{V}'_{1\cdot0}[= (0\cdot866 - j0\cdot5)220/\sqrt{3}]$ denotes the potential difference between line 1 and the neutral point of the supply system.

Whence

$$V_0 = \sqrt{(78\cdot7^2 + 207\cdot5^2)} = 221\cdot5 \text{ V}.$$

Calculation of Currents in Three-phase, Three-wire System with Delta-connected Generator and Star-connected Load. Let the no-

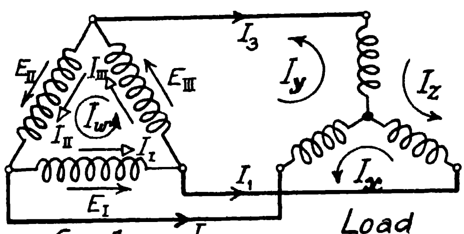


FIG. 300. CIRCUIT DIAGRAM FOR DELTA/STAR THREE-PHASE SYSTEM

load phase E.M.Fs. of the generator be denoted by E_{I}, E_{II}, E_{III}; the phase currents by I_{I}, I_{II}, I_{III}; the corresponding line currents by I_1, I_2, I_3; and the impedances of the load, connecting line wires and generator, by Z_1, Z_{1l}, Z_{1g}, etc. The delta-star system is then equivalent to four meshes (Fig. 300), of which one mesh is formed by the delta-connected generator, and three meshes by the inter-connected phases of the load, the line wires, and the phases of the generator. Let the fictitious circulating currents in these meshes be denoted by I_w, I_x, I_y, I_z.

Then the E.M.F. equations for the meshes are

$$\dot{E}_{\text{I}} + \dot{E}_{\text{II}} + \dot{E}_{\text{III}} - (\dot{I}_w + \dot{I}_x)\dot{Z}_{1g} - (\dot{I}_w + \dot{I}_y)\dot{Z}_{2g} - (\dot{I}_w + \dot{I}_z)\dot{Z}_{3g}$$
$$= 0 \quad . \quad . \quad . \quad . \quad (\alpha)$$

$$\dot{E}_{\text{I}} - (\dot{I}_w + \dot{I}_x)\dot{Z}_{1g} - (\dot{I}_x - \dot{I}_z)(\dot{Z}_1 + \dot{Z}_{1l}) - (\dot{I}_x - \dot{I}_y)(\dot{Z}_2 + \dot{Z}_{2l}) = 0 \quad . \quad . \quad . \quad . \quad (\beta)$$

$$\dot{E}_{\text{II}} - (\dot{I}_w + \dot{I}_y)\dot{Z}_{2g} - (\dot{I}_y - \dot{I}_x)(\dot{Z}_2 + \dot{Z}_{2l}) - (\dot{I}_y - \dot{I}_z)(\dot{Z}_3 + \dot{Z}_{3l}) = 0 \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$\dot{E}_{\text{III}} - (\dot{I}_w + \dot{I}_z)\dot{Z}_{3g} - (\dot{I}_z - \dot{I}_y)(\dot{Z}_3 + \dot{Z}_{3l}) - (\dot{I}_z - \dot{I}_x)(\dot{Z}_1 + \dot{Z}_{1l}) = 0 \quad . \quad . \quad . \quad . \quad (\delta)$$

Now $\dot{I}_w + \dot{I}_x = \dot{I}_{\text{I}}$; $\dot{I}_w + \dot{I}_y = \dot{I}_{\text{II}}$; $\dot{I}_w + \dot{I}_z = \dot{I}_{\text{III}}$;
$\dot{I}_x - \dot{I}_z = \dot{I}_1 = \dot{I}_{\text{I}} - \dot{I}_{\text{III}}$; $\dot{I}_y - \dot{I}_x = \dot{I}_2 = \dot{I}_{\text{II}} - \dot{I}_{\text{I}}$;
$\dot{I}_z - \dot{I}_y = \dot{I}_3 = \dot{I}_{\text{III}} - \dot{I}_{\text{I}}$.

Substituting these values in the above equations, we obtain a set of equations containing the phase currents, I_{I}, I_{II}, I_{III}, as the unknown quantities. Thus

$$\dot{E}_{\text{I}} + \dot{E}_{\text{II}} + \dot{E}_{\text{III}} - \dot{I}_{\text{I}}\dot{Z}_{1g} - \dot{I}_{\text{II}}\dot{Z}_{2g} - \dot{I}_{\text{III}}\dot{Z}_{3g} = 0 \quad . \quad . \quad (\alpha')$$

$$\dot{E}_{\text{I}} - \dot{I}_{\text{I}}(\dot{Z}_{1g} + \dot{Z}_1 + \dot{Z}_{1l} + \dot{Z}_2 + \dot{Z}_{2l}) + \dot{I}_{\text{II}}(\dot{Z}_2 + \dot{Z}_{2l}) = 0 \quad . \quad (\beta')$$

$$\dot{E}_{\text{II}} + \dot{I}_{\text{I}}(\dot{Z}_2 + \dot{Z}_{2l}) - \dot{I}_{\text{II}}(\dot{Z}_{2g} + \dot{Z}_2 + \dot{Z}_{2l} + \dot{Z}_3 + \dot{Z}_{3l}) + \dot{I}_{\text{III}}(\dot{Z}_3 + \dot{Z}_{3l}) = 0. \quad . \quad (\gamma')$$

$$\dot{E}_{\text{III}} + \dot{I}_{\text{I}}(\dot{Z}_1 + \dot{Z}_{1l}) + \dot{I}_{\text{II}}(\dot{Z}_3 + \dot{Z}_{3l}) - \dot{I}_{\text{III}}(\dot{Z}_{3g} + \dot{Z}_3 + \dot{Z}_{3l} + \dot{Z}_1 + \dot{Z}_{1l}) = 0 \, . \quad (\delta')$$

The solution to these equations is easily obtained by means of determinants.

Note to the Solution of Simultaneous Equations by Determinants. If we have three simultaneous equations, each of which contains three unknown quantities, x, y, z, and if the equations are expressed in the form

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

the solution is given by

$$x = -(D_1/D); \qquad y = D_2/D; \qquad z = -(D_3/D),$$

where D_1, D_2, D_3, D denote the following determinants of the third order—

$$D_1 = \begin{vmatrix} d_1b_1c_1 \\ d_2b_2c_2 \\ d_3b_3c_3 \end{vmatrix} \qquad D_2 = \begin{vmatrix} d_1a_1c_1 \\ d_2a_2c_2 \\ d_3a_3c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} d_1a_1b_1 \\ d_2a_2b_2 \\ d_3a_3b_3 \end{vmatrix} \qquad D = \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} \neq 0$$

Now
$$D_1 = d_1(b_2c_3 - b_3c_2) + b_1(c_2d_3 - c_3d_2) + c_1(d_2b_3 - d_3b_2)$$
$$D_2 = d_1(a_2c_3 - a_3c_2) + a_1(c_2d_3 - c_3d_2) + c_1(d_2a_3 - d_3a_2)$$
$$D_3 = d_1(a_2b_3 - a_3b_2) + a_1(b_2d_3 - b_3d_2) + b_1(d_2a_3 - d_3a_2)$$
$$D = a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

Alternatively, a solution may be obtained by ordinary algebra by substituting for, say, I_I in terms of I_{II} and I_{III}, in two of the equations and solving these, as ordinary simultaneous equations. for I_{II} and I_{III}. I_I is then obtained by substituting for I_{II} and I_{III} in one of the original equations. This process, however, usually involves more labour than the solution by determinants.

Substituting Z_{1t}', Z_{2t}', Z_{3t}', for the quantities $(Z_1 + Z_{1l})$, (Z_2+Z_{2l}), $(Z_3 + Z_{3l})$, respectively, in the equations (β'), (γ'), (δ') for the delta-star circuit, and assuming the generator impedances to have equal values,$_g$, we obtain*

$$-I_I(Z_g + Z_{1t}' + Z_{2t}') + I_{II}Z_{2t}' + I_{III}Z_{1t}' + E_I = 0 \quad . \quad . \quad . \quad (\beta'')$$

$$+ I_IZ_{2t}' - I_{II}(Z_g + Z_{2t}' + Z_{3t}') + I_{III}Z_{3t}' + E_{II} = 0 \quad . \quad . \quad . \quad (\gamma'')$$

$$+ I_IZ_{1t}' + I_{II}Z_{3t}' + I_{III}(Z_g + Z_{3t}' + Z_{1t}') + E_{III} = 0 \quad . \quad . \quad . \quad . \quad (\delta'')$$

The determinants to these equations are—

$$D_1 = \begin{vmatrix} E_I & + Z_{2t}' & + Z_{1t}' \\ E_{II} & -(Z_g + Z_{2t}' + Z_{3t}') & + Z_{3t}' \\ E_{III} & + Z_{3t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$= E_I[(Z_g + Z_{2t}' + Z_{3t}')(Z_g + Z_{3t}' + Z_{1t}') - Z_{3t}'^2]$$
$$+ Z_{2t}'[Z_{3t}'E_{III} + E_{II}(Z_g + Z_{3t}' + Z_{1t}')]$$
$$+ Z_{1t}'[E_{II}Z_{3t}' + E_{III}(Z_g + Z_{2t}' + Z_{3t}')]$$
$$= (E_I + E_{II} + E_{III})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')$$
$$+ Z_g[E_I(Z_g + 2Z_{3t}') + Z_{1t}'(E_I + E_{III}) + Z_{2t}'(E_I + E_{II})]$$

$$D_2 = \begin{vmatrix} E_I & -(Z_g + Z_{1t}' + Z_{2t}') & + Z_{1t}' \\ E_{II} & + Z_{2t}' & + Z_{3t} \\ E_{III} & + Z_{1t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$= E_I[-Z_{2t}'(Z_g + Z_{3t}' + Z_{1t}') - Z_{1t}'Z_{3t}']$$
$$-(Z_g + Z_{1t}' + Z_{2t}')[Z_{3t}'E_{III} + (Z_g + Z_{3t}' + Z_{1t}')E_{II}]$$
$$+ Z_{1t}'[E_{II}Z_{1t}' - E_{III}Z_{2t}']$$
$$= -\{(E_I + E_{II} + E_{III})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t} + Z_{3t}'Z_{1t}')$$
$$+ Z_g[E_{II}(Z_g + 2Z_{1t}') + Z_{2t}'(E_{II} + E_I) + Z_{3t}'(E_{II} + E_{III}) +]\}$$

* Since equation (α'), p. 486, represents the sum of equations (β'), (γ'), (δ'), its further consideration is unnecessary for the solution by determinants.

$$D_3 = \begin{vmatrix} E_{\text{I}} & -(Z_g + Z_{1t}' + Z_{2t}') & + Z_{2t}' \\ E_{\text{II}} & + Z_{2t}' & -(Z_g + Z_{2t}' + Z_{3t}') \\ E_{\text{III}} & + Z_{1t}' & + Z_{3t}' \end{vmatrix}$$

$$\begin{aligned} &= E_{\text{I}}[Z_{2t}'Z_{3t}' + Z_{1t}'(Z_g + Z_{2t}' + Z_{3t}')] \\ &\quad - (Z_g + Z_{1t}' + Z_{2t}')[-(Z_g + Z_{2t}' + Z_{3t}')E_{\text{III}} - Z_{3t}'E_{\text{II}}] \\ &\quad + Z_{2t}'[E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'] \\ &= (E_{\text{I}} + E_{\text{II}} + E_{\text{III}})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}') \\ &\quad + Z_g[E_{\text{III}}(Z_g + 2Z_{2t}') + Z_{3t}'(E_{\text{III}} + E_{\text{II}}) + Z_{1t}'(E_{\text{III}} + E_{\text{I}})] \end{aligned}$$

$$D = \begin{vmatrix} -(Z_g + Z_{1t}' + Z_{2t}') + Z_{2t}' & & + Z_{1t}' \\ + Z_{2t}' & -(Z_g + Z_{2t}' + Z_{3t}') + Z_{3t}' & \\ + Z_{1t}' & + Z_{3t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$\begin{aligned} &= -(Z_g + Z_{1t}' + Z_{2t}')[(Z_g + Z_{2t}' + Z_{3t}')(Z_g + Z_{3t}' + Z_{1t}') - Z_{3t}'^2] \\ &\quad + Z_{2t}'[Z_{3t}'Z_{1t}' + (Z_g + Z_{3t}' + Z_{1t}')Z_{2t}'] \\ &\quad + Z_{1t}'[Z_{2t}'Z_{3t}' + Z_{1t}'(Z_g + Z_{2t}' + Z_{3t}')] \\ &= -\{Z_g[Z_g^2 + 2Z_g(Z_{1t}' + Z_{2t}' + Z_{3t}')] \\ &\quad + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')\} \end{aligned}$$

Hence $\quad I_{\text{I}} = -(D_1/D), \qquad I_{\text{II}} = (D_2/D), \qquad I_{\text{III}} = -(D_3/D).$

In the case of a *symmetrical system* the expressions for the determinants may be considerably simplified, since $E_{\text{I}} + E_{\text{II}} + E_{\text{III}} = 0$; $E_{\text{I}} + E_{\text{II}} = -E_{\text{III}}$; $E_{\text{II}} + E_{\text{III}} = -E_{\text{I}}$; $E_{\text{III}} + E_{\text{I}} = -E_{\text{II}}$. Thus

$$\begin{aligned} D_1 &= Z_g[E_{\text{I}}(Z_g + 2Z_{3t}') - E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'] \\ D_2 &= -Z_g[E_{\text{II}}(Z_g + 2Z_{1t}') - E_{\text{III}}Z_{2t}' - E_{\text{I}}Z_{3t}'] \\ D_3 &= Z_g[E_{\text{III}}(Z_g + 2Z_{2t}') - E_{\text{I}}Z_{3t}' - E_{\text{II}}Z_{1t}'] \end{aligned}$$

Whence

$$I_{\text{I}} = \frac{E_{\text{I}}(Z_g + 2Z_{3t}') - E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \qquad (219)$$

$$I_{\text{II}} = \frac{E_{\text{II}}(Z_g + 2Z_{1t}') - E_{\text{III}}Z_{2t}' - E_{\text{I}}Z_{3t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \qquad (220)$$

$$I_{\text{III}} = \frac{E_{\text{III}}(Z_g + 2Z_{2t}') - E_{\text{I}}Z_{3t}' - E_{\text{II}}Z_{1t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \qquad (221)$$

The line currents I_1, I_2, I_3 are readily obtained from the phase currents, since $I_1 = I_{\text{I}} - I_{\text{III}}$, $I_2 = I_{\text{II}} - I_{\text{I}}$, $I_3 = I_{\text{III}} - I_{\text{II}}$.

In the *special case* when the system is symmetrical and the impedances of the branches of the load and the connecting line wires have the same value, Z_t', equations (219 to 221) reduce to

$$I_{\text{I}} = \frac{E_{\text{I}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad (219a)$$

$$I_{\text{II}} = \frac{E_{\text{II}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad (220a)$$

$$I_{\text{III}} = \frac{E_{\text{III}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad (221a)$$

Again, if the impedance of the generator is ignored and the no-load phase E.M.Fs. are replaced by the potential differences $V_{1\text{-}2}$, $V_{2\text{-}3}$, $V_{3\text{-}1}$ between the terminals of the generator, we have

$$I_{\text{I}} = \frac{3V_{1\text{-}2}Z_t'}{9Z_t'^2} = \frac{V_{1\text{-}2}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (219b)$$

$$I_{\text{II}} = \frac{3V_{2\text{-}3}Z_t'}{9Z_t'^2} = \frac{V_{2\text{-}3}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (220b)$$

$$I_{\text{III}} = \frac{3V_{3\text{-}1}Z_t'}{9Z_t'^2} = \frac{V_{3\text{-}1}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (221b)$$

Numerically, if the terminal pressure of the generator is V, the current (per phase) in the generator is given by

$$I_{ph} = V/3Z_t',$$

and the line (or load) current is given by

$$I = I_{ph}\sqrt{3} = (V/\sqrt{3})/Z_t'.$$

Calculation of Currents in Three-phase, Three-wire System in which Both Generator and Load are Delta-connected. This system may be considered as a network containing five meshes (Fig. 301), and accordingly five E.M.F. equations may be formed. Instead of assuming fictitious currents in each of the meshes, as hitherto, it will be more convenient in the present case to consider, as far as possible, the actual currents in the conductors forming the meshes, and obtain the E.M.F. equations for these conditions. By these means the number of unknown quantities in the equations are

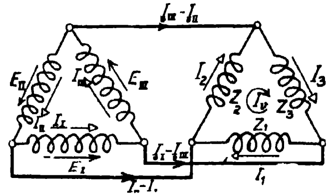


Fig. 301. Circuit Diagram for Delta/Delta Three-phase System

reduced and expressions are obtained directly for the actual currents.

For example, if the generator currents are denoted by I_{I}, I_{II}, I_{III}, the currents in the line wires will be given by $I_{\mathrm{I}} - I_{\mathrm{III}}$, $I_{\mathrm{II}} - I_{\mathrm{I}}$, $I_{\mathrm{III}} - I_{\mathrm{II}}$, and the currents in the branches of the load by $(I_{\mathrm{I}} - I_{\mathrm{III}} + I_v)$ $(I_{\mathrm{II}} - I_{\mathrm{I}} + I_v)$, $(I_{\mathrm{III}} - I_{\mathrm{II}} + I_v)$, where I_v is the assumed fictitious current circulating in the load mesh.

The five E.M.F. equations for the network are

$$E_{\mathrm{I}} - I_{\mathrm{I}} Z_{1g} - (I_{\mathrm{I}} - I_{\mathrm{III}})Z_{1l} - (I_{\mathrm{I}} - I_{\mathrm{III}} + I_v)Z_1 + (I_{\mathrm{II}} - I_{\mathrm{I}})Z_{2l} = 0 \quad . \qquad (\alpha)$$

$$E_{\mathrm{II}} - I_{\mathrm{II}} Z_{2g} - (I_{\mathrm{II}} - I_{\mathrm{I}})Z_{2l} - (I_{\mathrm{II}} - I_{\mathrm{I}} + I_v)Z_2 + (I_{\mathrm{III}} - I_{\mathrm{II}})Z_{3l} = 0 \quad . \qquad (\beta)$$

$$E_{\mathrm{III}} - I_{\mathrm{III}} Z_{3g} - (I_{\mathrm{III}} - I_{\mathrm{II}})Z_{3l} - (I_{\mathrm{III}} - I_{\mathrm{II}} + I_v)Z_3 + (I_{\mathrm{I}} - I_{\mathrm{III}})Z_{1l} = 0 \quad . \qquad (\gamma)$$

$$E_{\mathrm{I}} + E_{\mathrm{II}} + E_{\mathrm{III}} - I_{\mathrm{I}} Z_{1g} - I_{\mathrm{II}} Z_{2g} - I_{\mathrm{III}} Z_{3g} = 0 \quad . \quad . \qquad (\delta)$$

$$(I_{\mathrm{I}} - I_{\mathrm{III}} + I_v)Z_1 + (I_{\mathrm{II}} - I_{\mathrm{I}} + I_v)Z_2 + (I_{\mathrm{III}} - I_{\mathrm{II}} + I_v)Z_3 = 0 \quad . \qquad (\varepsilon)$$

which, when re-arranged, give

$$E_{\mathrm{I}} - I_{\mathrm{I}}(Z_{1g} + Z_{1l}' + Z_{2l}) + I_{\mathrm{II}} Z_{2l} + I_{\mathrm{III}} Z_{1l}' - I_v Z_1 = 0 \qquad (\alpha')$$

$$E_{\mathrm{II}} + I_{\mathrm{I}} Z_{2l}' - I_{\mathrm{II}}(Z_{2g} + Z_{2l}' + Z_{3l}) + I_{\mathrm{III}} Z_{3l} - I_v Z_2 = 0 \qquad (\beta')$$

$$E_{\mathrm{III}} + I_{\mathrm{I}} Z_{1l} + I_{\mathrm{II}} Z_{3l}' - I_{\mathrm{III}}(Z_{3g} + Z_{3l}' + Z_{1l}) - I_v Z_3 = 0 \qquad (\gamma')$$

$$E_{\mathrm{I}} + E_{\mathrm{II}} + E_{\mathrm{III}} - I_{\mathrm{I}} Z_{1g} - I_{\mathrm{II}} Z_{2g} - I_{\mathrm{III}} Z_{3g} = 0 \qquad (\delta')$$

$$I_{\mathrm{I}}(Z_1 + Z_2) + I_{\mathrm{II}}(Z_2 + Z_3) + I_{\mathrm{III}}(Z_3 + Z_1) + I_v(Z_1 + Z_2 + Z_3) = 0 \qquad (\varepsilon')$$

From equation (ε') we obtain I_v in terms of I_{I}, I_{II}, I_{III}, Z_1, Z_2, Z_3, thus*

$$I_v = \frac{I_{\mathrm{I}}(Z_1 - Z_2) + I_{\mathrm{II}}(Z_2 - Z_3) + I_{\mathrm{III}}(Z_3 - Z_1)}{Z_1 + Z_2 + Z_3} \qquad (\varepsilon'')$$

* Observe that $I_v = 0$ when $Z_1 = Z_2 = Z_3$.

and if this value be substituted in equations (α'), (β'), (γ'), we have then only three unknown quantities I_{I}, I_{II}, I_{III}. Thus

$$-I_{\text{I}}\left(Z_{1g} + Z_{1t}' + Z_{2l} + \frac{Z_1(Z_1 - Z_2)}{Z_1 + Z_2 + Z_3}\right)$$
$$+ I_{\text{II}}\left(Z_{2l} + \frac{Z_1(Z_2 - Z_3)}{Z_1 + Z_2 + Z_3}\right)$$
$$+ I_{\text{III}}\left(Z_{1t}' + \frac{Z_1(Z_3 - Z_1)}{Z_1 + Z_2 + Z_3}\right) + E_{\text{I}} = 0 \quad . \quad . \quad (\alpha'')$$

$$+ I_{\text{I}}\left(Z_{2t}' + \frac{Z_2(Z_1 - Z_2)}{Z_1 + Z_2 + Z_3}\right)$$
$$- I_{\text{II}}\left(Z_{2g} + Z_{2t}' + Z_{3l} + \frac{Z_2(Z_2 - Z_3)}{Z_1 + Z_2 + Z_3}\right)$$
$$+ I_{\text{III}}\left(Z_{3l} + \frac{Z_2(Z_3 - Z_1)}{Z_1 + Z_2 + Z_3}\right) + E_{\text{II}} = 0 \quad . \quad (\beta'')$$

$$+ I_{\text{I}}\left(Z_{1l} + \frac{Z_3(Z_1 - Z_2)}{Z_1 + Z_2 + Z_3}\right)$$
$$+ I_{\text{II}}\left(Z_{3t}' + \frac{Z_3(Z_2 - Z_3)}{Z_1 + Z_2 + Z_3}\right)$$
$$- I_{\text{III}}\left(Z_{3g} + Z_{3t}' + Z_{1l} + \frac{Z_3(Z_3 - Z_1)}{Z_1 + Z_2 + Z_3}\right) + E_{\text{III}} = 0 \quad . \quad (\gamma'')$$

These equations may be solved in a similar manner to those of the preceding section, but in view of the number of terms in each of the coefficients it is desirable to evaluate these before the general solution is attempted. For this reason no general expressions are given here for the generator phase currents I_{I}, I_{II}, I_{III}.

When the generator phase currents have been determined, the fictitious current, I_v, circulating in the load is calculated by means of equation (ε''), and the actual load currents are then readily determined, since

$$I_1 = I_{\text{I}} - I_{\text{III}} + I_v\,; \quad I_2 = I_{\text{II}} - I_{\text{I}} + I_v\,; \quad I_3 = I_{\text{III}} - I_{\text{II}} + I_v$$

Calculation of Currents in Three-phase, Four-wire System (i.e. Three-phase, Star-connected System with Neutral Wire). This system may be considered as a network containing three meshes

(Fig. 302), and accordingly three E.M.F. equations may be formed. Adopting the same symbols as in the previous cases, and denoting the line currents by I_1, I_2, I_3, I_0, and the impedance of the neutral wire by Z_{0l}, we have

$$E_{\mathrm{I}} - I_1(Z_{1g} + Z_{1l} + Z_1) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\alpha)$$

$$E_{\mathrm{II}} - I_2(Z_{2g} + Z_{2l} + Z_2) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\beta)$$

$$E_{\mathrm{III}} - I_3(Z_{3g} + Z_{3l} + Z_3) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\gamma)$$

Applying Kirchhoff's first law to the neutral point of the generator, we have

$$I_1 + I_2 + I_3 + I_0 = 0$$

whence $\qquad I_0 = -(I_1 + I_2 + I_3)$

Re-arranging terms and replacing the quantities $(Z_{1g} + Z_{1l} + Z_1)$,

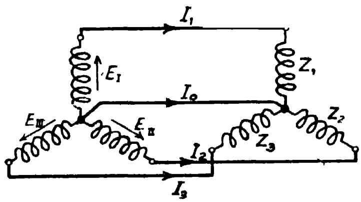


FIG. 302. CIRCUIT DIAGRAM FOR THREE-PHASE FOUR-WIRE SYSTEM

$(Z_{2g} + Z_{2l} + Z_2)$, $(Z_{3g} + Z_{3l} + Z_3)$ by the quantities Z_{1t}, Z_{2t}, Z_{3t}, respectively, we have

$$-I_1(Z_{1t} + Z_{0l}) \quad I_2Z_{0l} \quad - I_3Z_{0l} \quad + E_{\mathrm{I}} = 0 \quad . \quad (\alpha')$$

$$-I_1Z_0 \quad - I_2(Z_{2t} + Z_{0l}) - I_3Z_{0l} \quad + E_{\mathrm{II}} = 0 \quad . \quad (\beta')$$

$$-I_1Z_{0l} \quad - I_2Z_{0l} \quad - I_3(Z_{3t} + Z_{0l}) + E_{\mathrm{III}} = 0 \quad . \quad (\gamma')$$

The solution to these equations is easily obtained by determinants. The expressions for the line currents are

$$I_1 = \frac{E_{\mathrm{I}}Z_{2t}Z_{3t} + (E_{\mathrm{I}} - E_{\mathrm{III}})Z_{0l}Z_{2t} + (E_{\mathrm{I}} - E_{\mathrm{II}})Z_{0l}Z_{3t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (222)$$

$$I_2 = \frac{E_{\mathrm{II}}Z_{3t}Z_{1t} + (E_{\mathrm{II}} - E_{\mathrm{I}})Z_{0l}Z_{3t} + (E_{\mathrm{II}} - E_{\mathrm{III}})Z_{0l}Z_{1t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (223)$$

$$I_3 = \frac{E_{\mathrm{III}}Z_{1t}Z_{2t} + (E_{\mathrm{III}} - E_{\mathrm{II}})Z_{0l}Z_{1t} + (E_{\mathrm{III}} - E_{\mathrm{I}})Z_{0l}Z_{2t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (224)$$

$$I_0 = -(I_1 + I_2 + I_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Example. Calculate the line currents in a three-phase, four-wire system in which the branches of the load have resistances of 1·0, 1·3, and 1·85 ohms, the power-factor of all branches being unity. The load is supplied from a symmetrical three-phase star-connected generator through a four-core cable, the resistances of the principal conductors being 0·1 Ω., and the resistance of the neutral conductor being 0·2 Ω. The no-load E.M.F. of the generator per phase is 240 V.; the phase rotation is counter-clockwise, and the impedance per phase is $0{\cdot}13/\underline{76^\circ}$ ohm.

[NOTE. A full solution to this problem, by a method which does not involve the application of Kirchhoff's laws, has been given on pp. 472–474. The present solution will be obtained by the application of the equations deduced above.]

From the data of the example on p. 270 we have

$$Z_{1t} = 1{\cdot}1314 + j0{\cdot}1262, \quad Z_{2t} = 1{\cdot}4314 + j0{\cdot}1262,$$

$$Z_{3t} = 1{\cdot}9814 + j0{\cdot}1262, \quad Z_{0t} = 0{\cdot}2 + j0$$

$$E_{\text{I}} = 240 + j0, \quad E_{\text{II}} = -120 - j208, \quad E_{\text{III}} = -120 + j208$$

Hence,

$$E_{\text{I}} - E_{\text{III}} = 360 - j208, \quad E_{\text{I}} - E_{\text{II}} = 360 + j208$$

$$E_{\text{II}} - E_{\text{I}} = -360 - j208, \quad E_{\text{II}} - E_{\text{III}} = 0 - j416$$

$$E_{\text{III}} - E_{\text{I}} = -360 + j208, \quad E_{\text{III}} - E_{\text{II}} = 0 + j416$$

$$Z_{1t}Z_{2t} = (1{\cdot}1314 + j0{\cdot}1262)\,(1{\cdot}4314 + j0{\cdot}1262) = 1{\cdot}606 + j0{\cdot}324$$

$$Z_{2t}Z_{3t} = (1{\cdot}4314 + j0{\cdot}1262)\,(1{\cdot}9814 + j0{\cdot}1262) = 2{\cdot}824 + j0{\cdot}4313$$

$$Z_{3t}Z_{1t} = (1{\cdot}9814 + j0{\cdot}1262)\,(1{\cdot}1314 + j0{\cdot}1262) = 2{\cdot}226 + j0{\cdot}3934$$

$$Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t} = 6{\cdot}656 + j1{\cdot}1487$$

$$Z_{0t}Z_{1t} = 0{\cdot}2(1{\cdot}1314 + j0{\cdot}1262) = 0{\cdot}2263 + j0{\cdot}02524$$

$$Z_{0t}Z_{2t} = 0{\cdot}2(1{\cdot}4314 + j0{\cdot}1262) = 0{\cdot}2863 + j0{\cdot}02524$$

$$Z_{0t}Z_{3t} = 0{\cdot}2(1{\cdot}9814 + j0{\cdot}1262) = 0{\cdot}3963 + j0{\cdot}02524$$

$$Z_{0t}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t}) = 0{\cdot}2(6{\cdot}656 + j1{\cdot}1149)$$
$$= 1{\cdot}331 + j0{\cdot}2297$$

$$Z_{1t}Z_{2t}Z_{3t} = (1{\cdot}1314 + j0{\cdot}1262)\,(2{\cdot}824 + j0{\cdot}4313) = 3{\cdot}1456 + j0{\cdot}8495$$

$$I_1 = \frac{E_{\text{I}}Z_{2t}Z_{3t} + (E_{\text{I}} - E_{\text{III}})Z_{0t}Z_{2t} + (E_{\text{I}} - E_{\text{II}})Z_{0t}Z_{3t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0t}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{240(2{\cdot}824 + j0{\cdot}4313) + (360 - j208)\,(0{\cdot}2863 + j0{\cdot}02524) + (360 + j208)\,(0{\cdot}3963 + j0{\cdot}02524)}{3{\cdot}1456 + j0{\cdot}8495 + 1{\cdot}331 + j0{\cdot}2297}$$

$$= 202{\cdot}8 - j16{\cdot}55$$

$$I_1 = \sqrt{(202{\cdot}8^2 + 16{\cdot}55^2)} = 203{\cdot}3 \text{ A.}$$

$$I_2 = \frac{E_{\text{II}}Z_{3t}Z_{1t} + (E_{\text{II}} - E_{\text{I}})Z_{0t}Z_{3t} + (E_{\text{II}} - E_{\text{III}})Z_{0t}Z_{1t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0t}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{(-120 - j208)\,(2{\cdot}226 + j0{\cdot}3934) + (-360 - j208)\,(0{\cdot}3963 + j0{\cdot}02524) - j416(0{\cdot}2263 + j0{\cdot}0252)}{3{\cdot}1456 + j0{\cdot}8495 + 1{\cdot}331 + j0{\cdot}2297}$$

$$= -101{\cdot}6 - j131{\cdot}5$$

$$I_2 = \sqrt{(101{\cdot}6^2 + 131{\cdot}5^2)} = 166 \text{ A.}$$

$$I_3 = \frac{E_{II}Z_{1t}Z_{2t} + (E_{III} - E_{II})Z_{0l}Z_{1t} + (E_{III} - E_I)Z_{0l}Z_{2t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{\begin{array}{c}(-120 + j208)(1{\cdot}606 + j0{\cdot}324) + j416(0{\cdot}2263 + j0{\cdot}02524) \\ + (-360 + j208)(0{\cdot}2863 + j0{\cdot}02524)\end{array}}{3{\cdot}1456 + j0{\cdot}8495 + 1{\cdot}331 + j0{\cdot}2297}$$

$$= -57{\cdot}8 + j112$$

$$I_3 = \sqrt{(57{\cdot}8^2 + 112^2)} = 126 \text{ A.}$$

These values agree with those obtained on p. 473.

CHAPTER XXI

SYMMETRICAL COMPONENTS AND THEIR APPLICATION TO THREE-PHASE CIRCUITS

THE direct method of calculating three-phase circuits by the application of Kirchhoff's laws, discussed in the preceding chapter, while satisfactory for straightforward unbalanced circuits, becomes rather involved when complex circuits, networks with line-to-line and earth faults, and machines operating under unbalanced conditions have to be dealt with. Such cases, however, may be calculated without difficulty by an indirect method in which the unbalanced or unsymmetrical system is replaced by equivalent component systems, each of which is symmetrical and balanced. The calculation of the currents and voltages in these symmetrical systems is a simple process, and the superposition or vector addition of these currents is easily carried out to obtain the actual currents in the original unsymmetrical system.

As the procedure in the case of a four-wire system, or its equivalent (e.g. a three-wire system with earthed neutral point and earth faults), differs from that for a three-wire system, we shall discuss each system separately, and shall deal first with the simpler case of the three-wire system.

I. SYMMETRICAL COMPONENTS APPLIED TO THREE-WIRE SYSTEMS

Principle of Symmetrical Components. In the general case of an unsymmetrical three-wire system, the three line voltages, or currents, have unequal magnitudes. Moreover, the mutual phase differences between these voltages, or currents, will, in general, be unequal. But the vector sum of the three line voltages, or currents, is, in every case, zero.

Let the currents in such a system be represented by the vectors OI_1, OI_2, OI_3 (Fig. 303). Then each vector may be resolved into two components, and by this process the original three vectors may be replaced by six component vectors. If the resolution is carried out in a particular manner, the six component vectors will form two symmetrical three-phase groups, which are called the symmetrical components of the original vectors.

Fig. 304 shows the two groups of symmetrical components for the vector diagram of Fig. 303. One group of components (a) is

denoted by OI_{1p}, OI_{2p}, OI_{3p}; and the other group (*b*) by OI_{1n}, OI_{2n}, OI_{3n}. Observe that the phase sequence of one group (*a*) is the same as that of the original vectors, while the phase sequence of the other group (*b*) is opposite to that of the original vectors. In the diagram shown at (*c*), the two groups of components are compounded, and the vectors of Fig. 303 are obtained.

The group of component vectors having counter-clockwise, or positive, phase sequence is called the *positive-sequence component*,

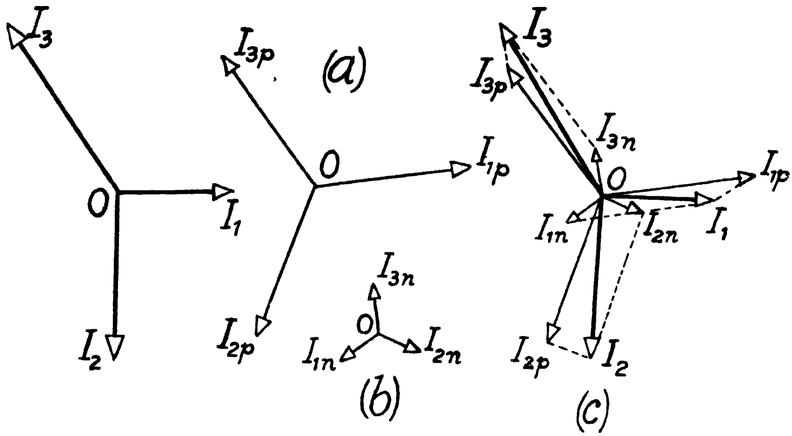


FIG. 303 FIG. 304

VECTOR DIAGRAMS FOR AN UNBALANCED, THREE-WIRE, THREE-PHASE SYSTEM (FIG. 303) AND ITS SYMMETRICAL COMPONENTS [FIG. 304 (*a*), (*b*)]

Diagram (*c*) shows that the original vector system (Fig. 303) is obtained when the symmetrical components are compounded.

and the group having clockwise phase sequence is called the *negative-sequence component*.

These components may be obtained from the original vectors either by a graphical construction or by calculation. But as the direct graphical method is based upon the analytical relationships between the components and the original vectors, it will be desirable at this stage to consider the theory of symmetrical components and to deduce their analytical relationship.

Theory of Symmetrical Components for the Three-phase, Three-wire System. The basis of the theory of symmetrical components is the vector relationship shown in Fig. 304 and the symbolic representation of these quantities.

Symbolic expressions in the rectangular form for symmetrical three-phase systems have already been developed (p. 442). In the

present case, however, the polar form is more convenient at this stage. Thus, for a symmetrical system of positive phase sequence, if I denotes the numerical value of the current in each line conductor, we have, if the current in line No. 1 is the reference quantity,

$$I_1 = I(1 + j0) = I\varepsilon^{j0}$$
$$I_2 = I(-0{\cdot}5 - j0{\cdot}866) = I\varepsilon^{-j2\pi/3}$$
$$I_3 = I(-0{\cdot}5 + j0{\cdot}866) = I\varepsilon^{j2\pi/3}$$

Now since $\varepsilon^{j0} = 1$, and $\varepsilon^{-j2\pi/3} = \varepsilon^{j4\pi/3} = \varepsilon^{2(j2\pi/3)}$, the expressions for the currents may be written in the form—

$$I_1 = I\varepsilon^{j0},$$
$$I_2 = I\varepsilon^{j0}\varepsilon^{2(j2\pi/3)} = I_1\varepsilon^{2(j2\pi/3)},$$
$$I_3 = I\varepsilon^{j0}\varepsilon^{j2\pi/3} = I_1\varepsilon^{j2\pi/3},$$

the interpretation of which is that the vector I_2 is equivalent to the rotation of the reference vector I_1 through an angle $\frac{4}{3}\pi$ radians in the positive (counter-clockwise) direction, and that the vector I_3 is equivalent to the rotation of the reference vector through an angle $\frac{2}{3}\pi$ radians in the same direction.

If, then, we regard $\varepsilon^{j2\pi/3}$ as a 120-degree turning operator (positive direction) and denote it by λ, the vectors I_1, $\lambda^2 I_1$, λI_1, taken in order, represent a symmetrical three-phase system of positive phase-sequence.

Similarly, the vectors I_1, λI_1, $\lambda^2 I_1$, taken in order, represent a symmetrical three-phase system of negative phase-sequence.

With this nomenclature, the vectors in Fig. 304 may be represented by—

$$\left.\begin{aligned} I_1 &= I_{1p} + I_{1n} \\ I_2 &= \lambda^2 I_{1p} + \lambda I_{1n} \\ I_3 &= \lambda I_{1p} + \lambda^2 I_{1n} \end{aligned}\right\} \qquad (226)$$

Properties of the Operator λ. By successive applications of the operator λ to a unit reference vector, we obtain the following relationships—

$$\lambda = \varepsilon^{j2\pi/3};\ \lambda^2 = \varepsilon^{2(j2\pi/3)};\ \lambda^3 = \varepsilon^{3(j2\pi/3)} = \varepsilon^{j2\pi} = 1;$$
$$\lambda^4 = \varepsilon^{4(j2\pi/3)} = \lambda;\ \lambda^5 = \varepsilon^{5(j2\pi/3)} = \lambda^2;\ \text{and so on.}$$

Hence $\lambda^3 = 1$, or $\lambda = \sqrt[3]{1}$, i.e. λ is numerically equivalent to the cube root of unity.

Now algebraically,

$$\varepsilon^{(j2\pi/3)}\ (= \lambda) = \cos\tfrac{2}{3}\pi + j\sin\tfrac{2}{3}\pi = -0{\cdot}5 + j0{\cdot}866, \text{ and}$$
$$\varepsilon^{(j4\pi/3)}\ (= \lambda^2) = \cos\tfrac{4}{3}\pi + j\sin\tfrac{4}{3}\pi = -0{\cdot}5 - j0{\cdot}866.$$

Hence $\lambda + \lambda^2 = -1$, or $1 + \lambda + \lambda^2 = 0$. Since $\lambda^3 = 1$, we may write $\lambda + \lambda^2 + \lambda^3 = 0$.

Calculation of the Components. With a knowledge of the properties of λ, we are now able to obtain the relationship between each symmetrical component and the original vectors.

For example, from the equations (226) we obtain an equation for I_{1p} by first multiplying the equations for I_2 and I_3 by λ and λ^2 respectively, and then adding the three equations. Thus,

$$I_1 = I_{1p} + I_{1n}$$
$$\lambda I_2 = I_{1p} + \lambda^2 I_{1n}$$
$$\lambda^2 I_3 = I_{1p} + \lambda I_{1n}$$

Whence $I_1 + \lambda I_2 + \lambda^2 I_3 = 3I_{1p} + I_{1n}(1 + \lambda^2 + \lambda)$

or
$$I_{1p} = \tfrac{1}{3}(I_1 + \lambda I_2 + \lambda^2 I_3) \qquad (227)$$

since $1 + \lambda^2 + \lambda = 0$.

Hence the positive sequence components are—

$$\left.\begin{aligned} I_{1p} &= \tfrac{1}{3}(I_1 + \lambda I_2 + \lambda^2 I_3) \\ I_{2p} &= \lambda^2 I_{1p} = \tfrac{1}{3}(I_2 + \lambda I_3 + \lambda^2 I_1) \\ I_{3p} &= \lambda I_{1p} = \tfrac{1}{3}(I_3 + \lambda I_1 + \lambda^2 I_2) \end{aligned}\right\} \qquad (228)$$

The negative sequence component (I_{1n}) may be obtained in a similar manner. Thus,

$$I_1 = I_{1p} + I_{1n}$$
$$\lambda^2 I_2 = \lambda I_{1p} + I_{1n}$$
$$\lambda I_3 = \lambda_2 I_{1p} + I_{1n}$$

Whence $I_1 + \lambda^2 I_2 + \lambda I_3 = I_{1p}(1 + \lambda + \lambda^2) + 3I_{1n}$

or
$$I_{1n} = \tfrac{1}{3}(I_1 + \lambda^2 I_2 + \lambda I_3) \qquad (229)$$

Hence the negative sequence components are—

$$\left.\begin{aligned} I_{1n} &= \tfrac{1}{3}(I_1 + \lambda^2 I_2 + \lambda I_3) \\ I_{2n} &= \lambda I_{1n} = \tfrac{1}{3}(I_2 + \lambda^2 I_3 + \lambda I_1) \\ I_{3n} &= \lambda^2 I_{1n} = \tfrac{1}{3}(I_3 + \lambda^2 I_1 + \lambda I_2) \end{aligned}\right\} \qquad (230)$$

Example. The currents in a three-phase, three-wire system are $I_1 = 100 + j0$; $I_2 = -40 - j80$; $I_3 = -60 + j80$. Calculate the symmetrical components.

Applying equations (228), the positive components are—

$$I_{1p} = \tfrac{1}{3}[100 + j0 + (-0{\cdot}5 + j0{\cdot}866)(-40 - j80) + (-0{\cdot}5 - j0{\cdot}666)(-60 + j80)]$$
$$= 96{\cdot}2 + j5{\cdot}8$$
$$I_{2p} = (-0{\cdot}5 - j0{\cdot}866)(96{\cdot}2 + j5{\cdot}8)$$
$$= -43{\cdot}1 - j86{\cdot}2$$
$$I_{3p} = (-0{\cdot}5 + j0{\cdot}866)(96{\cdot}2 + j5{\cdot}8)$$
$$= -53{\cdot}1 + j80{\cdot}4$$

Applying equations (230), the negative components are—

$$\dot{I}_{1n} = \tfrac{1}{3}[100 + j0 + (-0{\cdot}5 - j0{\cdot}866)(-40 - j80) + (-0{\cdot}5 + j0{\cdot}866)(-60 + j80)]$$
$$= 3{\cdot}8 - j5{\cdot}8$$
$$\dot{I}_{2n} = (-0{\cdot}5 + j0{\cdot}866)(3{\cdot}8 - j5{\cdot}8)$$
$$= 3{\cdot}1 + j6{\cdot}2$$
$$\dot{I}_{3n} = (-0{\cdot}5 - j0{\cdot}866)(3{\cdot}8 - j5{\cdot}8)$$
$$= -6{\cdot}9 - j0{\cdot}4$$

As a check upon the results, we will calculate $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$ from their components. Thus,

$$\dot{I}_1 = \dot{I}_{1p} + \dot{I}_{1n} = 96{\cdot}2 + j\,5{\cdot}8 + 3{\cdot}8 - j5{\cdot}8 = 100 + j0$$
$$\dot{I}_2 = \dot{I}_{2p} + \dot{I}_{2n} = -43{\cdot}1 - j86{\cdot}2 + 3{\cdot}1 + j6{\cdot}2 = -40 - j80$$
$$\dot{I}_3 = \dot{I}_{3p} + \dot{I}_{3n} = -53{\cdot}1 + j80{\cdot}4 - 6{\cdot}9 - j0{\cdot}4 = -60 + j80$$

Relationship between Numerical Values of Positive and Negative Symmetrical Components in a Three-wire System. In an unbalanced three-wire system, the following relationships between the expressions for the numerical values of the symmetrical components hold

$$I_p^2 + I_n^2 = \tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2) \quad . \quad . \quad . \quad . \quad (231)$$
$$I_p^2 - I_n^2 = 4\triangle/\sqrt{3} \quad . \quad . \quad . \quad . \quad . \quad (232)$$

where I_p, I_n denote the numerical values of the positive and negative symmetrical components respectively, I_1, I_2, I_3 denote the numerical values of the currents, and $\triangle$ denotes the area enclosed by the triangle formed by the vectors $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$ when drawn in triangular formation.

Note. $\triangle = \sqrt{[s(s - I_1)(s - I_2)(s - I_3)]}$ where $s = \tfrac{1}{2}(I_1 + I_2 + I_3)]$

Proof. The relationships follow directly from the vector equations (228), (230), already developed. Thus expanding the equations for $\dot{I}_{1p}$, $\dot{I}_{1n}$ by introducing numerical values for λ, λ^2, and horizontal and vertical components for $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$ (i.e. $\boldsymbol{a}_1$ for the horizontal component of $\dot{I}_1$—the reference vector—$\boldsymbol{b}_1$, $\boldsymbol{b}_2$, the horizontal and vertical components of $\dot{I}_2$; $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, the horizontal and vertical components of $\dot{I}_3$), we have

$$3\dot{I}_{1p} = \boldsymbol{a}_1 + \tfrac{1}{2}(\boldsymbol{b}_1 + \boldsymbol{c}_1) + \frac{\sqrt{3}}{2}(\boldsymbol{b}_2 + \boldsymbol{c}_2) + j[\tfrac{1}{2}(\boldsymbol{b}_2 - \boldsymbol{c}_2) - \frac{\sqrt{3}}{2}(\boldsymbol{b}_1 - \boldsymbol{c}_1)]$$

$$3\dot{I}_{1n} = \boldsymbol{a}_1 + \tfrac{1}{2}(\boldsymbol{b}_1 + \boldsymbol{c}_1) - \frac{\sqrt{3}}{2}(\boldsymbol{b}_2 + \boldsymbol{c}_2) + j[\tfrac{1}{2}(\boldsymbol{b}_2 - \boldsymbol{c}_2) + \frac{\sqrt{3}}{2}(\boldsymbol{b}_1 - \boldsymbol{c}_1)]$$

Whence,

$$(3I_p)^2 = [a_1 + \tfrac{1}{2}(b_1 + c_1) + \frac{\sqrt{3}}{2}(b_2 + c_2)]^2 + [\tfrac{1}{2}(b_2 - c_2) - \frac{\sqrt{3}}{2}(b_1 - c_1)]^2$$
$$= a_1^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 + a_1(b_1 + c_1) - b_1c_1 + b_2c_2,$$
$$+ \sqrt{3}a_1(b_2 + c_2) + \sqrt{3}(b_2c_1 + b_1c_2)$$

and

$$(3I_n)^2 = [a_1 + \tfrac{1}{2}(b_1 + c_1) - \frac{\sqrt{3}}{2}(b_2 + c_2)]^2 + [\tfrac{1}{2}(b_2 - c_2) + \frac{\sqrt{3}}{2}(b_1 - c_1)]^2$$
$$= a_1^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 + a_1(b_1 + c_1) - b_1c_1 + b_2c_2$$
$$- \sqrt{3}a_1(b_2 + c_2) - \sqrt{3}(b_2c_1 + b_1c_2)$$

Now, since we are considering a three-wire system, $(b_1 + c_1) = a_1$ and

$b_2 = c_2$, so that when these substitutions are made and the sum of the above expressions is taken, we have

$$(3I_p)^2 + (3I_n)^2 = 3(a_1^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2)$$
$$= 3(I_1^2 + I_2^2 + I_3^2)$$

Hence, $$I_p^2 + I_n^2 = \tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2)$$

Similarly, by taking the difference of the expressions for $(3I_p)^2$ and $(3I_n^2)$ we obtain

$$I_p^2 - I_n^2 = \tfrac{1}{9}\sqrt{3}[2a_1(b_2 + c_2) + 2(b_2c_1 + b_1c_2)]$$
$$= \tfrac{1}{9}\sqrt{3}[2b_2(a_1 + c_1 + a_1 + b_1)]$$
$$= \tfrac{1}{9}\sqrt{3}(2b_2 \times 3a_1)$$
$$= \tfrac{2}{3}(2b_2a_1)/\sqrt{3}$$
$$= 4\triangle/\sqrt{3}$$

where $\triangle = \tfrac{1}{2}a_1b_2$ = area enclosed by triangle formed by the vector I_1, I_2, I_3, when drawn in triangular formation.

Rigorous Expressions for Calculating Numerical Values of Components. These follow readily from equations (231), (232). Thus, by addition, we have

$$2I_p^2 = \tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2) + 4\triangle/\sqrt{3}$$

whence $$I_p = \sqrt{[\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2) + 2\triangle/\sqrt{3}]} \qquad (233)$$

By subtraction we have

$$2I_n^2 = \tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2) - 4\triangle/\sqrt{3}$$

whence $$I_n = \sqrt{[\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2) - 2\triangle/\sqrt{3}]} \qquad (234)$$

Special Case of Symmetrical Systems. Let the currents in a symmetrical system be denoted by I. Then the term $\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2)$ in equation (233) reduces to $\tfrac{1}{2}I^2$, and the term $2\triangle/\sqrt{3}$ reduces to $\tfrac{1}{2}I^2$, so that I_p is given by

$$I_p = \sqrt{(\tfrac{1}{2}I^2 + \tfrac{1}{2}I^2)} = I.$$

Hence $$I_n = \sqrt{(\tfrac{1}{2}I^2 - \tfrac{1}{2}I^2)} = 0.$$

Therefore in symmetrical systems no negative sequence components of current or voltage are present. The positive sequence components are, of course, identical with the vectors of the system.

Approximate Expressions for Calculating Numerical Values of Components. In a system in which the dissymmetry is small (i.e. the system is only slightly unbalanced or unsymmetrical), the negative sequence component will be small in comparison with the positive sequence component. Hence for approximate calculations, I_n^2 can be ignored in comparison with I_p^2, so that to a close approximation

$$I_p \cong \sqrt{[\tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2)]} \qquad (235)$$

Since in the case under consideration I_1, I_2, I_3 are nearly equal, a rough approximation to I_p is given by

$$I_p \cong \sqrt[3]{(I_1I_2I_3)} \qquad (236)$$

Equation (234) for the negative sequence components can, in cases where the system is only slightly unsymmetrical, be reduced to a form which closely approximates to the rigorous equation and which is very convenient for calculation, viz.

$$I_n \cong \frac{\sqrt{2}}{3}\sqrt{[(I_1 - I_2)^2 + (I_2 - I_3)^2 + (I_3 - I_1)^2]} \quad . \quad (237)$$

Proof. For brevity, write a, b, c for I_1, I_2, I_3 respectively. Then if a, b, c are nearly equal and m is the mean value, i.e. $m = \frac{1}{3}(a + b + c)$, we can write $a = m(1 \pm \alpha)$; $b = m(1 \pm \beta)$; $c = m(1 \pm \gamma)$, where α, β, γ are each small in comparison with unity and $\pm \alpha \pm \beta \pm \gamma = 0$.

Hence, considering only positive signs for convenience,

$$\begin{aligned}\tfrac{1}{3}(a^2 + b^2 + c^2) &= \tfrac{1}{3}m^2[(1 + \alpha)^2 + (1 + \beta)^2 + (1 + \gamma)^2] \\ &= m^2[1 + \tfrac{1}{3}(\alpha^2 + \beta^2 + \gamma^2)].\end{aligned}$$

Since

$$\Delta = \sqrt{[s(s - a)(s - b)(s - c)]},$$

where $2s = a + b + c = 3m$,

therefore

$$\begin{aligned}2\Delta/\sqrt{3} = \tfrac{1}{2}(4\Delta/\sqrt{3}) &= \tfrac{1}{2}\sqrt{\tfrac{1}{3}[16s(s - a)(s - b)(s - c)]} \\ &= \tfrac{1}{2}\sqrt{\tfrac{1}{3}[3m^4(1 - 2\alpha)(1 - 2\beta)(1 - 2\gamma)]} \\ &= \tfrac{1}{2}m^2\sqrt{[1 + 4(\alpha\beta + \beta\gamma + \gamma\alpha) - 8\alpha\beta\gamma]} \\ &= \tfrac{1}{2}m^2[1 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)] \text{ approximately.}\end{aligned}$$

$$\begin{aligned}\text{Hence, } I_n^2 &= \tfrac{1}{2}m^2[\tfrac{1}{3}(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] \\ &= \tfrac{2}{3}m^2(\alpha^2 + \beta^2 + \gamma^2),\end{aligned}$$

since as

$$\begin{aligned}\alpha + \beta + \gamma = 0, \text{ and } (\alpha + \beta + \gamma)^2 &= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= 0, \text{ we have } -2(\alpha\beta + \beta\gamma + \gamma\alpha) = \alpha^2 + \beta^2 + \gamma^2.\end{aligned}$$

$$\begin{aligned}\text{Now } (a - b)^2 + (b - c)^2 + (c - a)^2 &= m^2[(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2] \\ &= m^2[2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] = 3m^2(\alpha^2 + \beta^2 + \gamma^2).\end{aligned}$$

$$\text{Therefore} \quad I_n^2 = \tfrac{2}{9}[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

$$\begin{aligned}\text{i.e.} \quad I_n &= \tfrac{1}{3}\sqrt{2[(a - b)^2 + (b - c)^2 + (c - a)^2]} \\ &= \tfrac{1}{3}\sqrt{2[(I_1 - I_2)^2 + (I_2 - I_3)^2 + (I_3 - I_1)^2]}.\end{aligned}$$

Examples. The above rigorous and approximate methods will be applied to the calculation of the symmetrical components of the currents in a slightly unsymmetrical three-wire system. The line currents in this system are: $I_1 = 50$ A., $I_2 = 45$ A., $I_3 = 55$ A.

Solution by rigorous method

$$\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2) = \tfrac{1}{6}(50^2 + 45^2 + 55^2) = 1258$$

$$\tfrac{1}{2}(I_1 + I_2 + I_3) = \tfrac{1}{2}(50 + 45 + 55) = 75 \ (= s)$$

$$\begin{aligned}\therefore \quad 2\Delta/\sqrt{3} &= \frac{2}{\sqrt{3}}\sqrt{[s(s - I_1)(s - I_2)(s - I_3)]} \\ &= 1{\cdot}155\sqrt{(75 \times 25 \times 30 \times 20)} = 1290\end{aligned}$$

$$\begin{aligned}\therefore \quad I_p &= \sqrt{[\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2) + 2\Delta/\sqrt{3}]} = \sqrt{(1258 + 1225)} \\ &= 50{\cdot}8 \text{ A.}\end{aligned}$$

$$\begin{aligned}I_n &= \sqrt{[\tfrac{1}{6}(I_1^2 + I_2^2 + I_3^2) - 2\Delta/\sqrt{3}]} = \sqrt{(1258 - 1225)} \\ &= 5{\cdot}77 \text{ A.}\end{aligned}$$

Solution by approximate method

$$I_p = \sqrt{\tfrac{1}{3}(I_1^2 + I_2^2 + I_3^2)} = 50{\cdot}2 \text{ A.}$$

$$I_n = \tfrac{1}{3}\sqrt{2[(50 - 45)^2 + (45 - 55)^2 + (55 - 50)^2]} = 5{\cdot}77 \text{ A.}$$

The roughly approximate method gives

$$I_p = I\sqrt[3]{I_1 I_2 I_3} = \sqrt[3]{50 \times 45 \times 55} = 49{\cdot}8 \text{ A.}$$

Graphical Construction for Determining the Symmetrical Components. A number of simple geometrical methods are available, which are based either upon the fundamental equations (228), (230), or upon the geometrical properties of triangles.

Direct Method. The construction for this method is based upon the fundamental relationships—given in equations (228), (230)—between the original vectors and their symmetrical components.

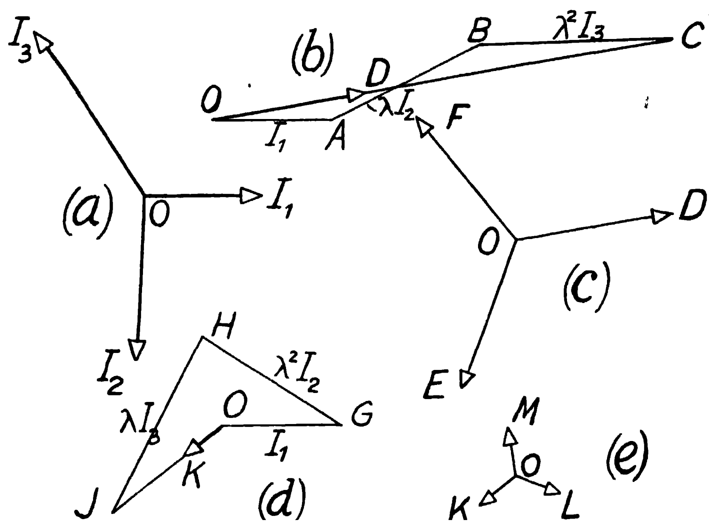


FIG. 305. DIRECT GRAPHICAL METHOD OF DETERMINING SYMMETRICAL COMPONENTS FOR THREE-WIRE, THREE-PHASE SYSTEM

For example, the positive sequence component, $\dot{I}_{1p}$, is determined by adding geometrically to the vector $\dot{I}_1$ the 120°-turned vector $\dot{I}_2$ and the 240°-turned vector $\dot{I}_3$, and taking one-third of the result.

Thus in diagram (*b*), Fig. 305, *OA* represents the vector $\dot{I}_1$ [diagram (*a*)], *AB* represents the vector $\dot{I}_2$ turned through 120° (i.e. $\lambda\dot{I}_2$), and *BC* represents the vector $\dot{I}_3$ turned through 240° (i.e. $\lambda^2\dot{I}_3$). The vector sum is represented by *OC*, and one-third of this is *OD*. The positive sequence components are, therefore, *OD*, *OE*, *OF*, as shown in diagram (*c*); *OE* and *OF* being equal in magnitude to *OD*, and having phase differences (lagging) of 120° and 240° respectively from *OD*.

To determine the negative sequence component $\dot{I}_{1n}$, we add to the vector $\dot{I}_1$ the 240°-turned vector $\dot{I}_2$, then add the 120°-turned vector $\dot{I}_3$, and finally take one-third of the result.

Thus in diagram (*d*), Fig. 305, *OG* represents the vector $\dot{I}_1$, *GH* represents the vector $\dot{I}_2$ turned through 240°, and *HJ* represents the vector $\dot{I}_3$ turned through 120°. The vector sum is represented by *OJ*, and one-third of this is *OK*. The negative sequence components are therefore *OK*, *OL*, *OM*, as shown in diagram (*e*).

Indirect Method No. 1, *suitable for Vectors Drawn in Radial Formation.* To determine $\dot{I}_{1p}$, join the extremities (I_2I_3) of the

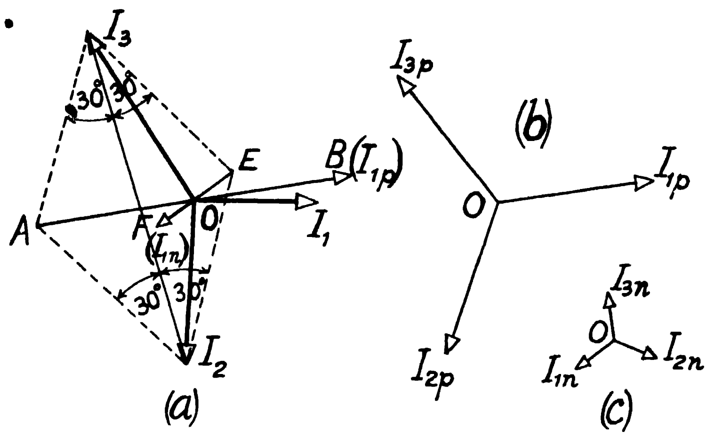


FIG. 306. INDIRECT GRAPHICAL METHOD OF DETERMINING SYMMETRICAL COMPONENTS FOR THREE-WIRE, THREE-PHASE SYSTEM

vectors $\dot{I}_2$ and $\dot{I}_3$, diagram (*a*), Fig. 306, and on the side of this line remote from *O*, erect an isosceles triangle I_2AI_3 with 30° base angles. Join *OA* and produce backwards to *B*, making $OB = OA$. Then *OB* represents $\dot{I}_{1p}$, and the positive sequence components are, therefore, represented by OI_{1p}, OI_{2p}, OI_{3p}, diagram (*b*).

To determine $\dot{I}_{1n}$, erect the isosceles triangle I_2EI_3, with 30° base angles, on the side of I_2I_3 adjacent to *O*. Join *OE* and produce backwards to *F*, making $OF = OE$. Then *OF* represents $\dot{I}_{1n}$, and the negative sequence components are therefore represented by OI_{1n}, OI_{2n}, OI_{3n}, diagram (*c*).

Indirect Method 2, *suitable for Vectors Drawn in Triangular Formation.* To determine the positive sequence components, construct isosceles triangles, with 30° base angles, on each side of the vector triangle, the apexes of the isosceles triangles being *outside* the vector triangle. The lines joining these apexes represent the three positive sequence components.

Thus in Fig. 307, diagram (*a*), if the triangle *ABC* represents the line voltages of an unsymmetrical system, the equilateral triangle

DEF (which is formed by joining the apexes of the 30° isosceles triangles *ABE*, *BCF*, *CAD*) will represent the positive sequence components. Of these components, *DE* is the component (${}_pV_{1-2}$) corresponding to *CB* (which represents the line voltage V_{1-2}), *EF* is the component (${}_pV_{2-3}$) corresponding to *AC* (which represents the line voltage V_{2-3}), and *FD* is the component (${}_pV_{3-1}$) corresponding to *BA* (which represents the line voltage V_{3-1}).

The negative sequence components are determined by a similar

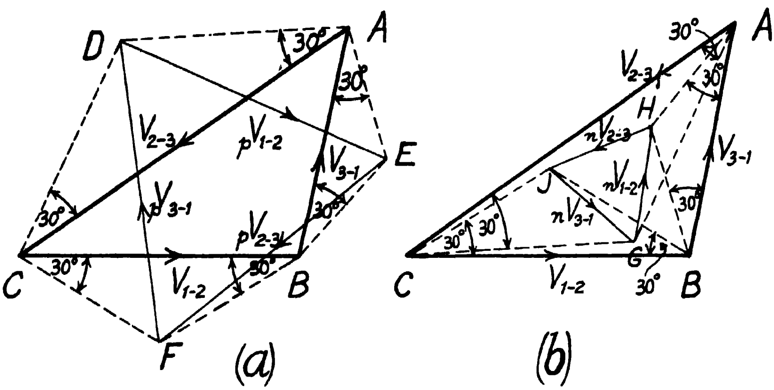


FIG. 307. ALTERNATIVE INDIRECT GRAPHICAL METHOD OF DETERMINING SYMMETRICAL COMPONENTS FOR THREE-WIRE, THREE-PHASE SYSTEM

(*a*) Construction for positive-sequence components; (*b*) Construction for negative-sequence components

method, but the isosceles triangles are now constructed on the *inside* of the vector triangle.

Thus in Fig. 307, diagram (*b*), *ABC* is the original vector triangle and *ABH*, *BCJ*, *CAG* are the 30°-isosceles triangles. The equilateral *GJH*—which is formed by joining the apexes of the isosceles triangles—represents the negative sequence components. Of these components, *GH* represents ${}_nV_{1-2}$, *HJ* represents ${}_nV_{2-3}$, and *JG* represents ${}_nV_{3-1}$.

Special Case of Symmetrical System. The method of determining the symmetrical components by the construction shown in Fig. 307 gives a very clear idea of the relative magnitudes of the positive and negative sequence components. When applied to a symmetrical system, it shows conclusively that such a system contains only positive sequence components. Thus in Fig. 308 let the equilateral triangle *ABC* represent the line voltages of a symmetrical system. Constructing 30° isosceles triangles on the outside of each side, we

obtain the equilateral triangle DEF which is identical with ABC. The apexes of the inside triangles ABG, BCG, CAG meet at a point (G), and therefore there are no negative sequence components.

Symmetrical Components for Star and Delta Circuits Supplied from a Three-wire System. In the preceding sections we have dealt with systems in which the line voltages and currents were known. This was done in order that expressions could be derived for the symmetrical components of these voltages and currents. We have

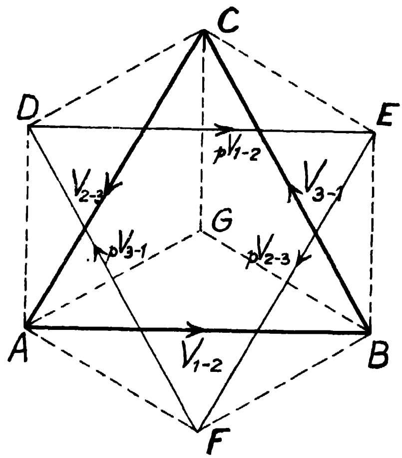


FIG. 308. SYMMETRICAL COMPONENTS FOR SYMMETRICAL THREE-PHASE SYSTEM

now to consider how the method of symmetrical components can be applied to the calculation of the currents in unsymmetrical circuits when the line voltages and the impedances of the circuits are given.

Two cases will need consideration, viz. the delta-connected circuit and the star-connected circuit. In these circuits unsymmetrical conditions may result in either—

(1) the vector sum of the phase currents (in the delta-connected circuit) being not equal to zero, although (a) the vector sum of the phase voltages is equal to zero, and (b) the vector sum of the line currents is equal to zero; or

(2) the vector sum of the phase voltages (in the star-connected circuit) being not equal to zero, although (c) the vector sum of the line voltages is equal to zero, and (d) the vector sum of the line currents is equal to zero.

Obviously the vectors representing the phase currents in an unsymmetrical delta-connected circuit, or those representing the phase voltages in an unsymmetrical star-connected circuit, cannot be replaced by two systems of symmetrical vectors, because the geometric sum of the vectors is not equal to zero. But they can be replaced by *three* systems of symmetrical vectors, in which one system consists of three equal co-phasal vectors—the sum of which is equal to the vector sum of original vectors—and the other two systems consist of symmetrical three-phase vectors of positive and negative phase sequences.

The three co-phasal vectors are called the *zero phase-sequence components*. They will be denoted by the subscript z. Thus, in the case of a delta-connected circuit, if the phase currents are denoted by $\dot{I}_{\mathrm{I}}$, $\dot{I}_{\mathrm{II}}$, $\dot{I}_{\mathrm{III}}$, their symmetrical components are—

Zero phase-sequence components—

$$\dot{I}_z = \tfrac{1}{3}(\dot{I}_{\mathrm{I}} + \dot{I}_{\mathrm{II}} + \dot{I}_{\mathrm{III}}).$$

[NOTE. The zero sequence components may be considered as representing a fictitious single-phase current circulating in the closed delta of the circuit.]

Positive phase-sequence components—

$$\dot{I}_{\mathrm{I}p},\ \dot{I}_{\mathrm{II}p}(=\lambda^2\dot{I}_{\mathrm{I}p}),\ \dot{I}_{\mathrm{III}p}(=\lambda\dot{I}_{\mathrm{I}p}),$$

of which $\dot{I}_{\mathrm{I}p} = \tfrac{1}{3}(\dot{I}_{\mathrm{I}} + \lambda\dot{I}_{\mathrm{II}} + \lambda^2\dot{I}_{\mathrm{III}})$.

Negative phase-sequence components—

$$\dot{I}_{\mathrm{I}n}.\ \dot{I}_{\mathrm{II}n}(=\lambda\dot{I}_{\mathrm{I}n}),\ \dot{I}_{\mathrm{III}n}(=\lambda^2\dot{I}_{\mathrm{I}n}),$$

of which $\dot{I}_{\mathrm{I}n} = \tfrac{1}{3}(\dot{I}_{\mathrm{I}} + \lambda^2\dot{I}_{\mathrm{II}} + \lambda\dot{I}_{\mathrm{III}})$.

Similarly, in the case of a star-connected circuit, if the phase voltages are denoted by $\dot{V}_{1\text{-}0}$, $\dot{V}_{2\text{-}0}$, $\dot{V}_{3\text{-}0}$, their symmetrical components are—

Zero phase-sequence components—

$$\dot{V}_z = \tfrac{1}{3}(\dot{V}_{1\text{-}0} + \dot{V}_{2\text{-}0} + \dot{V}_{3\text{-}0}).$$

Positive phase-sequence components—

$${}_p\dot{V}_{1\text{-}0},\ {}_p\dot{V}_{2\text{-}0}(=\lambda^2{}_p\dot{V}_{1\text{-}0}),\ {}_p\dot{V}_{3\text{-}0}(=\lambda{}_p\dot{V}_{1\text{-}0}),$$

of which ${}_p\dot{V}_{1\text{-}0} = \tfrac{1}{3}(\dot{V}_{1\text{-}0} + \lambda\dot{V}_{2\text{-}0} + \lambda^2\dot{V}_{3\text{-}0})$.

Negative phase-sequence components—

$${}_n\dot{V}_{1\text{-}0},\ {}_n\dot{V}_{2\text{-}0}(=\lambda{}_n\dot{V}_{1\text{-}0}),\ {}_n\dot{V}_{3\text{-}0}(=\lambda^2{}_n\dot{V}_{1\text{-}0}),$$

of which ${}_n\dot{V}_{1\text{-}0} = \tfrac{1}{3}(\dot{V}_{1\text{-}0} + \lambda^2\dot{V}_{2\text{-}0} + \lambda\dot{V}_{3\text{-}0})$.

Example 1. Resolve into symmetrical components the currents in a delta-connected circuit, the values of which are—

$$I_{\mathrm{I}} = 9{\cdot}66 - j2{\cdot}59;\ I_{\mathrm{II}} = -17{\cdot}3 - j10;\ I_{\mathrm{III}} = -2{\cdot}5 + j4{\cdot}33.$$

We have—

$$I_{z} = \tfrac{1}{3}(I_{\mathrm{I}} + I_{\mathrm{II}} + I_{\mathrm{III}}) = \tfrac{1}{3}(9{\cdot}66 - j2{\cdot}59 - 17{\cdot}3 - j10 - 2{\cdot}5 + j4{\cdot}33)$$
$$= -3{\cdot}38 - j2{\cdot}75$$

$$I_{\mathrm{I}p} = \tfrac{1}{3}(I_{\mathrm{I}} + \lambda I_{\mathrm{II}} + \lambda^2 I_{\mathrm{III}}) = \tfrac{1}{3}[9{\cdot}66 - j2{\cdot}59 + (-0{\cdot}5 + j0{\cdot}866)$$
$$(-17{\cdot}3 - j10) + (-0{\cdot}5 - j0{\cdot}866)(-2{\cdot}5 + j4{\cdot}33)]$$
$$= 10{\cdot}65 - j4{\cdot}2$$

$$I_{\mathrm{I}n} = \tfrac{1}{3}(I_{\mathrm{I}} + \lambda^2 I_{\mathrm{II}} + \lambda I_{\mathrm{III}}) = \tfrac{1}{3}[9{\cdot}66 - j2{\cdot}59 + (-0{\cdot}5 - j0{\cdot}866)$$
$$(-17{\cdot}3 - j10) + (-0{\cdot}5 + j0{\cdot}866)(-2{\cdot}5 + j4{\cdot}33)]$$
$$= 2{\cdot}39 + j4{\cdot}36.$$

Check—

$$I_z + I_{\mathrm{I}p} + I_{\mathrm{I}n} = -3{\cdot}38 - j2{\cdot}75 + 10{\cdot}65 - j4{\cdot}2 + 2{\cdot}39 + j4{\cdot}36$$
$$= 9{\cdot}66 - j2{\cdot}59\ (= I_{\mathrm{I}}).$$

The remaining positive and negative phase-sequence components are obtained from $I_{\mathrm{I}p}$ and $I_{\mathrm{I}n}$. Thus,

$$I_{\mathrm{II}p} = \lambda^2 I_{\mathrm{I}p} = (-0{\cdot}5 - j0{\cdot}866)(10{\cdot}65 - j4{\cdot}2)$$
$$= -8{\cdot}96 - j7{\cdot}12$$
$$I_{\mathrm{III}p} = \lambda I_{\mathrm{I}p} = (-0{\cdot}5 + j0{\cdot}866)(10{\cdot}65 - j4{\cdot}2)$$
$$= -1{\cdot}68 + j11{\cdot}32$$
$$I_{\mathrm{II}n} = \lambda I_{\mathrm{I}}n = (-0{\cdot}5 + j0{\cdot}866)(2{\cdot}39 + j4{\cdot}36)$$
$$= -4{\cdot}92 - j0{\cdot}11$$
$$I_{\mathrm{III}n} = \lambda^2 I_{\mathrm{I}n} = (-0{\cdot}5 - j0{\cdot}866)(2{\cdot}39 + j4{\cdot}36)$$
$$= 2{\cdot}64 - j4{\cdot}25.$$

Example 2. Determine the symmetrical components of the voltages across the branches of an unsymmetrical star-connected load, the values of which are—

$$V_{1\text{-}0} = 188{\cdot}8 + j144,\ V_{2\text{-}0} - 31{\cdot}5 + j144,\ V_{3\text{-}0} = 78{\cdot}7 + j334{\cdot}5.$$

We have

$$V_z = \tfrac{1}{3}(V_{1\text{-}0} + V_{2\text{-}0} + V_{3\text{-}0}) = \tfrac{1}{3}(188{\cdot}8 + j144 - 31{\cdot}5 + j144 + 78{\cdot}7 + j334{\cdot}5)$$
$$= 78{\cdot}7 + j207{\cdot}5$$

$$_pV_{1\text{-}0} = \tfrac{1}{3}(V_{1\text{-}0} + \lambda V_{2\text{-}0} + \lambda^2 V_{3\text{-}0}) = \tfrac{1}{3}[188{\cdot}8 + j144 + (-0{\cdot}5 + j0{\cdot}866)]$$
$$(-31{\cdot}5 + j144) + (-0{\cdot}5 - j0{\cdot}866)(78{\cdot}7 + j334{\cdot}5)$$
$$= 110 - j63{\cdot}5$$

$$_nV_{1\text{-}0} = \tfrac{1}{3}(V_{1\text{-}0} + \lambda^2 V_{2\text{-}0} + \lambda V_{3\text{-}0}) = \tfrac{1}{3}[188{\cdot}8 + j144 + (-0{\cdot}5 - j0{\cdot}866)]$$
$$(-31{\cdot}5 + j144) + (-0{\cdot}5 + j0{\cdot}86)(78{\cdot}7 + j334{\cdot}5)$$
$$= 0 + j0.$$

Check—

$$V_z + {_pV_{1\text{-}0}} + {_nV_{1\text{-}0}} = 78{\cdot}7 + j207{\cdot}5 + 110 - j63{\cdot}5$$
$$= 188{\cdot}7 + j144 (= V_{1\text{-}0})$$

The remaining positive phase-sequence components are obtained from ${}_pV_{1-0}$. Thus,

$$\begin{aligned}{}_pV_{2-0} = \lambda^2{}_pV_{1-0} &= (-0{\cdot}5 - j0{\cdot}866)(110 - j63{\cdot}5)\\ &= -110 - j63{\cdot}5\\ {}_pV_{3-0} = \lambda_pV_{1-0} &= (-0{\cdot}5 + j0{\cdot}866)(110 - j63{\cdot}5)\\ &= 0 + j127.\end{aligned}$$

Since ${}_nV_{1-0}$ is zero, the phase voltages have no negative phase-sequence components. This is because the supply system is symmetrical (see p. 482).

Calculation of Currents in a Delta-connected Circuit. Let three unsymmetrical impedances, Z_a, Z_b, Z_c, the values of which are the same for currents of any phase-sequence, be connected in delta to a three-wire system, and let the line voltages be denoted by V_{1-2}, V_{2-3}, V_{3-1}. Then if I_{I}, I_{II}, I_{III} denote the phase currents, we have

$$V_{1-2} = I_{\text{I}}Z_a;\ V_{2-3} = I_{\text{II}}Z_b;\ V_{3-1} = I_{\text{III}}Z_c.$$

Therefore, the phase currents I_{I}, I_{II}, I_{III} can be determined directly when the line voltages and impedances are given.

Hence for delta-connected circuits the direct method of solution is always employed, as the method of solution by symmetrical components would involve unnecessary labour.

Calculation of Currents in an Unsymmetrical Star-connected Circuit Supplied from an Unsymmetrical Three-phase, Three-wire System. In this case the symmetrical components of the line voltages and the load currents will consist of both positive and negative phase-sequence components, but those of the load voltages will consist of positive, negative, and zero phase-sequence components.

Denoting the line voltages by V_{1-2}, V_{2-3}, V_{3-1}; the line and load currents by I_1, I_2, I_3; the load voltages by V_{1-0}, V_{2-0}, V_{3-0}; and the impedances of the load by Z_1, Z_2, Z_3 (the values of which are the same for currents of any sequence), then $V_{1-0} = Z_1 I_1$; $V_{2-0} = Z_2I_2$; $V_{3-0} = Z_3I_3$.

Hence, if ${}_pV_{1-0}$, ${}_nV_{1-0}$, and V_z are the symmetrical components of V_{1-0}, we have

$$\begin{aligned}V_z &= \tfrac{1}{3}(V_{1-0} + V_{2-0} + V_{3-0}) = \tfrac{1}{3}(Z_1I_1 + Z_2I_2 + Z_3I_3)\\ &= \tfrac{1}{3}[Z_1(I_{1p} + I_{1n}) + Z_2(I_{2p} + I_{2n}) + Z_3(I_{3p} + I_{3n})]\\ &= \tfrac{1}{3}[Z_1(I_{1p} + I_{1n}) + Z_2(\lambda^2I_{1p} + \lambda I_{1n}) + Z_3(\lambda I_{1p} + \lambda^2I_{1n})]\\ &= I_{1p}\tfrac{1}{3}(Z_1 + \lambda^2Z_2 + \lambda Z_3) + I_{1n}\tfrac{1}{3}(Z_1 + \lambda Z_2 + \lambda^2Z_3)\\ &= I_{1p}Z_{1n} + I_{1n}Z_{1p}. \qquad (238)\end{aligned}$$

$$\begin{aligned}{}_pV_{1-0} &= \tfrac{1}{3}(V_{1-0} + \lambda V_{2-0} + \lambda^2V_{3-0}) = \tfrac{1}{3}(Z_1I_1 + \lambda Z_2I_2 + \lambda^2Z_3I_3)\\ &= \tfrac{1}{3}[Z_1(I_{1p} + I_{1n}) + \lambda Z_2(I_{2p} + I_{2n}) + \lambda^2Z_3(I_{3p} + I_{3n})]\\ &= I_{1p}\tfrac{1}{3}(Z_1 + Z_2 + Z_3) + I_{1n}\tfrac{1}{3}(Z_1 + \lambda^2Z_2 + \lambda Z_3)\\ &= I_{1p}Z_z + I_{1n}Z_{1n} \qquad (239)\end{aligned}$$

$$_{n}\dot{V}_{1-0} = \dot{I}_{1n}\dot{Z}_{z} + \dot{I}_{1p}\dot{Z}_{1p} \qquad (240)$$

where $\dot{Z}_{1p} = \frac{1}{3}(\dot{Z}_1 + \lambda\dot{Z}_2 + \lambda^2\dot{Z}_3)$, $\dot{Z}_{1n} = \frac{1}{3}(\dot{Z}_1 + \lambda^2\dot{Z}_2 + \lambda\dot{Z}_3)$, $\dot{Z}_z = \frac{1}{3}(\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3)$.

Similarly,

$$_{p}\dot{V}_{2-0} = \dot{I}_{2p}\dot{Z}_z + \dot{I}_{2n}\dot{Z}_{2n} = \lambda^2(\dot{I}_{1p}\dot{Z}_z + \dot{I}_{1n}\dot{Z}_{1n}) = \lambda^2{}_{p}\dot{V}_{1-0}$$
$$_{p}\dot{V}_{3-0} = \dot{I}_{3p}\dot{Z}_z + \dot{I}_{3n}\dot{Z}_{3n} = \lambda(\dot{I}_{1p}\dot{Z}_z + \dot{I}_{1n}\dot{Z}_{1n}) = \lambda{}_{p}\dot{V}_{1-0}$$
$$_{n}\dot{V}_{2-0} = \dot{I}_{2n}\dot{Z}_z + \dot{I}_{2p}\dot{Z}_{2p} = \lambda(\dot{I}_{1n}\dot{Z}_z + \dot{I}_{1p}\dot{Z}_{1p}) = \lambda{}_{n}\dot{V}_{1-0}$$
$$_{n}\dot{V}_{3-0} = \dot{I}_{3n}\dot{Z}_z + \dot{I}_{3p}\dot{Z}_{3p} = \lambda^2(\dot{I}_{1n}\dot{Z}_z + \dot{I}_{1p}\dot{Z}_{1p}) = \lambda^2{}_{n}\dot{V}_{1-0}$$

Now $_{p}\dot{V}_{1-0}$ and $_{n}\dot{V}_{1-0}$ can be determined from the relationship between the line and phase voltages. Thus,

$$\begin{aligned}
\dot{V}_{1-2} &= \dot{V}_{1-0} - \dot{V}_{2-0} \\
&= \dot{V}_z + {}_{p}\dot{V}_{1-0} + {}_{n}\dot{V}_{1-0} - (\dot{V}_z + {}_{p}\dot{V}_{2-0} + {}_{n}\dot{V}_{2-0}) \\
&= {}_{p}\dot{V}_{1-0}(1 - \lambda^2) + {}_{n}\dot{V}_{1-0}(1 - \lambda) \\
&= \tfrac{1}{2}\sqrt{3}[{}_{p}\dot{V}_{1-0}(\sqrt{3} + j1) + {}_{n}\dot{V}_{1-0}(\sqrt{3} - j1)] \,. \qquad (240a) \\
\dot{V}_{2-3} &= \dot{V}_{2-0} - \dot{V}_{3-0} \\
&= \dot{V}_z + {}_{p}\dot{V}_{2-0} + {}_{n}\dot{V}_{2-0} - (\dot{V}_z + {}_{p}\dot{V}_{3-0} + {}_{n}\dot{V}_{3-0}) \\
&= {}_{p}\dot{V}_{1-0}(\lambda^2 - \lambda) - {}_{n}\dot{V}_{1-0}(\lambda - \lambda^2) \\
&= -j\sqrt{3}{}_{p}\dot{V}_{1-0} + j\sqrt{3}{}_{n}\dot{V}_{1-0}
\end{aligned}$$

Whence

$$_{n}\dot{V}_{1-0} = {}_{p}\dot{V}_{1-0} - j\dot{V}_{2-3}/\sqrt{3}$$

Substituting this value of $_{n}\dot{V}_{1-0}$ into equation (240a) and simplifying, we have

$$_{p}\dot{V}_{1-0} = \tfrac{1}{3}[\dot{V}_{1-2} + \tfrac{1}{2}\dot{V}_{2-3}(1 + j\sqrt{3})] \qquad (241)$$
$$_{n}\dot{V}_{1-0} = \tfrac{1}{3}[\dot{V}_{1-2} + \tfrac{1}{2}\dot{V}_{2-3}(1 - j\sqrt{3})] \qquad (242)$$

When $_{p}\dot{V}_{1-0}$ and $_{n}\dot{V}_{1-0}$ have been determined, $\dot{I}_{1p}$ and $\dot{I}_{1n}$ are calculated by means of the following equations. Thus,

$$\dot{I}_{1p} = ({}_{p}\dot{V}_{1-0}\dot{Z}_z - {}_{n}\dot{V}_{1-0}\dot{Z}_{1n})/(\dot{Z}_z^2 - \dot{Z}_{1p}\dot{Z}_{1n}) \qquad (243)$$
$$\dot{I}_{1n} = ({}_{n}\dot{V}_{1-0}\dot{Z}_z - {}_{p}\dot{V}_{1-0}\dot{Z}_{1p})/(\dot{Z}_z^2 - \dot{Z}_{1p}\dot{Z}_{1n}) \qquad (244)$$

or, alternatively, when $\dot{I}_{1p}$ has been determined, $\dot{I}_{1n}$ can be obtained from the simpler equation

$$\dot{I}_{1n} = ({}_{n}\dot{V}_{1-0} - \dot{I}_{1p}\dot{Z}_{1p})/\dot{Z}_z \qquad (245)$$

The symmetrical components for the currents in the other phases are determined with very little mental effort by the application of the relationships already developed [equations (228), (230)]. With these components determined, the actual load currents are obtained simply by the addition of the appropriate components.

The voltages across the branches of the load may be determined

either directly, by calculating the products $\dot{I}_1\dot{Z}_1$, $\dot{I}_2\dot{Z}_2$, $\dot{I}_3\dot{Z}_3$, or indirectly by first calculating the zero phase-sequence component, $\dot{V}_z$, of the load voltages, and then adding this to the appropriate positive and negative phase-sequence components (e.g. $\dot{V}_{1-0} = \dot{V}_z + {}_p\dot{V}_{1-0} + {}_n\dot{V}_{1-0}$, etc.).

Calculation of the Currents in an Unsymmetrical Star-connected Load Supplied from a Three-wire, Three-phase System, the Voltages of which are Symmetrical. This is a special case of the general case just considered. The symmetry of the voltages of the supply system enables the above equations for the symmetrical components of the currents to be simplified, since the symmetrical components of the voltages across the branches of the load now consist only of positive and zero phase-sequence components. This fact can easily be verified by substituting the appropriate value of $\dot{V}_{2-3}$ in equation (242). Thus, in a symmetrical system of positive phase-sequence, $\dot{V}_{2-3} = \dot{V}_{1-2}(-0{\cdot}5 - j0{\cdot}866)$, and on substituting this into equation (242), we have

$$\begin{aligned} {}_n\dot{V}_{1-0} &= \tfrac{1}{3}[\dot{V}_{1-2} + \tfrac{1}{2}\dot{V}_{2-3}(1 - j\sqrt{3})] \\ &= \tfrac{1}{3}[\dot{V}_{1-2} + \tfrac{1}{2}\dot{V}_{1-2}(-0{\cdot}5 - j0{\cdot}866)(1 - j\sqrt{3})] \\ &= \tfrac{1}{3}(\dot{V}_{1-2} - \dot{V}_{1-2}) = 0. \end{aligned}$$

Hence, making this substitution in equations (241), (243), we have

$${}_p\dot{V}_{1-0} = j\dot{V}_{2-3}/\sqrt{3} = \tfrac{1}{2}\dot{V}_{1-2} - j\tfrac{1}{2}\dot{V}_{1-2}/\sqrt{3} \qquad (245)$$

$$\begin{aligned} \dot{I}_{1p} &= {}_p\dot{V}_{1-0}\dot{Z}_z/(\dot{Z}_z^2 - \dot{Z}_{1p}\dot{Z}_{1n}) \\ &= {}_p\dot{V}_{1-0}\frac{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}{\dot{Z}_1\dot{Z}_2 + \dot{Z}_2\dot{Z}_3 + \dot{Z}_3\dot{Z}_1} \end{aligned} \qquad (246)$$

$$\begin{aligned} \dot{I}_{1n} &= -\dot{I}_{1p}\dot{Z}_{1p}/\dot{Z}_z \\ &= -\dot{I}_{1p}\frac{\dot{Z}_1 + \lambda\dot{Z}_2 + \lambda^2\dot{Z}_3}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3} \end{aligned} \qquad (247)$$

The load currents and voltages are obtained by the same methods as in the general case.

The following examples illustrate the procedure in calculating the currents in an unsymmetrical star-connected load for the cases of a given load supplied from (1) a symmetrical three-wire system, (2) an unsymmetrical three-wire system.

Example 1. The three branches of a star-connected load are made up as follows—Branch 0–1, an impedance, $17/\underline{30°}$ ohms; branch 0–2, a condenser, reactance = 5 ohms; branch 0–3, an impedance $22/\underline{60°}$ ohms. The load is supplied from a 220-V. three-wire system, the phase sequence being 1–2–3. Calculate the current in, and the voltage across, each branch of the load.

This problem was calculated on pp. 482–485 by the analytical method based upon Kirchhoff's laws, and the present working by the method of symmetrical components will form an interesting comparison of the two methods.

Denoting the impedances of the branches 0–1, 0–2, 0–3 by Z_1, Z_2, Z_3, we have (see p. 482).

$$Z_1 = 14{\cdot}74 + j8{\cdot}5\,;\ Z_2 = 0 - j5\,;\ Z_3 = 11 + j19{\cdot}06.$$

The symmetrical component impedances required in the calculations are—

$$Z_s = \tfrac{1}{3}(Z_1 + Z_2 + Z_3) = \tfrac{1}{3}(25{\cdot}74 + j22{\cdot}56) = 8{\cdot}58 + j7{\cdot}52$$

$$Z_{1p} = \tfrac{1}{3}(Z_1 + \lambda Z_2 + \lambda^2 Z_3) = \tfrac{1}{3}(30{\cdot}07 - j8{\cdot}06) = 10{\cdot}02 - j2{\cdot}69$$

$$Z_{1n} = \tfrac{1}{3}(Z_1 + \lambda^2 Z_2 + \lambda Z_3) = \tfrac{1}{3}(-11{\cdot}59 + j11) = -3{\cdot}86 + j3{\cdot}66$$

$$Z_s^2 - Z_{1p}Z_{1n} = \tfrac{1}{3}(Z_1Z_2 + Z_2Z_3 + Z_3Z_1) = \tfrac{1}{3}(137{\cdot}8 + j245{\cdot}3) = 45{\cdot}93 + j81{\cdot}77$$

Denoting the voltage applied to the terminals 1 and 2 by V_{1-2}, that applied to terminals 2 and 3 by V_{2-3}, and that applied to terminals 3 and 1 by V_{3-1}, and taking V_{1-2} as the reference vector, we have from equation (245)

$$\begin{aligned} {}_pV_{1-0} &= \tfrac{1}{2}V_{1-2} - j\tfrac{1}{2}V_{1-2}/\sqrt{3} \\ &= \tfrac{1}{2}(220 + j0) - j\tfrac{1}{2}(220 + j0)/\sqrt{3} \\ &= 110 - j63{\cdot}5 \end{aligned}$$

Whence, from equation (245),

$$\begin{aligned} I_{1p} &= {}_pV_{1-0}Z_s/(Z_s^2 - Z_{1p}Z_{1n}) \\ &= \frac{(110 - j63{\cdot}5)\,(8{\cdot}58 + j7{\cdot}52)}{45{\cdot}93 + j81{\cdot}77} \\ &= 10{\cdot}04 - j11{\cdot}74 \end{aligned}$$

and from equation (247),

$$\begin{aligned} I_{1n} &= -I_{1p}Z_{1p}/Z_s \\ &= \frac{-(10{\cdot}04 - j11{\cdot}74)\,(10{\cdot}02 - j2{\cdot}69)}{8{\cdot}58 + j7{\cdot}52} \\ &= 3{\cdot}78 + j13{\cdot}52. \end{aligned}$$

Therefore,

$$I_1 = I_{1p} + I_{1n} = 13{\cdot}82 + j1{\cdot}78$$

Whence $I_1 = \sqrt{(13{\cdot}82^2 + 1{\cdot}78^2)} = 13{\cdot}92$

Now

$$\begin{aligned} I_{2p} &= \lambda^2 I_{1p} = (-0{\cdot}5 - j0{\cdot}866)\,(10{\cdot}04 - j11{\cdot}74) \\ &= -15{\cdot}22 - j2{\cdot}86 \\ I_{2n} &= \lambda I_{1n} = (-0{\cdot}5 + j0{\cdot}866)\,(3{\cdot}78 + j13{\cdot}52) \\ &= -13{\cdot}61 - j3{\cdot}5 \\ I_{3p} &= \lambda I_{1p} = (-0{\cdot}5 + j0{\cdot}866)\,(10{\cdot}04 - j11{\cdot}74) \\ &= 5{\cdot}18 + j14{\cdot}6 \\ I_{3n} &= \lambda^2 I_{1n} = (-0{\cdot}5 - j0{\cdot}866)\,(3{\cdot}78 + j13{\cdot}52) \\ &= 9{\cdot}83 - j10{\cdot}03. \end{aligned}$$

Therefore

$$I_2 = I_{2p} + I_{2n} = -28{\cdot}83 - j6{\cdot}36$$

$$I_3 = I_{3p} + I_{3n} = 15{\cdot}01 + j4{\cdot}57$$

Whence

$$I_2 = \sqrt{(28{\cdot}83^2 + 6{\cdot}36^2)} = 29{\cdot}5$$

$$I_3 = \sqrt{(15{\cdot}01^2 + 4{\cdot}57^2)} = 15{\cdot}37$$

Check—

$$\begin{aligned} I_1 + I_2 + I_3 &= 13{\cdot}82 + j1{\cdot}78 - 28{\cdot}83 - j6{\cdot}36 + 15{\cdot}01 + j4{\cdot}57 \\ &= 0{\cdot}00 - j0{\cdot}01. \end{aligned}$$

These values are in close agreement with those calculated on pp. 483, 484. As all calculations were made by means of a slide rule, extreme accuracy of agreement is not to be expected.

The load voltages V_{1-0}, V_{2-0}, V_{3-0} will be calculated by the indirect method employing symmetrical components.

Now, in the present case, $V_{1-0} = V_z + {}_pV_{1-0}$; $V_{2-0} = V_z + {}_pV_{2-0} = V_z + \lambda^2{}_pV_{1-0}$; $V_{3-0} = V_z + {}_pV_{3-0} = V_z + \lambda_p V_{1-0}$.

From equation (238) we have

$$V_z = I_{1p}Z_{1n} + I_{1n}Z_{1p}$$
$$= [(10{\cdot}04 - j11{\cdot}74)(-3{\cdot}86 + j3{\cdot}66) + (3{\cdot}78 + j13{\cdot}52)(10{\cdot}02 - j2{\cdot}69)]$$
$$= 78{\cdot}5 + j208$$

Therefore,

$$V_{1-0} = 78{\cdot}5 + j208 + 110 - j63{\cdot}5$$
$$= 188{\cdot}5 + j144{\cdot}2$$
$$V_{2-0} = 78{\cdot}5 + j208 + (-0{\cdot}5 - j0{\cdot}866)(110 - j63{\cdot}5)$$
$$= -31{\cdot}5 + j144{\cdot}4$$
$$V_{3-0} = 78{\cdot}5 + j208 + (-0{\cdot}5 + j0{\cdot}866)(110 - j63{\cdot}5)$$
$$= 78{\cdot}5 + j335.$$

These values check with those calculated by the direct method on pp. 484, 485.

Example 2. The star-connected load in the preceding example is supplied from an unsymmetrical three-wire system, the line voltages being—

$$V_{1-2} = 220 + j0;\ V_{2-3} = -120 - j172;\ V_{3-1} = -100 + j172.$$

Calculate the current in each branch of the load.

The first step is to calculate, by means of equations (241), (242) the positive and negative phase-sequence components of the voltage V_{1-0}.

Thus,
$$_pV_{1-0} = \tfrac{1}{3}[V_{1-2} + \tfrac{1}{2}V_{2-3}(1 + j\sqrt{3})]$$
$$= \tfrac{1}{3}[220 + j0 + \tfrac{1}{2}(-120 - j172)(1 + j\sqrt{3})]$$
$$= 103 - j63{\cdot}3$$
$$_nV_{1-0} = \tfrac{1}{3}[V_{1-2} + \tfrac{1}{2}V_{2-3}(1 - j\sqrt{3})]$$
$$= 3{\cdot}67 + j6.$$

The second step is to calculate by means of equations (243), (244) the positive and negative phase-sequence components of the current I_1. Thus,

$$I_{1p} = ({}_pV_{1-0}Z_z - {}_nV_{1-0}Z_{1n})/(Z_z^2 - Z_{1p}Z_{1n})$$
$$= \frac{(103 - j63{\cdot}3)(8{\cdot}58 + j7{\cdot}52) - (3{\cdot}67 + j6)(-3{\cdot}86 + j3{\cdot}66)}{45{\cdot}93 + j81{\cdot}77}$$
$$= 9{\cdot}17 - j11{\cdot}92$$
$$I_{1n} = ({}_nV_{1-0} - I_{1p}Z_{1p})/Z_z$$
$$= \frac{(3{\cdot}67 + j6) - (9{\cdot}17 - j11{\cdot}92)(10{\cdot}02 - j2{\cdot}69)}{8{\cdot}58 + j7{\cdot}52}$$
$$= 4{\cdot}98 + j13{\cdot}16.$$

Therefore,

$$I_1 = I_{1p} + I_{1n} = 14{\cdot}15 + j1{\cdot}24$$

whence

$$I_1 = \sqrt{(14{\cdot}15^2 + 1{\cdot}24^2)} = 14{\cdot}2 \text{ A}.$$

Now,

$$I_{2p} = \lambda^2 I_{1p} = (-0{\cdot}5 - j0{\cdot}866)(9{\cdot}17 - j11{\cdot}92) = -15{\cdot}03 - j2{\cdot}08$$
$$I_{3p} = \lambda I_{1p} = (-0{\cdot}5 + j0{\cdot}866)(9{\cdot}17 - j11{\cdot}92) = 5{\cdot}86 + j14$$
$$I_{2n} = \lambda I_{1n} = (-0{\cdot}5 + j0{\cdot}866)(4{\cdot}98 + j13{\cdot}16) = -14{\cdot}03 - j2{\cdot}21$$
$$I_{3n} = \lambda^2 I_{1n} = (-0{\cdot}5 - j0{\cdot}866)(4{\cdot}98 + j13{\cdot}16) = 9{\cdot}05 - j10{\cdot}95$$

Therefore,

$$I_2 = -29{\cdot}06 - j4{\cdot}29$$
$$I_3 = 14{\cdot}91 + j3{\cdot}05$$

Whence

$$I_2 = \sqrt{(29{\cdot}06^2 + 4{\cdot}29^2)} = 30{\cdot}9 \text{ A.}$$
$$I_3 = \sqrt{(14{\cdot}92^2 + 3{\cdot}05^2)} = 15{\cdot}22 \text{ A.}$$

Phase-sequence Impedances of a Three-phase System. In the above treatment of three-phase loads, and the development of the phase-sequence components of their impedances, it was specifically assumed that the values of these impedances were the same for currents of any phase sequence. In practice, such an assumption is quite in order in the case of "static" circuits which have no mutual inductance between the circuits of the several phases. Examples of such circuits are resistances, condensers, single-phase reactance coils and transformers (i.e. reactance coils and transformers in which the magnetizing windings are not magnetically interlinked with adjacent windings connected to other phases of the system), or a combination of any of these.

With three-phase transformers, transmission lines, and three-core cables, the impedance will have the same value for currents of positive or negative phase sequence, but will have a different value for currents of zero phase-sequence owing to the different values of the mutual inductance of these circuits under (1) normal conditions (i.e. with symmetrical three-phase currents in the conductors), and (2) zero phase-sequence conditions (i.e. with single-phase currents in all conductors).

With rotating machinery, e.g. alternators, induction motors, synchronous motors, etc., the impedance will have different values for currents of different phase sequences. Thus one value of impedance will have to be used with currents of positive phase-sequence, another value with currents of negative phase-sequence, and a third value with currents of zero phase-sequence. In general, the impedance to currents of negative phase-sequence will be lower than that to currents of positive phase-sequence.

Equivalent Symmetrical Circuit. In the solution of unsymmetrical star-connected circuits, comprising a generator and load networks, by the method of symmetrical components it is advantageous to be able to replace each of the unsymmetrical star-connected loads by two equivalent symmetrical loads, one carrying currents of positive phase-sequence and the other carrying currents of negative phase-sequence. By means of this artifice, the neutral points of all the (equivalent) loads are at the same potential, and therefore the calculation of the currents in the equivalent circuits is easily effected.

Thus in the general case of an unsymmetrical star-connected load having impedances $\underset{\cdot}{Z}_1$, $\underset{\cdot}{Z}_2$, $\underset{\cdot}{Z}_3$, let the phase voltages be denoted

by V_{1-0}, V_{2-0}, V_{3-0} and the currents by I_1, I_2, I_3. The symmetrical components corresponding to V_{1-0} are V_z, ${}_pV_{1-0}$, ${}_nV_{1-0}$; those corresponding to V_{2-0} are V_z, $\lambda^2{}_pV_{1-0}$, λ_nV_{1-0}; and those corresponding to V_{3-0} are V_z, λ_pV_{1-0}, $\lambda^2{}_nV_{1-0}$. The symmetrical components of the three currents are I_{1p}, I_{1n}; $\lambda^2 I_{1p}$, λI_{1n}; λI_{1p}, $\lambda^2 I_{1n}$.

Since there are no currents of zero phase-sequence (i.e. $I_z = 0$), only two equivalent symmetrical impedances can be obtained, viz.

$$Z_P = {}_pV_{1-0}/I_{1p} = {}_pV_{2-0}/I_{2p} = {}_pV_{3-0}/I_{3p}$$

and

$$Z_N = {}_nV_{1-0}/I_{1n} = {}_nV_{2-0}/I_{2n} = {}_nV_{3-0}/I_{3n}$$

These quantities will be called the *positive sequence symmetrical impedance* and the *negative sequence symmetrical impedance* respectively.

Hence the unsymmetrical load Z_1, Z_2, Z_3 can be replaced by two symmetrical star-connected loads (connected in parallel), one Z_P carrying currents of positive phase-sequence, and the other Z_N carrying currents of negative phase-sequence.

The values of Z_P and Z_N can be determined by the aid of equations (239), (240). Thus,

$$Z_P = {}_pV_{1-0}/I_{1p} = Z_z + Z_{1n}(I_{1n}/I_{1p})$$
$$= Z_z + Z_{1n}\left(\frac{{}_nV_{1-0}Z_z - {}_pV_{1-0}Z_{1p}}{{}_pV_{1-0}Z_z - {}_nV_{1-0}Z_{1n}}\right). \qquad (248)$$

$$Z_N = {}_nV_{1-0}/I_{1n} = Z_z + Z_{1p}(I_{1p}/I_{1n})$$
$$= Z_z + Z_{1p}\left(\frac{{}_pV_{1-0}Z_z - {}_nV_{1-0}Z_{1n}}{{}_nV_{1-0}Z_z - {}_pV_{1-0}Z_{1p}}\right). \qquad (249)$$

Example. Calculate the positive sequence and negative sequence symmetrical equivalent impedances for the unsymmetrical circuit in the example on p. 510.

The branch circuit impedances are $Z_1 = \mathbf{14{\cdot}74 + j8{\cdot}5}$; $Z_2 = \mathbf{0 - j5}$; $Z_3 = \mathbf{11 + j19{\cdot}06}$. The symmetrical components of the voltage V_{1-0} are ${}_pV_{1-0} = \mathbf{103 - j63{\cdot}5}$ and ${}_nV_{1-0} = \mathbf{3{\cdot}67 + j6}$. From p. 511 we have

$$Z_z = \mathbf{8{\cdot}58 + j7{\cdot}52};\ Z_{1p} = \mathbf{10{\cdot}02 - j2{\cdot}69};\ Z_{1n} = \mathbf{-3{\cdot}86 + j3{\cdot}66}.$$

Whence
$$Z_P = Z_z + Z_{1n}\left(\frac{{}_nV_{1-0}Z_z - {}_pV_{1-0}Z_{1p}}{{}_pV_{1-0}Z_z - {}_nV_{1-0}Z_{1n}}\right)$$
$$= \mathbf{7{\cdot}57 + j2{\cdot}61}$$
$$Z_N = Z_z + Z_{1p}\left(\frac{{}_pV_{1-0}Z_z - {}_nV_{1-0}Z_{1n}}{{}_nV_{1-0}Z_z - {}_pV_{1-0}Z_{1p}}\right)$$
$$= \mathbf{0{\cdot}52 - j0\ 1}.$$

Calculation of Terminal Voltage of Star-connected Generator Supplying a Three-wire System on which the Load is Unsymmetrical. The generated E.M.Fs. are assumed to be symmetrical, although the currents are unsymmetrical. This condition of operation is possible in machines built with cylindrical (i.e. non-salient-pole)

rotors. Hence the symmetrical components of the generated E.M.Fs. will consist only of positive phase-sequence components, but those of the currents will consist of both positive and negative phase-sequence components.

Let $\dot{E}_{\mathrm{I}}$, $\dot{E}_{\mathrm{II}}$, $\dot{E}_{\mathrm{III}}$ denote the generated phase E.M.Fs. (which are symmetrical, i.e. $\dot{E}_{\mathrm{II}} = \lambda^2 \dot{E}_{\mathrm{I}}$, $\dot{E}_{\mathrm{III}} = \lambda \dot{E}_{\mathrm{I}}$); $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$, the currents; $\dot{Z}_{gp}$, the impedance per phase of the generator to currents of positive phase-sequence; $\dot{Z}_{gn}$, the impedance per phase of the generator to currents of negative phase-sequence.

The load will be assumed to be star-connected, and the impedance between the neutral point of the load and each terminal of the generator will be denoted by $\dot{Z}_1$, $\dot{Z}_2$, $\dot{Z}_3$. Each of these impedances, therefore, includes the impedance of the appropriate line conductor, and the values are assumed to be the same for currents of either positive or negative phase-sequence.

Since the symmetrical components of the currents in the system consist of positive and negative phase-sequence components, all voltage drops in the system will be produced by these components. Therefore, for the purpose of calculating the voltages, the actual system of symmetrical generator and unsymmetrical load may be replaced by *two symmetrical systems* (each consisting of the symmetrical generator and a symmetrical load), in one of which only symmetrical currents of positive phase-sequence circulate, and in the other only currents of negative phase-sequence circulate.

Since each of these systems is symmetrical, there will be *no difference of potential between the neutral point of each load and that of the generator*. Hence a simple solution may be obtained for the voltages in each phase, as the voltage drops due to each current may be equated to the corresponding generated E.M.Fs. Thus, considering phase I, we have for the positive phase-sequence system

$$\dot{E}_{\mathrm{I}} = \dot{I}_{1p}\dot{Z}_{gp} + {}_p\dot{V}_{1-0} = \dot{I}_{1p}\dot{Z}_{gp} + \dot{I}_{1p}\dot{Z}_z + \dot{I}_{1n}\dot{Z}_{1n},$$

and for the negative phase-sequence system, since the generated negative phase-sequence E.M.F. is zero, we have

$$0 = \dot{I}_{1n}\dot{Z}_{gn} + {}_n\dot{V}_{1-0} = \dot{I}_{1n}\dot{Z}_{gn} + \dot{I}_{1n}\dot{Z}_z + \dot{I}_{1p}\dot{Z}_{1p},$$

where ${}_p\dot{V}_{1-0}$ and ${}_n\dot{V}_{1-0}$ are the positive and negative phase-sequence components of the voltage between terminal No. 1 of the generator and the neutral point of the load.

From the second of these equations we obtain

$$\dot{I}_{1n} = -\dot{I}_{1p}\dot{Z}_{1p}/(\dot{Z}_{gn} + \dot{Z}_z),$$

and, on substituting this into the first equation, we have

$$\dot{E}_{\mathrm{I}} = \dot{I}_{1p}[\dot{Z}_{gp} + \dot{Z}_z - \dot{Z}_{1p}\dot{Z}_{1n}/(\dot{Z}_{gn} + \dot{Z}_z)]$$

Hence, $$I_{1p} = \frac{E_{\mathrm{I}}}{Z_{gp} + Z_z - Z_{1p}Z_{1n}/(Z_{gn} + Z_z)}$$

and $$I_{1n} = -\frac{E_{\mathrm{I}}Z_{1p}/(Z_{gn} + Z_z)}{Z_{gp} + Z_z - Z_{1p}Z_{1n}/(Z_{gn} + Z_z)}$$

Whence the generator terminal voltages per phase are

$$V_{1-0} = E_{\mathrm{I}} - I_{1p}Z_{gp} - I_{1n}Z_{gn}$$
$$V_{2-0} = \lambda^2 E_{\mathrm{I}} - \lambda^2 I_{1p}Z_{gp} - \lambda I_{1n}Z_{gn}$$
$$V_{3-0} = \lambda E_{\mathrm{I}} - \lambda I_{1p}Z_{gp} - \lambda^2 I_{1n}Z_{gn}$$

and the generator line voltages are

$$V_{1-2} = V_{1-0} - V_{2-0} = \tfrac{1}{2}\sqrt{3}[(\sqrt{3} + j1)(E_{\mathrm{I}} - I_{1p}Z_{gp}) - (\sqrt{3} - j1)I_{1n}Z_{gn}]$$
$$V_{2-3} = V_{2-0} - V_{3-0} = -j\sqrt{3}(E_{\mathrm{I}} - I_{1p}Z_{gp} + I_{1n}Z_{gn})$$
$$V_{3-1} = V_{3-0} - V_{1-0} = \tfrac{1}{2}\sqrt{3}[(-\sqrt{3} + j1)(E_{\mathrm{I}} - I_{1p}Z_{gp}) - (-\sqrt{3} - j1)I_{1n}Z_{gn}]$$

II. Symmetrical Components for Four-wire Systems

The four-wire, three-phase system is used most extensively in practice for combined power and lighting distribution at low voltage (400/230 V.). The three-wire, three-phase system with earthed neutral point is the standard system for power transmission and bulk distribution. But when earth faults occur on any of the lines, this system becomes virtually a four-wire system, as fault currents now flow *via* earth to the neutral point of the generator or transformer supplying the transmission or distribution lines.

Vector Diagram for a Four-wire System. In the general case of an unbalanced four-wire system, the vector sum of the currents in the three principal line wires will not be zero. Such a system may, therefore, be represented by the vectors OI_1, OI_2, OI_3 (Fig. 309, diagram (*a*)), which represent the currents (I_1, I_2, I_3) in the three principal line wires, and the vector OI_0 which represents the current (I_0) in the neutral wire. Since, at the neutral point, the vector sum of all the currents must be zero, we have for the conventional circuit diagram of Fig. 310, with the positive directions of currents as shown (viz. outwards from the neutral point of the generator)—

$$I_1 + I_2 + I_3 + I_0 = 0,$$

whence $$I_1 + I_2 + I_3 = -I_0.$$

Resolution of Vector Diagram for Four-wire System into Symmetrical Components. If the current in the neutral is considered to divide equally among the three phases of the system, then these phase currents (each of which is equal in magnitude to one-third of

the neutral current) may be considered as the symmetrical components of the neutral current. Hence the vector OI_0 (Fig. 309)

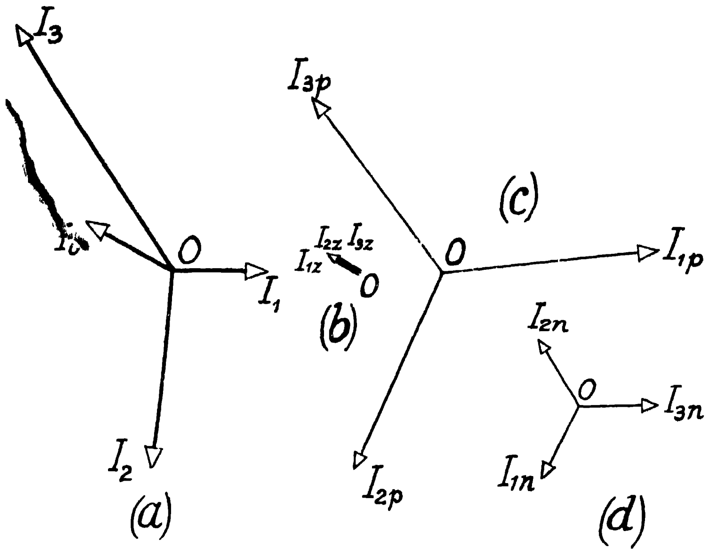


Fig. 309. Vector Diagrams for an Unbalanced Four-wire, Three-phase System (*a*), and Its Symmetrical Components (*b*) (*c*), (*d*)

may be replaced by the equivalent of three equal co-phasal vectors OI_{1z}, OI_{2z}, OI_{3z}, as indicated in diagram (*b*). As these vectors are

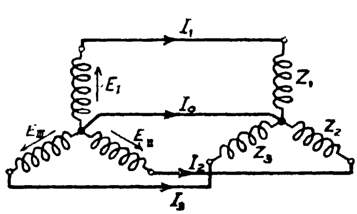


Fig. 310. Circuit Diagram of Four-wire, Three-phase System

all in phase with one another, the symmetrical components they represent are therefore called *zero-sequence components.*

The vectors OI_1, OI_2, OI_3, considered by themselves, may be replaced by two symmetrical three-phase vector groups, of positive and negative sequence respectively, in exactly the same manner as in the case of a three-wire system. These positive sequence and

negative sequence components are shown in Fig. 309, diagrams (*c*) and (*d*) respectively.

Symmetrical Components for Four-wire System. Denoting the zero-sequence components by I_{1z}, I_{2z}, I_{3z}; the positive-sequence components by I_{1p}, I_{2p}, I_{3p}; and the negative-sequence components by I_{1n}, I_{2n}, I_{3n}, we have

$$\left.\begin{aligned} I_{1_z} + I_{1p} + I_{1n} &= I_1 \\ I_{2_z} + I_{2p} + I_{2n} &= I_2 \\ I_{3_z} + I_{3p} + I_{3n} &= I_3 \end{aligned}\right\} \quad (250)$$

Now $I_{1_z} = I_{2_z} = I_{3_z}$; $I_{2p} = \lambda^2 I_{1p}$; $I_{2n} = \lambda I_{1n}$; $I_{3p} = \lambda I_{1p}$; $I_{3n} = \lambda^2 I_{1n}$, so that the equations (250) may be written in the form

$$\left.\begin{aligned} I_{1_z} + I_{1p} + I_{1n} &= I_1 \\ I_{1_z} + \lambda^2 I_{1p} + \lambda I_{1n} &= I_2 \\ I_{1_z} + \lambda I_{1p} + \lambda^2 I_{1n} &= I_3 \end{aligned}\right\} \quad (250a)$$

Whence, by addition,

$$3I_{1_z} = I_1 + I_2 + I_3$$

or

$$I_{1_z} = \tfrac{1}{3}(I_1 + I_2 + I_3) \quad (251)$$

which determines the value of the zero-sequence component.

The values of the positive-sequence and negative-sequence components are obtained in the same manner as for the three-wire system, and are

$$I_{1p} = \tfrac{1}{3}(I_1 + \lambda I_2 + \lambda^2 I_3)$$
$$I_{1n} = \tfrac{1}{3}(I_1 + \lambda^2 I_2 + \lambda I_3)$$

Graphical Construction for Determining the Symmetrical Components. The zero-sequence component is obtained by determining the vector sum of the three line-current or phase-voltage vectors, as the case may be, and taking one-third of the result.

The positive-sequence and negative-sequence components are determined in exactly the same manner as for a three-wire system, using either the direct method (p. 502) or the indirect method for radial vectors (p. 504).

Component Impedances. Let three unsymmetrical impedances Z_1, Z_2, Z_3 be connected in star to a four-wire system. Let the line currents be denoted by I_1, I_2, I_3, and the phase voltages by V_{1-0}, V_{2-0}, V_{3-0}. Then $V_{1-0} = I_1 Z_1$, $V_{2-0} = I_2 Z_2$, $V_{3-0} = I_3 Z_3$.

Since we are dealing with a four-wire system, each of these voltages and currents will have symmetrical components of zero, positive, and negative sequences. Denoting the symmetrical

components of V_{1-0} and I_1 by ${}_zV_{1-0}$, ${}_pV_{1-0}$ ${}_nV_{1-0}$ and I_z, I_{1p}, I_{1n} respectively, we have—

$$
\begin{aligned}
{}_zV_{1-0} &= \tfrac{1}{3}(V_{1-0} + V_{2-0} + V_{3-0}) = \tfrac{1}{3}(I_1Z_1 + I_2Z_2 + I_3Z_3) \\
&= \tfrac{1}{3}[Z_1(I_{1z} + I_{1p} + I_{1n}) + Z_2(I_{2z} + I_{2p} + I_{2n}) \\
&\qquad + Z_3(I_{3z} + I_{3p} + I_{3n})] \\
&= \tfrac{1}{3}[Z_1(I_{1z} + I_{1p} + I_{1n}) + Z_2(I_{1z} + \lambda^2 I_{1p} + \lambda I_{1n}) \\
&\qquad + Z_3(I_{1z} + \lambda I_{1p} + \lambda^2 I_{1n})] \\
&= I_{1z}\tfrac{1}{3}(Z_1 + Z_2 + Z_3) + I_{1p}\tfrac{1}{3}(Z_1 + \lambda^2 Z_2 + \lambda Z_3) \\
&\qquad + I_{1n}\tfrac{1}{3}(Z_1 + \lambda Z_2 + \lambda^2 Z_3)] \\
&= I_{1z}Z_z + I_{1p}Z_{1p} + I_{1n}Z_{1n} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (252)
\end{aligned}
$$

where $Z_z[= \tfrac{1}{3}(Z_1 + Z_2 + Z_3)]$, $Z_{1p}[= \tfrac{1}{3}(Z_1 + \lambda^2 Z_2 + \lambda Z_3)]$, and $Z_{1n}[= \tfrac{1}{3}(Z_1 + \lambda Z_2 + \lambda^2 Z_3)]$ are called zero, positive, and negative component impedances.

Similarly,

$$
\begin{aligned}
{}_pV_{1-0} &= \tfrac{1}{3}(V_{1-0} + \lambda V_{2-0} + \lambda^2 V_{3-0}) \\
&= \tfrac{1}{3}(I_1Z_1 + \lambda I_2Z_2 + \lambda^2 I_3Z_3) \\
&= I_{1p}Z_z + I_{1n}Z_{1n} + I_{1z}Z_{1p} \quad . \quad . \quad . \qquad (253)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_nV_{1-0} &= \tfrac{1}{3}(V_{1-0} + \lambda^2 V_{2-0} + \lambda V_{3-0}) \\
&= \tfrac{1}{3}(I_1Z_1 + \lambda^2 I_2Z_2 + \lambda^3 I_3Z_3) \\
&= I_{1n}Z_z + I_{1z}Z_{1n} + I_{1p}Z_{1p} \quad . \quad . \quad . \qquad (254)
\end{aligned}
$$

The equations (252), (253), (254), therefore, enable the symmetrical components of the load voltages to be calculated.

Calculation of the Currents in an Unsymmetrical Star-connected Load Supplied from a Four-wire System. If the load voltages are known. the problem is so simple that the solution by the method of symmetrical components is unwarranted.

If only the generator no-load voltage is known, the problem may be treated on similar principles to those employed for the corresponding three-wire problem. Thus the system is replaced by three equivalent symmetrical systems, viz. (1) a system in which only currents of positive phase-sequence circulate; (2) a system in which only currents of negative phase-sequence circulate; (3) a system in which only currents of zero phase-sequence circulate. The equivalent zero phase-sequence system is a single-phase system which consists of the neutral wire in series with three parallel-connected circuits, each of which contains one phase of the generator winding, a line wire, and one branch of the load (as shown in Fig. 311).

Obviously the direct connection between the neutral points of generator and load will not affect the currents in the systems of positive and negative phase-sequence, and, therefore, these currents are obtained by the application of the same equations as were developed for the three-wire system (p. 510).

The current in the equivalent zero phase-sequence system is obtained by equating to zero the voltage drops due to the zero phase-sequence currents. Observe that the current in the neutral is I_z and that the current in each line wire is $\frac{1}{3}I_z$. Observe, also, that the currents, $\frac{1}{3}I_z$, in the line wires are in phase with one another, and, therefore, the value of reactance used in the calculations must correspond to these operating conditions.

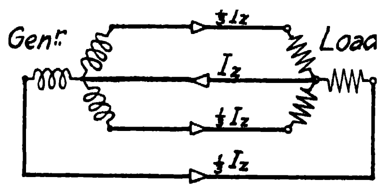


FIG. 311. DISTRIBUTION OF ZERO SEQUENCE CURRENTS IN THREE-PHASE SYSTEM

Calculation of Currents in a Three-wire System with Earthed Neutral Point and Earth Faults on Line Conductors. Under earth-fault conditions this system is equivalent to a four-wire system, the earth acting as the neutral conductor. The currents can, therefore, be calculated by employing the same methods as for a four-wire system. Observe that the fault currents are of zero phase-sequence.

III. MEASUREMENT OF SYMMETRICAL-COMPONENT VOLTAGES AND CURRENTS—APPLICATIONS OF SYMMETRICAL COMPONENT PRINCIPLES IN PRACTICE

Measurement of the Symmetrical Components of the Voltages of a Three-wire System. The symmetrical components of the voltages of a three-wire system may be determined by means of a voltmeter (preferably an electrostatic instrument), and a combination of resistances and reactances connected as shown in Fig. 312 (*b*).* The circuit is based upon the geometrical method (Fig. 307) of determining the positive and negative phase-sequence components, the principle utilized being that the lines joining the apexes of the positive and negative phase-sequence triangles are perpendicular to and bisect the sides of the line-voltage triangle. For example, when the positive and negative phase-sequence triangles are drawn in one diagram, Fig. 312 (*a*), then the line joining the apexes D and

* This scheme is due to Prof. Catterson-Smith.

G bisects AC perpendicularly. Similarly, EH and FJ bisect AB and BC perpendicularly.

The values of the resistances R_1, R_2 for use with a condenser of capacitance C, may be readily calculated from the vector diagram, Fig. 312 (c).

For a 50-cycle supply system and $C = 1\mu$F., $R_1 = 3675\Omega$., $R_2 = 1837{\cdot}5\Omega$.

Measurement of the Symmetrical Components of the Currents in a Three-wire System. The positive and negative components are

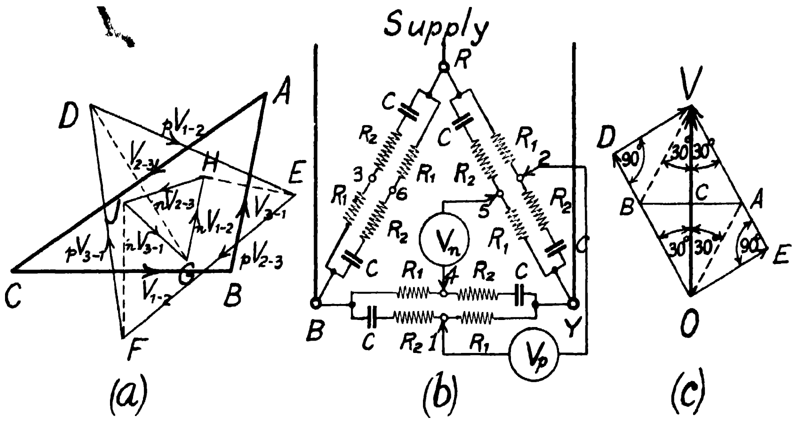


FIG. 312. VECTOR AND CIRCUIT DIAGRAMS FOR MEASUREMENT OF SYMMETRICAL COMPONENTS (VOLTAGE) OF THREE-WIRE, THREE-PHASE SYSTEM

(a) Vector diagram for system and symmetrical components; (b) Circuit diagram of testing network; (c) Vector diagram of voltages for one "leg" of network

measured separately, using current transformers with inter-connected secondary windings loaded on specially-adjusted impedances.

A number of schemes can be devised from a study of the fundamental equations (227), (229). For example, from equation (227) for the positive-sequence component, I_{1p}, we have

$$\begin{aligned} I_{1p} &= \tfrac{1}{3}(I_1 + \lambda I_2 + \lambda^2 I_3) \\ &= \tfrac{1}{3}[I_2(\lambda - 1) + I_3(\lambda^2 - 1)] \\ &= \tfrac{1}{3}\sqrt{3}[I_2(-0{\cdot}866 + j0{\cdot}5) + I_3(-0{\cdot}866 - j0{\cdot}5)] \\ &= (1/\sqrt{3})\,[-I_2\underline{/-30^\circ} + (-I_3\underline{/+30^\circ})] \end{aligned}$$

Thus if we measure the sum of the currents $(I_2/\sqrt{3})\underline{/-30^\circ}$ and $(I_3/\sqrt{3})\underline{/+30^\circ}$ we shall obtain the positive-sequence component, I_{1p}.

Similarly,

$$\dot{I}_{1n} = (1/\sqrt{3})[-\dot{I}_2/\underline{+30°} + (-\dot{I}_3/\underline{-30°})].$$

Thus in this case we measure the sum of the currents

$$(\dot{I}_2/\sqrt{3})/\underline{+30°} \text{ and } (\dot{I}_3/\sqrt{3})/\underline{-30°}.$$

The measurements are made by means of two ammeters connected, as shown in Fig. 313, to a special network of resistances and impedances supplied from two current transformers.

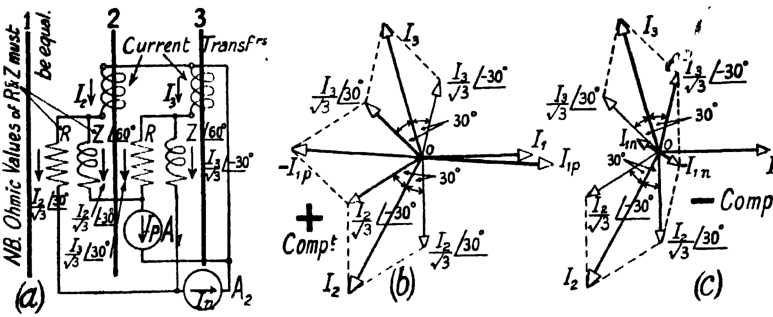


FIG. 313. CIRCUIT AND VECTOR DIAGRAMS FOR THE MEASUREMENT OF SYMMETRICAL COMPONENTS (CURRENT) OF THREE-WIRE, THREE-PHASE SYSTEM

Measurement of the Symmetrical Components of the Currents in a Four-wire System. The positive-sequence and negative-sequence components are measured in exactly the same manner as in the case of a three-wire system. The zero sequence component is obtained by measuring the current in the neutral and dividing by 3.

Application of the Principles of Symmetrical Components in Practice. The principles of symmetrical components have a number of practical applications in protective-relay schemes for the protection of transmission lines against earth faults. In these schemes the zero-sequence currents are utilized. The principles are also employed in connection with relays which are required to give discriminative action when unbalanced conditions occur. In these cases the positive- and negative-sequence components of the line currents or the line voltages are utilized.

Two examples will be considered—(1) the control relay of an automatic voltage regulator for an alternator; (2) a protective relay for protecting a machine against unbalanced operating conditions.

(1) *Application of Principles of Symmetrical Components to Automatic Voltage Regulator.* The control relay of an automatic voltage regulator for a three-phase alternator is excited by the voltage

across two of the line wires of the system (see *Power Wiring Diagrams*, p. 197). Under abnormal conditions, such as a single-phase fault or a heavy unbalanced load, this voltage may be higher or lower than the normal value, and if the control relay were controlled solely by this voltage, the regulator may act in an undesirable manner (e.g. reduce the excitation of the alternator under fault conditions when the excitation should be maintained or increased). If, however, the relay were controlled by the positive

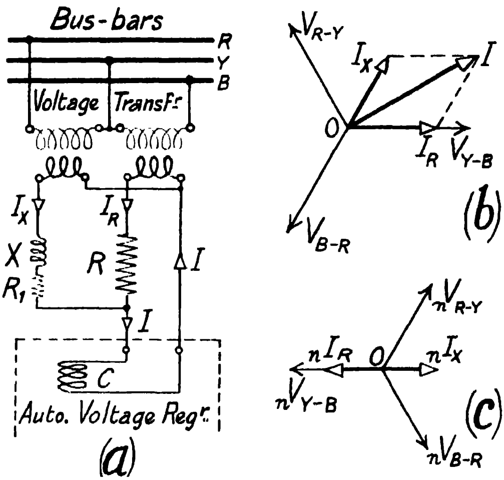


FIG. 314. CIRCUIT AND VECTOR DIAGRAMS FOR POSITIVE SEQUENCE CONTROL OF AUTOMATIC VOLTAGE REGULATOR
(*Metropolitan-Vickers Electrical Co.*)

sequence voltage, the control of the voltage would be unaffected by unbalanced-load or fault conditions, since the positive-sequence voltage is always symmetrical.

Fig. 314 (*a*) shows the connections employed for the control relay of the Metropolitan-Vickers' voltage regulator. The current in the coil of this relay is the vector sum of the currents in the circuits XR_1 and R connected to the secondary windings of two voltage transformers, the primary windings of which are connected in open-delta to the bus-bars of the supply system. Circuit XR_1 is inductive, and is so adjusted that the current in it has a phase difference of 60 degrees (lagging) with reference to the voltage between the lines R and Y. Circuit R is non-inductive: its resistance is so adjusted that the current is equal to that in circuit XR_1, this current being in phase with the voltage between the lines Y and B.

For positive phase-sequence voltages, the current in circuit R is represented, in Fig. 314 (b), by OI_R, and that in circuit XR_1 by OI_X. The vector sum of these currents is represented by OI.

For negative phase-sequence voltages, the current in circuit R is represented, in Fig. 314 (c), by $O\,{}_nI_R$, and that in circuit X by $O\,{}_nI_X$. Since the current in X lags 60 degrees from the R–Y voltage this current is in opposition to the current in circuit R, when the

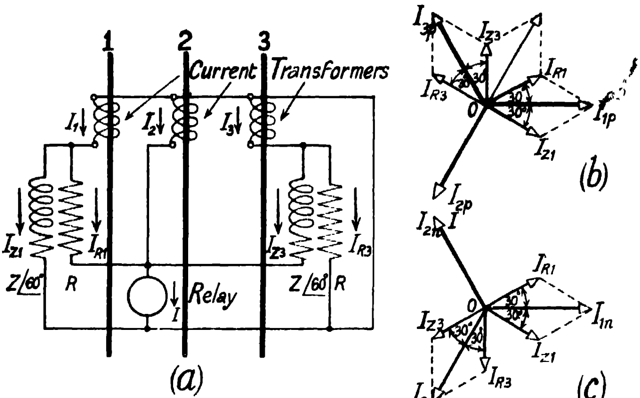


Fig. 315. Circuit and Vector Diagrams for Negative Sequence Protective Relay
(*Metropolitan-Vickers Electrical Co.*)

phase sequence is negative. Hence the control relay is operative only to positive phase-sequence voltages.

(2) *Application of the Principles of Symmetrical Components to Protective Relay.* Heavy unbalanced loads on alternators result in increased heating compared with that due to balanced loads of equal kVA., and therefore it may be desirable to give warning when the unbalance of the load exceeds a predetermined amount. This can be effected by means of a relay, the current in which is proportional to the negative sequence component of the generator current.

Fig. 315 (a) shows the scheme of connections (due to The Metropolitan-Vickers Electrical Co.). A bridge network of non-inductive resistances and impedances, of equal values, is connected to three current transformers in the generator circuit. The relay is connected across opposite corners of this network, and the relay current is the vector sum of the currents I_2, I_{R1}, I_{z3}.

For currents of positive phase-sequence, the relay current is zero, diagram (b); but for currents of negative phase-sequence, the relay current is equal to I_{2n}, as shown in diagram (c).

CHAPTER XXII

THE LOAD DIAGRAM FOR THE GENERAL CIRCUIT

In Chapter VII the no-load and short-circuit diagrams, and the generalized vector, or load, diagram were deduced for simple series, parallel, and series-parallel circuits. It was shown that the load diagram, which was developed from the no-load and short-circuit diagrams, had important practical applications, as it enabled the performance of the circuit under variable circuit conditions to be determined by a simple graphical process.

The circuits considered in Chapter VII were of the simplest type, viz. (1) a series circuit consisting of fixed and variable impedances (which is the equivalent circuit of a short transmission line with negligible capacitance); (2) a parallel circuit, one branch of fixed impedance and the other branch of variable impedance (which is an approximation to the equivalent circuit of the transformer); (3) a series-parallel circuit consisting of two parallel branches, one of fixed impedance and the other of variable impedance, in series with a fixed impedance.

The more general case of a series-parallel circuit is shown in Fig. 316, in which one of the variable branches contains both a fixed impedance as well as variable impedance. This type of circuit is representative of the equivalent circuit of a transmission line, a transformer, and a polyphase induction motor as will be apparent by comparing Fig. 316 with Figs. 87 (p. 159), 159 (p. 230), 174 (p. 260). When applied to polyphase systems and apparatus, the circuit refers to one phase, and the quantities represent "phase quantities."

Although inductive resistances are shown in Fig. 316, it must be understood that each branch of this circuit may consist of any combination of resistance, inductance and capacitance, or of one of these quantities alone. Thus, the branch Z_3, Fig. 316, is inductive in the circuits of transformers and induction motors, but is capacitive in the circuit of a long transmission line. Again, the "load" impedance, Z_2, is non-inductive in the circuit of induction motors, but may be reactive or non-reactive in the circuits of transformers and transmission lines.

Although the no-load and short-circuit diagram for a series-parallel circuit of the type shown in Fig. 316 may be constructed without difficulty, the performance of the circuit cannot be

Whence,

$$Y_o Z_s = 1 - 1/C_1 C_2{}^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

Let the load impedance now have a value Z_2, and let the supply voltage be adjusted so as to give a voltage of E_2 at the terminals of the load. Moreover, let the load current under these conditions be I_2, the supply voltage E_1, and the line current I_1. Then the line current may be considered to be made up of two components, one being the no-load current corresponding to the supply voltage E_1, and the other being the line short-circuit current corresponding to the load short-circuit current I_2. Similarly the supply voltage may be considered to be made up of two components, one being the supply voltage necessary to obtain a voltage E_2 at the terminals of the load at no-load, and the other being the supply voltage necessary to obtain a load current equal to I_2 when the load is short-circuited.

Hence,

$$\begin{aligned} I_1 &= I_{1s} + I_o \\ &= C_2 I_2 + E_{1o} Y_o \\ &= C_2 I_2 + C_1 E_2 Y_o \quad . \quad . \quad . \quad . \quad . \quad . \quad (261) \\ &= C_2 E_2/Z_2 + C_1 E_2 Y_o \\ &= E_2(C_2/Z_2 + C_1 Y_o) \quad . \quad . \quad . \quad . \quad (261a) \end{aligned}$$

since $I_2 = E_2/Z_2$.

Also,

$$E_1 = E_{1o} + E_{1s} = C_1 E_2 + C_2 I_2 Z_s \quad . \quad . \quad . \quad (262)$$

* This expression is obtained by substituting for $Y_0 Z_s$, reducing, and then substituting in terms of $C_1 C_2$. Thus

$$\begin{aligned} Y_0 Z_s &= \left(\frac{1}{Z_1 + Z_3}\right)\left(Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4}\right) \\ &= \frac{Z_1}{Z_1 + Z_3} + \frac{Z_3 Z_1}{(Z_1 + Z_3)(Z_3 + Z_4)} \end{aligned}$$

Now since $C_1 = 1 + Z_1/Z_3$, and $C_2 = 1 + Z_4/Z_3$, we have

$$Z_1 + Z_3 = C_1 Z_3, \qquad Z_1/Z_3 = C_1 - 1;$$
$$Z_3 + Z_4 = C_2 Z_3, \qquad Z_4/Z_3 = C_2 - 1.$$

Hence,

$$\begin{aligned} Y_0 Z_s &= \frac{Z_1}{C_1 Z_3} + \frac{Z_3 Z_4}{C_1 C_2 Z_3^2} \\ &= \frac{C_1 - 1}{C_1} + \frac{C_2 - 1}{C_1 C} \\ &= 1 - \frac{1}{C_1 C_2} \end{aligned}$$

If equation (261) be multiplied throughout by Z_s, and is then subtracted from equation (262), we obtain

$$\begin{aligned} E_1 - I_1 Z_s &= C_1 E_2 + C_2 I_2 Z_s - (C_2 I_2 Z_s + C_1 E_2 Y_o Z_s) \\ &= C_1 E_2 (1 - Y_o Z_s) \\ &= E_2 / C_2, \end{aligned}$$

since $(1 - Y_o Z_s) = 1/C_1 C_2$.

Whence,

$$E_2 = C_2 (E_1 - I_1 Z_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (263)$$

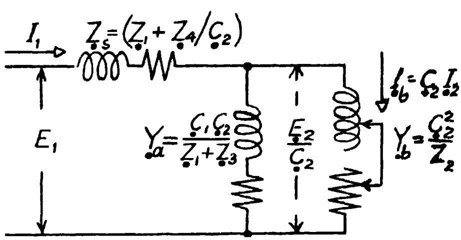


FIG. 318. SIMPLE SERIES-PARALLEL CIRCUIT EQUIVALENT TO THE GENERAL CIRCUIT OF FIG. 316

Substituting this value in equation (261*a*), we obtain

$$\begin{aligned} I_1 &= C_2 (E_1 - I_1 Z_s)\,(C_2/Z_2 + C_1 Y_o) \\ &= (E_1 - I_1 Z_s)\,(C_2^2/Z_2 + C_1 C_2 Y_o) \quad . \quad . \quad . \quad (261b) \end{aligned}$$

I.e. in the general circuit the line current, corresponding to a given supply voltage, E_1, and a particular value, Z_2, of the load impedance, is given by the product of two quantities; of which one—$(C_2^2/Z_2 + C_1 C_2 Y_o)$—represents the joint admittance of the two parallel branches of a simple series-parallel circuit, and the other—$(E_1 - I_1 Z_s)$—represents the voltage across these branches when the supply voltage is E_1, the line current is I_1 and the series impedance is Z_s.

Therefore the general series-parallel circuit of Fig 316 may be replaced by the equivalent simple series-parallel circuit shown in Fig. 318, in which the series impedance has a value equal to the short-circuit impedance of the general circuit, i.e. $Z_s = Z_1 + Z_4/C_2$; the fixed branch of the parallel portion of this circuit, Fig. 318, has an admittance equal to $C_1 C_2$ times the "no-load admittance" of the general circuit, i.e. $Y_a = C_1 C_2 Y_o = C_1 C_2/(Z_1 + Z_3) = C_1 C_2 Y_o \epsilon^{j(\psi_1 + \psi_2 - \varphi_o)}$; and the variable branch has an admittance equal to C_2^2 times the admittance of the "load" in the general circuit, i.e. $Y_b = C_2^2/Z_2 = C_2^2 Y_2 \epsilon^{j(\varphi_2 + 2\psi_2)}$.

In comparing the general circuit of Fig. 316 with its equivalent circuit, Fig. 318, it is found that, for any particular value of the load impedance in the general circuit, the same value is obtained for the joint impedance* of both circuits, and therefore if the same voltage is impressed on both circuits the line current and the power taken from the supply system will be the same for each circuit.

The voltage at the terminals of the parallel branches of the equivalent circuit is given by $(E_1 - I_1 Z_s)$, which, as is shown by equation (263), is equal to E_2/C_2, where E_2 is the voltage at the terminals of the load in the general circuit.

The current, I_b, in the variable branch of the equivalent circuit is given by

$$I_b = Y_b E_2/C_2 = E_2(C_2^2 Y_2)/C_2 = C_2 E_2 Y_2 = C_2 I_2 \quad . \quad (264)$$

Hence the power, P_b, supplied to the variable branch of the equivalent circuit is given by

$$\begin{aligned} P_b &= \frac{E_2}{C_2} C_2 I_2 \cos(\varphi_2 - 2\psi_2) \\ &= E_2 I_2 \cos(\varphi_2 - 2\psi_2) \\ &= P_2 \frac{\cos(\varphi_2 - 2\psi_2)}{\cos \varphi_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (265) \end{aligned}$$

where P_2 is the power supplied to the load in the general circuit. Thus, according to the signs and magnitudes of φ_2 and ψ_2 the

* The joint impedance of the general circuit of Fig. 316 is given by

$$Z = Z_1 + \frac{1}{(1/Z_3) + [1/(Z_2 + Z_4)]} = Z_1 + \frac{Z_3(Z_2 + Z_4)}{Z_2 + Z_3 + Z_4}$$

and that of the equivalent circuit is given by

$$Z_e = Z_s + [1/(C_2^2 Y_2 + C_1 C_2 Y_0)].$$

Substituting for Z_s, $C_2^2 Y_2$, $C_1 C_2 Y_0$ the values

$$Z_s = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4}; \quad C_2^2 Y_2 = \frac{1}{Z_2}\left(1 + \frac{Z_4}{Z_3}\right)^2;$$

$$C_1 C_2 Y_0 = \frac{Y_0}{(1 - Y_0 Z_s)} = Z_3 - \frac{Z_3 Z_4}{Z_3 + Z_4};$$

we have

$$\begin{aligned} Z_e = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4} + \frac{Z_3^2 Z_2}{(Z_3 + Z_4)^2 + Z_2(Z_3 + Z_4)} \\ = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4} + \frac{Z_2 Z_3^2}{(Z_3 + Z_4)(Z_2 + Z_3 + Z_4)} \\ = Z_1 + \frac{Z_3(Z_2 + Z_4)}{Z_2 + Z_3 + Z_4} \\ = Z \end{aligned}$$

power supplied to the variable branch of the equivalent circuit may be equal to, greater than, or less than the power supplied to the load in the general circuit, but the ratio of the quantities is constant for a given circuit.

The power, P_s, expended in the series impedance, Z_s, of the equivalent circuit is given by $P_s = I_1^2 R_s$, where R_s is the "short-circuit resistance" of the general circuit, and includes the resistance of the line impedance, Z_1, as well as the joint resistance of the branch impedances Z_3, Z_4, but not the resistance of the load. For a given value of the line current in both the equivalent and the general circuit, this power, P_s, closely approximates to the I^2R, or copper, losses in all parts, except the load, of the general circuit.

The power, P_a, supplied to the fixed branch of the parallel portion of the equivalent circuit is best expressed in symbolic notation and is given by

$$\dot{P}_a = [(E_2/C_2)^2/Z_a]\varepsilon^{j\varphi_a} = (E_2^2 Y_o C_1/C_2)\varepsilon^{j\varphi_a} \quad . \quad . \quad (266)$$

If the power supplied to the general circuit at no-load is denoted by P_o, then $\dot{P}_o = E_1^2 Y_o \varepsilon^{j\varphi_o}$. Therefore

$$\dot{P}_a(C_2/C_1) = \dot{P}_o(E_2/E_1)^2\varepsilon^{j(\varphi_a - \varphi_o)} = \dot{P}_a(E_2/E_1)^2\varepsilon^{-j(\psi_1 + \psi_2)}$$

or approximately, if $(\psi_1 + \psi_2)$ is a very small angle,

$$\dot{P}_a(C_2/C_1) = \dot{P}_o(E_2/E_1)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (266a)$$

Now the power supplied to the general circuit at no-load is expended in (1) the magnetic (hysteresis and eddy-current) losses in the iron portions of the circuit, (2) the losses in the dielectrics surrounding the conductors, and in the dielectrics of any condensers included in the circuit, (3) the I^2R losses due to the no-load current. Hence if the no-load I^2R losses are ignored, or P_o is corrected to allow for these losses, then the quantity $P_a(C_2/C_1)$ represents the magnetic and dielectric losses in the general circuit, since the dielectric losses vary as the square of the E.M.F. impressed when the frequency is constant, and the magnetic iron losses vary approximately as the square of the induced E.M.F.

[NOTE.—Magnetic losses are considered in detail in Chapter XI.]

Thus the performance of any circuit which can be reduced to a general series-parallel circuit may be obtained from the performance of the equivalent series-parallel circuit; the constants of which may be determined, as already shown, either directly from those of the general circuit, or indirectly from no-load and short-circuit tests as described later.

Deduction of Performance of General Circuit from the Load Diagram for the Equivalent Circuit. The no-load and short-circuit diagram

for the equivalent circuit is constructed in the same manner as that for the simple series-parallel circuit, but care must be exercised in obtaining the correct phase angles for the equivalent admittances and impedances. For example, the admittance vector, AB, Fig. 319, for the branch, Y_a, of constant admittance is set off at an angle equal to $(\pm \varphi_o \pm \psi_1 \pm \psi_2)$ with respect to the vertical axis, φ_o being the phase angle for the no-load admittance (i.e. cos φ_o is the no-load power factor of the general circuit), and ψ_1, ψ_2, are determined as shown later (p. 537). The length of this vector on the

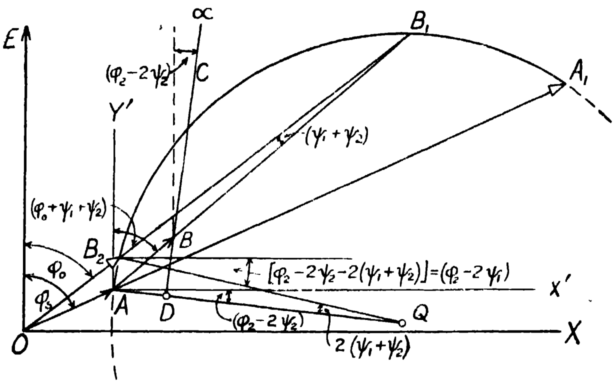


FIG. 319. NO-LOAD AND SHORT-CIRCUIT DIAGRAM FOR CIRCUIT OF FIG. 318

admittance scale is equal to $C_1C_2Y_o$. The admittance vector, $AB\infty$, Fig. 318, for the variable branch is then set off at an angle equal to $(\pm \varphi_2 \mp 2\psi_2)$ with respect to the vertical axis, cos φ_2 being the power factor of the load in the general circuit. The inversion of this line with respect to A gives, therefore, the locus of the joint impedance vector for the parallel branches of the equivalent circuit Observe that the line AQ, Fig. 319, is now inclined at an angle equal to $(\pm \varphi_2 \mp 2\psi_2)$ with respect to the horizontal axis.

The vector representing the series, or line, impedance in the equivalent circuit is then drawn in position, its inclination to the vertical axis being equal to $\pm \varphi_s$, where cos φ_s is the power factor for the general circuit when the load is short circuited. Finally, the inversion of the impedance circle with respect to the origin, O, is obtained, and the scale is changed so as to obtain the no-load and short-circuit points, B_2, A_1, respectively.

The load diagram, Fig. 320, is obtained by drawing the semi-polar, VU; the tangent, A_1V, at the short-circuit point; joining

the no-load and short-circuit points and producing this line so as to cut the horizontal axis at T; and drawing a line through the short-circuit point, A_1, and the point of intersection, U, of the semi-polar and the horizontal axis, exactly as in the case for the simple series-parallel circuit.

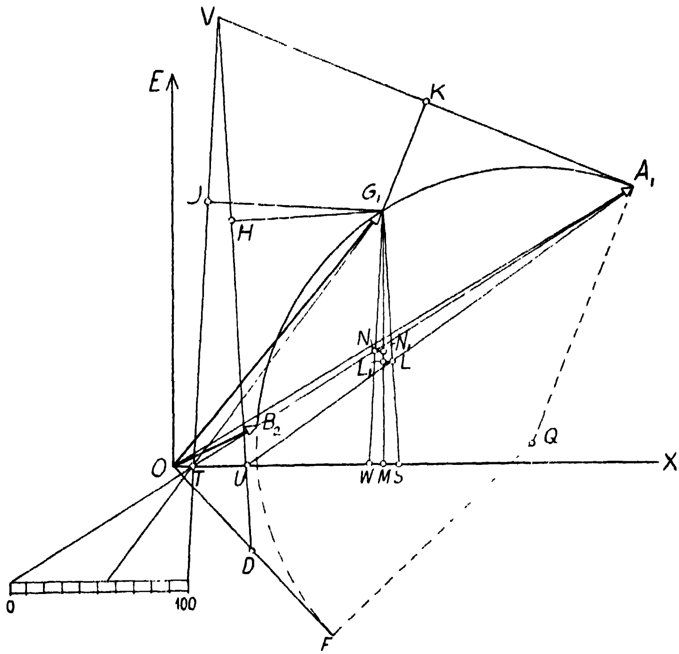


FIG. 320. LOAD DIAGRAM FOR CIRCUIT OF FIG. 318

The performance of the general circuit, when the line current is represented by the vector OG_1, is deduced from the load diagram as follows—

The *power input* from the supply system is given by the ordinate G_1M.

The *power supplied to the load* is given by $G_1N_1\,[\cos\varphi_2/\cos(\varphi_2 - 2\psi_2)]$ where G_1N_1 is the difference of the ordinates at G_1 and N; N being the point of intersection of the lines G_1W and A_1B_2, the former being drawn from G_1 parallel to VT and the latter being drawn through the no-load and short-circuit points.

The I^2R, or *copper losses,* in the circuit are given by L_1M, which is the ordinate at L; L being the point of intersection of the lines

G_1S and A_1U, the former being drawn from G_1 parallel to the semi-polar, and the latter being drawn through the short-circuit point and the point at which the semi-polar intersects the horizontal axis. This loss may be expressed as a percentage of the power input by constructing a "percentage I^2R loss" scale as follows: Draw any convenient line, parallel to the horizontal axis, to intersect the semi-polar, VU, and the line A_1U; divide the intercepted portion into 100 equal parts, the zero being at the point of intersection with the semi-polar. The percentage I^2R loss is then given by the scale reading at the point where the line joining the points G_1 and U intersects the scale.

The *magnetic, dielectric, and losses other than I^2R losses* are most conveniently obtained by calculating the difference of the power input and the power supplied to the load together with the I^2R losses. If the losses are to be obtained from the diagram, it is necessary to subtract from N_1L_1 (which is the difference of the ordinates at N and L) the quantity

$$G_1N_1\left(\frac{\cos\varphi_2}{\cos(\varphi_2 - 2\psi_2)} - 1\right)$$

The *scale* for the above quantities is the "ordinate power scale" which is E times the current scale of the diagram.

The *efficiency scale* is obtained by drawing any convenient line, parallel to the horizontal axis, to intersect the lines A_1T, VT, or their extensions. The portion thus intercepted is then divided into $100[\cos\varphi_2/\cos(\varphi_2 - 2\psi_2)]$ equal parts. The efficiency of the general circuit, i.e. the ratio

$$\frac{\text{power supplied to load}}{\text{power taken from supply system}}$$

is then given directly by the scale reading at the point where the line joining the points G_1 and T, produced if necessary, intersects the scale.

The *current in the load* is given by G_1B_2/C_2, where C_2 is the absolute value of the complex number $\underset{.}{C}_2$ The scale is the same as that for the line current.

The voltage at the load is given by $G_1A_1 \times C_2$.

Determination of the No-load and Short-circuit Constants of a General Circuit. The construction of the no-load and short-circuit diagram for an equivalent circuit involves a knowledge of the four quantities $\underset{.}{C}_1, \underset{.}{C}_2, \underset{.}{Y}_o, \underset{.}{Z}_s$. These quantities are called the "no-load" and "short-circuit" constants of the general circuit. As already

shown, they are not independent of one another, but are connected by equation (260), p. 528, which may be expressed in the form

$$C_1 C_2 = \frac{1}{1 - Y_o Z_s} = \frac{1/Z_s}{(1/Z_s) - Y_o}$$

If numerator and denominator are multiplied by E_1, we obtain

$$C_1 C_2 = \frac{E_1/Z_s}{(E_1/Z_s) - E_1 Y_o}$$

$$= I_s/(I_s - I_o) \quad . \quad . \quad . \quad . \quad . \quad (267)$$

where $I_s (= E_1/Z_s)$ is the short-circuit current, and $I_o (= E_1 Y_o)$ is the no-load current, taken by the general circuit when the supply voltage is E_1. Thus the product $C_1 C_2$ is equal to the quotient of the short-circuit current at normal supply voltage and the vector difference of the short-circuit current and the no-load current: it is therefore readily obtained from no-load and short-circuit tests. For example, in the no-load and short-circuit diagram, Fig. 318, the absolute value of the product $C_1 C_2$ is given by $C_1 C_2 = OA_1/A_1B_2$, and the argument, $\psi_1 + \psi_2$, is given by the angle OA_1B_2, the value of which depends upon the relative phase differences of the no-load and short-circuit currents, e.g.

$$\tan(\psi_1 + \psi_2) = [I_o \sin(\varphi_o - \varphi_s)]/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad (268)$$

In cases where the phase differences of the no-load and short-circuit currents, with respect to the supply voltage, are approximately equal, the angle OA_1B_2 will be small, and the value of the product C_1C_2 may be obtained by a simple calculation instead of by geometrical construction or complex algebra. For example, when φ_o is approximately equal to φ_s, the product C_1C_2 is given with sufficient accuracy for practical purposes by

$$C_1 C_2 = I_s/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad . \quad . \quad . \quad . \quad (269)$$

Moreover, with symmetrical circuits, i.e. those for which $Z_1 = Z_4$, $C_1 = C_2 = C$,* and $\psi_1 = \psi_2 = \psi$: whence, from equations (260), (267), we have

$$C = \sqrt{[1/(1 - Y_o Z_s)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (260a)$$

$$= \sqrt{[I_s/(I_s - I_o)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (267a)$$

$$C = \sqrt{[I_s/\{I_s - I_o \cos(\varphi_o - \varphi_s)\}]} \quad . \quad . \quad . \quad (269a)$$

$$\tan 2\psi = [I_o \sin(\varphi_o - \varphi_s)]/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad . \quad (268a)$$

* This follows from equation (258) by substituting $Z_4 = Z_1$, thus

$$C_2 = 1 + (Z_1/Z_3) = C_1$$

With unsymmetrical circuits, i.e. those in which Z_1 and Z_4 are unequal, the angles ψ_1, ψ_2 may, under suitable conditions, be determined by measuring directly the phase difference between the E.M.Fs. at the supply and load terminals at no-load, and the phase difference between the currents in the line and load at short circuit.* But if ψ_1 and ψ_2 are small it will be difficult to obtain these angles accurately by measurement. Hence in this case, and in other cases where it is impossible to measure these phase differences directly, the angles ψ_1, ψ_2, must be obtained indirectly by carrying out either a no-load or a short-circuit test from the load end of the circuit in addition to the ordinary no-load and short-circuit tests from the supply end.

For example, if in the circuit of Fig. 316 the supply and load are interchanged, i.e. the impedance Z_4 is placed in the supply circuit and the impedance Z_1 is placed in the load circuit, and if the no-load admittance and the short-circuit impedance under these conditions are denoted by Y'_o, Z'_s, respectively, we have

$$Y'_o = 1/(Z_3 + Z_4), \quad Z'_s = Z_4 + Z_1Z_3/(Z_1 + Z_3).$$

Now for the original circuit the no-load admittance and the short-circuit impedance are given by $Y_o = 1/(Z_1 + Z_3)$, $Z_s = Z_1 + Z_3Z_4/(Z_3 + Z_4)$, respectively.

Whence

$$\frac{Y'_o}{Y_o} = \frac{Z_1 + Z_3}{Z_3 + Z_4} = \frac{1 + Z_1/Z_3}{1 + Z_4/Z_3} = \frac{C_1}{C_2}$$

and

$$\frac{Z_s}{Z'_s} = \frac{Z_1 + [Z_3 Z_4/(Z_3 + Z_4)]}{Z_4 + [Z_1 Z_3/(Z_1 + Z_3)]} = \frac{[Z_1 (Z_3 + Z_4) + Z_3 Z_4] (Z_1 + Z_3)}{[Z_4 (Z_1 + Z_3) + Z_1 Z_3] (Z_3 + Z_4)}$$

$$= \frac{Z_1 + Z_3}{Z_3 + Z_4} = \frac{C_1}{C_2}$$

Therefore,

$$\frac{C_1}{C_2} = \frac{Y_o'}{Y_o} = \frac{Z_s}{Z_s'}$$

or

$$\frac{C_1}{C_2} = \frac{Y_o'}{Y_o} \varepsilon^{j(\varphi_o' - \varphi_o)} \frac{Z_s'}{Z_s} \varepsilon^{j(\varphi_s - \varphi_s')}$$

Whence

$$C_1/C_2 = Y_o'/Y_o = Z_s/Z_s' \quad \ldots \ldots \quad (270)$$

and

$$\psi_1 - \psi_2 = \varphi_o' - \varphi_o = \varphi_s - \varphi_s' \quad \ldots \ldots \quad (271)$$

* Methods of determining these quantities are described in Chapter XVIII.

or, since

$$2(\psi_1 - \psi_2) = \varphi_o' - \varphi_o + \varphi_s - \varphi_s',$$

$$\psi_1 - \psi_2 = \tfrac{1}{2}(\varphi_o' - \varphi_o + \varphi_s - \varphi_s') \quad . \quad . \quad . \quad . \quad (271a)$$

These equations and those (267), (268), given on page 535 enable the four quantities C_1, C_2, ψ_1, ψ_2, to be readily calculated. Thus

$$C_1 = \sqrt{[Y_o'/Y_o(1 - Y_o Z_s)]} = \sqrt{[I_o' I_s/I_o(I_s - I_o)]} \quad . \quad . \quad (272)$$

$$C_2 = \sqrt{[Y_o/Y_o' (1 - Y_o Z_s)]} = \sqrt{[I_o I_s/I_o' (I_s - I_o)]} \quad . \quad . \quad (273)$$

where I_o' is the no-load current, at normal voltage, when the supply and load are interchanged.*

$$\psi_1 = \tfrac{1}{2}[(\psi_1 + \psi_2) + (\psi_1 - \psi_2)]$$

$$= \tfrac{1}{2}\left[\tan^{-1}\frac{I_o \sin(\varphi_o - \varphi_s)}{I_s - I_o \cos(\varphi_o - \varphi_s)} + \tfrac{1}{2}(\varphi_s - \varphi_s' + \varphi_o' - \varphi_o)\right] \quad . \quad (274)$$

$$\psi_2 = \tfrac{1}{2}[(\psi_1 + \psi_2) - (\psi_1 - \psi_2)]$$

$$= \tfrac{1}{2}\left[\tan^{-1}\frac{I_o \sin(\varphi_o - \varphi_s)}{I_s - I_o \cos(\varphi_o - \varphi_s)} - \tfrac{1}{2}\{\varphi_s - \varphi_s' + \varphi_o - \varphi_o'\}\right] \quad . \quad (275)$$

The phase differences φ_o, φ_s, φ'_o, φ_s', are calculated from the power and volt-ampere inputs at no-load and short-circuit respectively, the power input being measured by a wattmeter and the volt-ampere input being measured by a voltmeter and ammeter. Thus—

$\cos\varphi_o = P_o/E_{1o}I_o$, where P_o is the power input, E_{1o} the supply voltage, and I_o the line current at no-load.

Similarly,

$\cos\varphi_s = P_s/E_{1s}I_{1s}$, where P_s is the power input, E_{1s} the supply voltage, and I_{1s} the line current at short-circuit.

Construction of the Load Diagram for a General Circuit from Test Data. If the impedances of the several branches of a general circuit are unknown, the no-load and short-circuit diagram must be constructed from data obtained from no-load and short-circuit tests. The data required are: (1) the no-load and short-circuit currents at normal supply voltage, (2) the phase differences of these currents

* Alternatively, if the voltages at the supply and load terminals can be measured accurately at no-load, and these voltages are denoted by E_{1o}, E_{2o}, respectively, then $C_1 = E_{10}/E_{20}$.

Again, if the currents in the line and load portions of the circuit can be measured under short-circuit conditions, and these currents are denoted by I_{1s}, I_{2s}, respectively, then $C_2 = I_{1s}/I_{2s}$.

with respect to the supply voltage, (3) either the phase difference, φ'_o, between the no-load current and the supply voltage when the supply and load terminals are interchanged, or the phase difference, φ'_s, between the short-circuit current and the supply voltage when the supply and load terminals are interchanged.

The vectors, OA_1, OB_2, Fig. 321 (*a*), representing the normal, no-load, and short-circuit currents are drawn in their correct positions with respect to the axis of reference, and the centre of the current circle is determined as follows: Join the no-load and short-circuit

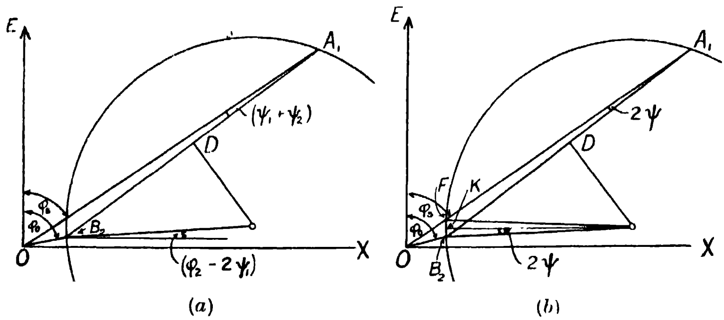


FIG. 321. CONSTRUCTION FOR DETERMINING CENTRE OF CURRENT CIRCLE

points, bisect this line at D, and draw the perpendicular DQ. From B_2 draw the line B_2Q inclined at the angle $(\pm\varphi_2 \mp 2\psi_1)$* with the horizontal axis B_2X'. Then the point, Q, where this line intersects the perpendicular DQ is the centre of the circle. Observe that the angle $(\varphi_2 - 2\psi_1)$, Fig. 321 (*a*), may be expressed as

$$[\varphi_2 - (\psi_1 + \psi_2) - (\psi_1 - \psi_2)] = \varphi_2 - \angle OA_1B_2 - (\psi_1 - \psi_2),$$

and if $(\psi_1 - \psi_2)$ is neglected the angle $X'B_2Q$ becomes equal to $(\varphi_2 - \angle OA_1B_2)$; the sign of the angle OA_1B_2 being determined with reference to the short-circuit line OA_1, i.e. the angle OA_1B_2 is positive when the no-load point lies above the short-circuit line, and *vice versâ*. Moreover, if $\varphi_2 = 0$, and $\psi_1 = \psi_2$, the angle $X'B_2Q$ is equal to the angle OA_1B_2, and the construction shown in Fig. 321 (*b*) may be adopted for obtaining the centre of the circle. In this case a vertical, B_2F, is erected from the no-load point to intersect the short-circuit line at F. The portion B_2F is bisected at K, and a perpendicular KQ is drawn to intersect the line DQ at Q, which is the centre of the circle.

* See remarks on p. 142 for meaning of signs.

Proof. In Fig. 319 the line AQ, which contains the centre of the circle, is inclined at the angle $-(\varphi_2 + 2\psi_2)$ with respect to the horizontal axis AX'. Now the angle AQB_2 is equal to twice the angle AB_1B_2, and since the latter is equal to $(\psi_1 + \psi_2)$, the angle AQB_2 is equal to $2(\psi_1 + \psi_2)$. Hence, since the angle $X''B_2Q$, Fig. 82, is equal to the difference of the angles $X'AQ$ and AQB_2, i.e. $\angle X''B_2Q = (\varphi_2 - 2\psi_2) - 2(\psi_1 + \psi_2)\ \varphi_2 - 2\psi_1$, a line drawn from B_2 at this angle with respect to the horizontal axis contains the centre of the circle.

With the construction shown in Fig. 321(*b*), which may be adopted when $\varphi_2 = 0$, and $\psi_1 = \psi_2$, we have

$$\angle B_2QK = \angle KQF = \tfrac{1}{2}\angle B_2QF = \angle B_2A_1O = 2\psi.$$

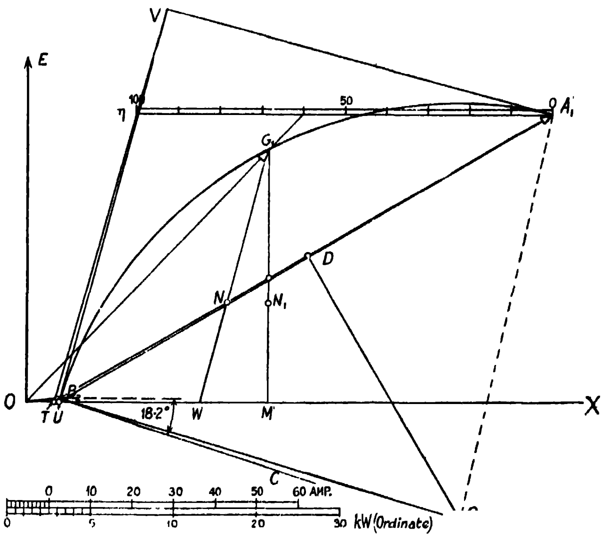


FIG. 322. LOAD DIAGRAM FOR SERIES-PARALLEL GENERAL CIRCUIT
(Data in Example and Table XIII)

Example. To illustrate the application of the above principles we will now calculate the performance of a symmetrical series-parallel general circuit for a variable load of 0·95 power factor (lagging), the no-load and short-test readings for the circuit being as follows—

No-load	Volts 500.	Amperes 8·25.	Watts 430.
Short-circuit	Volts 500.	Amperes 145.	Watts 34,800.

The performance will be calculated for a constant supply pressure of 500 volts

From the no-load and short-circuit readings we obtain

$$\cos \varphi_o = \frac{430}{500 \times 8{\cdot}25} = 0{\cdot}1044$$

$$\varphi_o = 84^\circ$$

$$\cos \varphi_s = \frac{34800}{500 \times 145} = 0{\cdot}485$$

$$\varphi_s = 61^\circ$$

Selecting a current scale of 1 cm. = 5 amperes, we commence the construction of the no-load and short-circuit diagram by drawing OB_2, OA_1, Fig. 322, to represent the no-load and short-circuit currents respectively: the length of OB_2 being 1·65 (= 8·25/5) cm. and its inclination to the vertical being 84°, the length of OA_1 being 29 (= 145/5) cm. and its inclination to the vertical being 61°.

Next join B_2A_1, bisect at D, and draw the perpendicular DQ.

Since the power factor of the load is 0·95, lagging, $\varphi_2 = \cos^{-1}0{\cdot}95 = 18{\cdot}2°$, and is negative. Also since the circuit is symmetrical and $\varphi_0 > \varphi_s$, the angle OA_1B_2 is equal to 2ψ and is positive. Hence from B_2 a line B_2Q is drawn inclined at the angle $-(\varphi_2 - 2\psi) = -(18{\cdot}2 - \angle OA_1B_2)°$ with respect

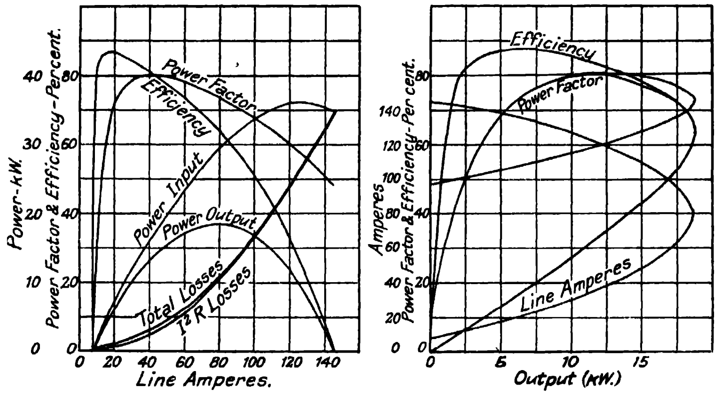


FIG. 323. PERFORMANCE CURVES OF GENERAL CIRCUIT (Determined from the Load Diagram of Fig. 322)

to the horizontal axis.* The point of intersection of this line with the perpendicular drawn from D is the centre of the current circle.

The diagram is completed by drawing the semi-polar, VU, the tangent, A_1V, at the short-circuit point, determining their point of intersection, V, and joining this point to the point, T, where the line joining the no-load and short-circuit points intersects the horizontal axis. The short-circuit point is also joined to the point, U, at which the semi-polar intersects the horizontal axis.

The power scales may be determined when the angles of inclination of VT and VU with respect to the vertical are known. By measurements on the diagram, Fig. 322, these angles are found to be 15·5° and 14·9° respectively. Hence the power scales are

1 cm. = 5 × 500 = 2500 watts, for measurements along the ordinate;

1 cm. = 2500 × cos 15·5° = 2408 watts, for measurements parallel to the "total loss line," VT;

1 cm. = 2500 × cos 14·9° = 2415 watts, for measurements parallel to the semi-polar, VU.

The value of 2ψ, obtained from equation (268a), is

$$\tan 2\psi = \frac{I_0 \sin(\varphi_0 - \varphi_s)}{I_s - I_0 \cos(\varphi_0 - \varphi_s)} = \frac{8{\cdot}25 \sin(84° - 61°)}{145 - 8{\cdot}25 \cos(84° - 61°)} = 0{\cdot}02345,$$

whence $2\psi = 1{\cdot}36$.

* This construction is best effected by drawing a line, B_2C, at an angle of 18·2° below the horizontal axis and then constructing the angle CB_2Q equal to the angle OA_1B_2.

The correction factor $[\cos\varphi_2/\cos(\varphi_2 - 2\psi)]$ for the efficiency scale and for obtaining the power supplied to the load from the intercept, G_1N, Fig. 322, is equal to

$$\cos 18{\cdot}2/\cos(18{\cdot}2 - 1{\cdot}36) = 0{\cdot}95/0{\cdot}9422 = 1{\cdot}008,$$

and is so close to unity that it may be neglected for practical purposes.

Finally, the value of C is calculated from equation (269a), thus

$$C = \sqrt{\frac{I_s}{I_s - I_0\cos(\varphi_0 - \varphi_s)}} = \sqrt{\frac{145}{145 - 8{\cdot}25\cos(84° - 61°)}} = \sqrt{1{\cdot}055} = 1{\cdot}027$$

The performance of the circuit as deduced from measurements on the load diagram, Fig. 322, is given in Table XIII, and in the curves of Fig. 323.

TABLE XIII

MEASURED AND CALCULATED QUANTITIES FROM FIG. 322 FOR PERFORMANCE OF SERIES-PARALLEL GENERAL CIRCUIT

Line current (amp.)		20	40	60	80	100	120
Length OG_1 (cm.)		4	8	12	16	20	24
Input	Length G_1W* (cm.)	3	6·64	9·73	12·28	14·07	15
	Power (kW.)	7·24	16·03	23·5	29·65	33·97	36·22
Output	Length G_1N (cm.)	2·6	5·42	7·12	7·76	7·05	5
	Power (kW.)	6·28	13·09	17·2	18·73	17·02	12·07
Total losses	Length NW (cm.)	0·4	1·22	2·61	4·52	7·02	10
	Power (kW.)	0·96	2·94	6·3	10·92	16·95	24·15
I^2R losses	Length LW† (cm.)	0·2	1·03	2·44	4·38	6·91	9·94
	Power (kW.)	0·48	2·49	5·89	10·57	16·68	24
Efficiency (per cent)		86·8	81·7	73·2	65·4	50·1	33·3
Power factor (per cent)		72·4	80	78·4	74·1	68	60·4

* Parallel to VU.

† L is point of intersection (not marked in Fig. 322) of GW and A_1U.

CHAPTER XXIII

INITIAL (TRANSIENT) CONDITIONS IN SIMPLE ELECTRIC CIRCUITS

In all the preceding discussions of electric circuits, particularly those given in Chapters III and IV, the relationship between impressed E.M.F. and current was obtained by *assuming* the law of variation of the current, e.g. by assuming a sinusoidal current to be flowing in the circuit. The conditions relating to the sudden application of a sinusoidal E.M.F. to circuits possessing resistance, inductance, and capacity will now be investigated.

Relationship Between Impressed E.M.F. and Initial Current for a Non-inductive Circuit. Let the impressed E.M.F. be represented by the equation $e = E_m \sin \omega t$. Then, if R denotes the resistance of the circuit, the internal E.M.F. (e_R) at any instant is given by $e_R = -Ri$, and this E M.F. must balance the impressed E.M.F.

Hence, for any value (e) of the impressed E.M.F. the current must equal e/R. Therefore, at whatever point on the E.M.F. wave the circuit is closed, the current rises instantly to the normal value corresponding to the value of the E.M.F. For example, if the circuit is closed at the instant the impressed E.M.F. attains its maximum value, the current rises immediately from zero to its maximum value. Thus the conditions become permanent as soon as the circuit is closed and there are no transient phenomena.

Relationship Between Impressed E.M.F. and Initial Current for a Purely Inductive Circuit. Let the inductance of the circuit be L and the resistance be zero. Then at every instant the impressed E.M.F. balances the E.M.F. of self-inductance, and the relationship between impressed E.M.F. and current is given by the equation

$$e = L\, di/dt = E_m \sin \omega t,$$

which may be re-arranged in the form

$$\frac{di}{dt} = \frac{e}{L} = \frac{E_m}{L} \sin \omega t.$$

Transposing, we have,

$$di = (E_m/L) \sin \omega t \,.\, dt,$$

and, upon integrating this expression, we obtain

$$i = -(E_m/\omega L) \cos \omega t + A \quad . \quad . \quad . \quad (276)$$

where A is the arbitrary constant of integration, the value of which is determined by the conditions existing at the instant the circuit is closed.

For example, let the circuit be closed when ωt has the value ψ. The current at this instant is zero. Hence, substituting for ωt and i in equation (276), we obtain

$$0 = - (E_m/\omega L) \cos \psi + A,$$

whence $\quad A = (E_m/\omega L) \cos \psi.$

Therefore, the general equation for the current is

$$\begin{aligned} i &= - (E_m/\omega L) \cos \omega t + (E_m/\omega L) \cos \psi \\ &= (E_m/\omega L)(\cos \psi - \cos \omega t) \quad . \quad . \quad . \quad (277) \end{aligned}$$

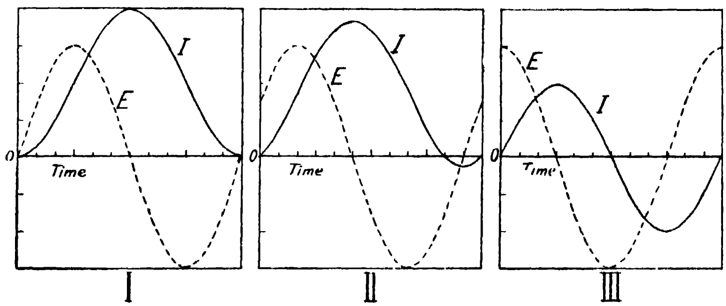


FIG. 324. WAVE-FORMS OF INITIAL CURRENT IN PURELY INDUCTIVE CIRCUIT

[NOTE.—The wave-form of the impressed E.M.F. is shown dotted.]

It can be shown that, except for the special cases when $\psi = 0°$ and $\psi = 90°$, this equation represents a sinusoidal curve with its axis displaced from the abscissa axis (*see* curves I, II, III, Fig. 324).

In the special case when $\psi = 0°$ the current curve does not cross the abscissa axis, and, therefore, the current never reverses (i.e. the current is pulsating instead of alternating).

In the other special case, when $\psi = 90°$, the current curve is symmetrical with respect to the abscissa axis, and the permanent conditions—as were considered in Chapter III—obtain from the instant at which the circuit is closed.

In Fig. 324 are shown the current wave-forms—calculated from equation (277)—corresponding to $\psi = 0°$, $\psi = 30°$, $\psi = 90°$, when the impressed E.M.F., at the instant of closing the circuit, has the values 0, 0·5 E_m, E_m.

Since equation (277) does not contain a transient term, the conditions represented by this equation are permanent, i.e. the succeeding current waves will be exact reproductions of the initial

wave, and the current will continue in the same form in which it started.

In practice, purely inductive circuits devoid of resistance and losses cannot be obtained, and with all highly inductive circuits the resistance, together with the losses in the iron core, produce damping effects, so that, a short time after the circuit is closed, the current wave-form assumes its normal shape.

Relationship Between Impressed E.M.F. and Initial Current for an Inductive Circuit (Resistance and Inductance in Series). The general equation connecting impressed E.M.F. and current is

$$e = Ri + L\,di/dt = E_m \sin \omega t,$$

which, when re-arranged, becomes

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_m}{L} \sin \omega t \qquad . \quad . \quad . \quad . \quad . \qquad (278)$$

This is a linear differential equation of the first order, the general form of which is

$$\frac{dy}{dx} + Py = Q.$$

Such an equation is integrated by the factor $\varepsilon^{\int P dx}$, and the general solution is

$$y\varepsilon^{\int P dx} = \int Q\varepsilon^{\int P dx}\, dx + c$$

or

$$y = \varepsilon^{-\int P dx} \int Q\varepsilon^{\int P dx}\, dx + c\varepsilon^{-\int P dx}$$

where c is the arbitrary constant of integration.

The general solution to equation (278) is therefore given by

$$i = \varepsilon^{-\int (R/L)dt} \frac{E_m}{L} \int \varepsilon^{\int (R/L)dt} \sin\ \omega t\ .\ dt + c\varepsilon^{-\int (R/L)dt}$$

Evaluating the supplementary integrals, we have

$$i = \varepsilon^{-(R/L)t} \frac{E_m}{L} \int \varepsilon^{(R/L)t} \sin \omega t\ .\ dt + c\varepsilon^{-(R/L)t}$$

The remaining integral is of the form

$$\int \varepsilon^{ax} \sin bx\ .\ dx,$$

which, when integrated, gives

$$\frac{(a \sin bx - b \cos bx)}{a^2 + b^2} \varepsilon^{ax}$$

Hence the solution to equation (278) is

$$i = \frac{E_m}{L[(R/L)^2 + \omega^2]}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right) + c\varepsilon^{(R/L)t}$$

Simplifying, we have

$$i = \frac{E_m}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t) + c\varepsilon^{-(R/L)t}$$

$$= \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}}\left\{\frac{R}{\sqrt{(R^2 + \omega^2 L^2)}}\sin \omega t - \frac{\omega L}{\sqrt{(R^2 + \omega^2 L^2)}}\cos \omega t\right\} + c\varepsilon^{(R/l)t}$$

$$= \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}}\sin(\omega t - \varphi) + c\varepsilon^{-(R/L)t} \quad . \quad . \quad . \quad (279)$$

$$= I_m \sin(\omega t - \varphi) + c\varepsilon^{-(R/L)t} \quad . \quad . \quad . \quad (279a)$$

where $\quad I_m = E_m/\sqrt{(R^2 + \omega^2 L^2)}$

and $\quad \tan \varphi = \omega L/R.$

The first portion of equation (279) is the *particular* solution of equation (278), which is obtained when t has large values. This (particular) solution corresponds to the permanent state of the current in the circuit.

The second, or transient, term in equation (279) is the solution of the equation $L\,di/dt + Ri = 0$. This term takes into account the initial conditions in the circuit, viz., that initially the current is zero, and, therefore, when an E.M.F. is applied suddenly to the circuit, the current must start from zero at whatever point on the E.M.F. wave the circuit is closed.

The value of the arbitrary constant of integration (c) is determined by the conditions existing when the circuit is closed.

If the circuit is closed at a time t_1 after the E.M.F. has passed through its zero value (which corresponds to $t = 0$), the current at this instant (t_1) is zero, and, on substituting in equation (279a), we obtain

$$0 = I_m \sin(\omega t_1 - \varphi) + c\varepsilon^{-(R/L)t_1}$$

whence $\quad c = -I_m \varepsilon^{(R/L)t_1} \sin(\omega t_1 - \varphi).$

Therefore the general equation for the current becomes

$$i = I_m \sin(\omega t - \varphi) - I_m \varepsilon^{(t_1 - t)R/L} \sin(\omega t_1 - \varphi) \quad . \quad (280)$$

Observe that at the instant of closing the circuit (i.e. $t = t_1$), the value of the exponential term is equal to that of the term representing the permanent conditions in the circuit.

This equation, except for the special case when $\omega t_1 = \varphi$, represents, when t is small, a series of unsymmetrical periodic curves. The dissymmetry gradually decreases with respect to time and ultimately vanishes (*see* Fig. 325).

When $\omega t_1 = \varphi$, the second, or exponential, term in equation (280) is zero and the permanent conditions exist (i.e. the current is sinusoidal) from the instant that the circuit is closed.

A graphical representation of equation (280), in the case of a particular circuit, is given in Fig. 325 (curve i), the applied E.M.F.

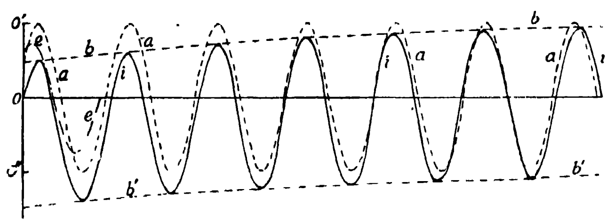


FIG. 325. WAVE-FORMS OF INITIAL CURRENT IN CIRCUIT CONTAINING RESISTANCE AND INDUCTANCE

being shown at e. The points on the current curve may be calculated directly by equation (280), but it is easier to obtain the current curve in the following manner—

First plot the current curve corresponding to the permanent conditions. This is a sine curve (a, Fig. 325) having a maximum value equal to $E_m/\sqrt{(R^2 + \omega^2 L^2)}$ and lagging φ with respect to the E.M.F. curve.

Next calculate a few points and draw the exponential curve (b). The calculated points are obtained from the equation

$$i' = I_m \varepsilon^{(t_1 - t)R/L} \sin(\omega t_1 - \varphi).$$

Finally, compound the curves according to equation (280)—i.e. subtract (b) from (a)—and obtain the resultant curve i.

[NOTE. Fig. 325 refers to the case in which L is constant. If, however, the magnetic circuit contains iron, the initial state of magnetization of the iron, together with the hysteresis loop, will have to be taken into consideration in determining the initial current and its wave-form. In this case a graphical construction, similar to that described in Chapter XV, will have to be employed.]

Relationship Between Impressed E.M.F. and Initial Current for a Circuit Containing only Capacitance. Let an E.M.F., represented by the equation $e = E_m \sin \omega t$, be applied to a condenser of capacitance

C. Then, since the charging current of the condenser at any instant is equal to $C\,de/dt$, we have

$$i = C\frac{de}{dt} = C\frac{d}{dt}E_m \sin \omega t$$
$$= \omega C\, E_m \cos \omega t$$
$$= \omega C\, E_m \sin(\omega t + \tfrac{1}{2}\pi)$$

which is the equation (25) deduced in Chapter IV for the permanent conditions in the circuit.

Hence in this case—i.e. if the resistance is zero and there are no losses—there are no transient phenomena on closing the circuit.

If, however, the switch makes bad contact, so that sparking occurs at the contacts, transient conditions will occur, as the circuit is then equivalent to a condenser and a resistance (which possesses unstable characteristics) in series.

Relationship Between Impressed E.M.F. and Initial Current for a Series Circuit Containing Capacitance and Resistance. Let a sinusoidal E.M.F.—represented by $e = E_m \sin \omega t$—be applied to a series circuit containing a condenser of capacitance C, and a resistance R. Then the general equation of the circuit is

$$e = Ri + (1/C)\int i\, dt.$$

Differentiating with respect to time, we obtain

$$\frac{de}{dt} = R\frac{di}{dt} + \frac{i}{C}$$

Transposing and re-arranging terms, we have

$$\frac{di}{dt} + \frac{1}{RC}i = \frac{1}{R}\frac{de}{dt} = \frac{\omega E_m}{R}\cos \omega t,$$

which is a linear differential equation of the first order.

The solution of this equation is obtained in exactly the same manner as that for an inductive circuit (*see* p. 544), and is

$$i = \frac{E_m}{\sqrt{[R^2 + (1/\omega C)^2]}}\sin(\omega t + \varphi) + A\varepsilon^{-t/RC} \quad . \quad . \quad (281)$$
$$= I_m \sin(\omega t + \varphi) + A\varepsilon^{-t/RC} \quad . \quad . \quad . \quad . \quad (281a)$$

This equation differs from equation (279*a*) in the time constant and the sign of the phase-angle φ. Except in the special case when the circuit is closed at the instant the impressed E.M.F. is passing through zero, the initial current waves are represented by a series of unsymmetrical waves similar to those shown in Fig. 325.

Relationship Between Impressed E.M.F. and Initial Current for a Series Circuit Containing Capacitance, Resistance, and Inductance. Let the applied E.M.F. be represented by the equation $e = E_m \sin \omega t$, and let R, L, C denote the resistance, inductance, and capacitance, respectively, of the circuit. Then the general equation of the circuit is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e = E_m \sin \omega t \quad . \quad . \quad . \quad (282)$$

Differentiating throughout in order to clear the integral, we obtain

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \frac{de}{dt} = \omega E_m \cos \omega t \quad . \quad . \quad . \quad (283)$$

Differentiating again, we obtain

$$R\frac{d^2i}{dt^2} + L\frac{d^3i}{dt^3} + \frac{1}{C}\frac{di}{dt} = -\omega^2 E_m \sin \omega t$$

Substituting from equation (282), we have

$$L\frac{d^3i}{dt^3} + R\frac{d^2i}{dt^2} + \frac{1}{C}\frac{di}{dt} = -\omega^2\left(Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt\right)$$

Differentiating again for the purpose of clearing the integral, and re-arranging, we have

$$CL\frac{d^4i}{dt^4} + CR\frac{d^3i}{dt^3} + (1 + \omega^2 CL)\frac{d^2i}{dt^2} + \omega^2 CR\frac{di}{dt} + \omega^2 i = 0$$

The solution to this equation may be expressed in the form

$$i = A_1\varepsilon^{m_1 t} + A_2\varepsilon^{m_2 t} + A_3\varepsilon^{m_3 t} + A_4\varepsilon^{m_4 t}$$

where A_1, A_2, A_3, A_4 are constants and m_1, m_2, m_3, m_4 are the roots of the equation

$$CLx^4 + CRx^3 + (1 + \omega^2 CL)x^2 + \omega^2 CRx + \omega^2 = 0$$

The roots are readily obtained, since this equation can be factorized thus

$$(x^2 + \omega^2)(CLx^2 + CRx + 1) = 0$$

Whence the roots are

$$m_1 = \omega\sqrt{-1}, \qquad m_2 = -\omega\sqrt{-1}$$

$$m_3 = -\frac{R}{2L} + \frac{\sqrt{(R^2 - 4L/C)}}{2L}, \quad m_4 = -\frac{R}{2L} - \frac{\sqrt{(R^2 - 4L/C)}}{2L}$$

Therefore the general solution is

$$= A_1\varepsilon^{j\omega t} + A_2\varepsilon^{-j\omega t} + \varepsilon^{(R/2L)t}\left(A_3\varepsilon^{t\sqrt{(R^2-4L/C)}/2L} + A_4\varepsilon^{-t\sqrt{(R^2-4L/C)}/2L}\right) \quad . \quad . \quad (284)$$

The first terms represent the particular solution and correspond to the permanent conditions in the circuit; the last term represents the transient conditions. This term corresponds to the solution of the equation

$$L\, di^2/dt^2 + R\, di/dt + i/C = 0$$

Observe that the transient term may take one of three forms, according to whether CR^2 is $> = < 4L$.

The particular solution—i.e. $i = A_1\varepsilon^{j\omega t} + A_2\varepsilon^{-j\omega t}$— reduces to

$$\begin{aligned} i &= A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t) \\ &= (A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t \\ &= A\cos\omega t + B\sin\omega t \quad . \quad . \quad . \quad . \quad (285) \end{aligned}$$

where $A = A_1 + A_2$, $B = j(B_1 + B_2)$.

The constants A and B are evaluated by substituting in equation (283) the value of i given by equation (285). Thus

$$\begin{aligned} di/dt &= -\omega A\sin\omega t + \omega B\cos\omega t, \\ d^2i/dt^2 &= -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t \end{aligned}$$

Whence, from equation (283), we have

$$\begin{aligned} &-\omega^2 L(A\cos\omega t + B\sin\omega t) + \omega R(B\cos\omega t - A\sin\omega t) \\ &+ (A\cos\omega t + B\sin\omega t)/C = \omega E_m\cos\omega t \end{aligned}$$

Equating sine and cosine terms we have

(i) $$-\omega^2 LB - \omega RA + B/C = 0$$

(ii) $$-\omega^2 LA + \omega RB + A/C = \omega E_m$$

from which $$A = \frac{E_m(1/\omega C - \omega L)}{(1/\omega C - \omega L)^2 + R^2}$$

$$B = \frac{E_m R}{(1/\omega C - \omega L)^2 + R^2}$$

Substituting these quantities in equation (285), we have

$$i = \frac{E_m}{R^2 + (1/\omega C - \omega L)^2}\left[\left(\frac{1}{\omega C} - \omega L\right)\cos\omega t + R\sin\omega t\right]$$

or, if $\omega L > 1/\omega C$, as is usually the case,

$$\begin{aligned} i &= \frac{E_m}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\sin(\omega t - \varphi) \quad . \quad (286) \\ &= I_m\sin(\omega t - \varphi) \quad . \quad . \quad . \quad . \quad . \quad (286a) \end{aligned}$$

where $\quad I_m = E_m/\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$

and $\quad \varphi = \tan^{-1}(\omega L - 1/\omega C)/R.$

This equation is, of course, identical with that deduced in Chapter VI by assuming permanent conditions in the circuit.

The general equation (284) may, therefore, be written in the form

$$i = I_m \sin(\omega t - \varphi) + \varepsilon^{-(R/2L)t}\left(A_3\varepsilon^{t\sqrt{(R^2-4L/C)}/2L} + A_4\varepsilon^{-t\sqrt{(R^2-4L/C)}/2L}\right) \quad . \quad . \quad (287)$$

The evaluation of the transient term of this equation is made according to whether R is $> = < 2\sqrt{(L/C)}$.* Each case will be considered separately.

Case I. $R < 2\sqrt{(L/C)}$. This case is more interesting than the others, as the conditions correspond to an oscillatory discharge when a condenser discharges through an inductive resistance, the constants of which satisfy the above relationship, i.e. $R^2 < 4L/C$.

Denote $\sqrt{(4L/C - R^2)}/2L$ by λ. Then we may write

$$\sqrt{(R^2 - 4L/C)}/2L = j\lambda.$$

Accordingly, the transient term becomes

$$\varepsilon^{-(R/2L)t}\left(A_3\varepsilon^{j\lambda t} + A_4\varepsilon^{-j\lambda t}\right) = A'\varepsilon^{-(R/2L)t}\sin(\lambda t + \varphi')$$

and equation (287) takes the form

$$i = I_m \sin(\omega t - \varphi) + A'\varepsilon^{-(R/2L)t}\sin(\lambda t + \varphi') \qquad (288)$$

where the constants A', φ', include the constants A_3, A_4, and are to be determined from the initial conditions.

As two constants are involved, we must have two equations connecting the initial conditions. One equation connects current and time; the other connects charge (q) and time. The latter is obtained from the former, since $q = \int i\,dt$. Hence, by integrating equation (288) we have†

$$q = -(I_m/\omega)\cos(\omega t - \varphi) + \varepsilon^{-(R/2L)t}A'\sqrt{(LC)}\sin(\lambda t + \varphi' + \varphi'') \quad . \quad (289)$$

where $\quad \varphi'' = \tan^{-1}\sqrt{(4L/C - R^2)}/R.$

* The quantity $\sqrt{(L/C)}$ is called the "surge impedance" (sometimes the "natural impedance") of the circuit, and is of importance in connection with surges and similar disturbances.

† NOTE. $\int e^{ax}\sin bx = e^{ax}\left(\dfrac{a\sin bx - b\cos bx}{a^2 + b^2}\right)$

$$= e^{ax}\left(\frac{\sin[bx - \tan^{-1}(b/a)]}{\sqrt{(a^2 + b^2)}}\right).$$

If the condenser is initially uncharged, then, at the instant (t_1) of closing the circuit, $i = 0$, $q = 0$. Hence, substituting in equations (288), (289), we have

$$0 = I_m \sin(\omega t_1 - \varphi) + A' \varepsilon^{-(R/2L)t_1} \sin(\lambda t_1 + \varphi')$$

and

$$0 = -(I_m/\omega)\cos(\omega t_1 - \varphi) + \varepsilon^{-(R/2L)t_1} A' \sqrt{(LC)} \sin(\lambda t_1 + \varphi' + \varphi'')$$

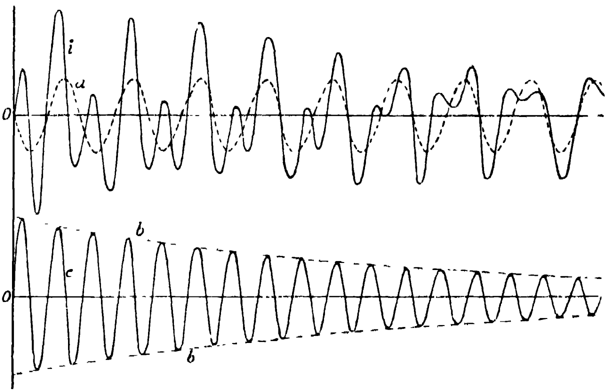


FIG. 326. WAVE-FORMS OF INITIAL CURRENT IN CIRCUIT, CONTAINING RESISTANCE, INDUCTANCE, AND CAPACITANCE IN SERIES

[NOTE.—The dotted sine curve, *a*, represents the current wave-form corresponding to the steady-state, or permanent, conditions.]

Whence

$$\varphi' = \cot^{-1} - \left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega \sqrt{(4LC - C^2R^2)}} \right) - \lambda t_1$$

$$A' = -I_m \varepsilon^{+(R/2L)t_1} \sin(\omega t_1 - \varphi) / \sin\left[\cot^{-1} - \left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega\sqrt{(4LC - C^2R^2)}} \right) \right]$$

$$= -\frac{2I_m \varepsilon^{(R/2L)t}}{\omega\sqrt{(4LC - C^2R^2)}} \sqrt{[(\omega^2 LC - 1)\sin^2(\omega t_1 - \varphi) + \tfrac{1}{2}\omega CR \sin 2(\omega t_1 - \varphi) + 1]}$$

Substituting these values in equation (288), we have

$$i = I_m \sin(\omega t - \varphi) - \frac{2I_m \sqrt{[(\omega^2 LC - 1)\sin^2(\omega t_1 - \varphi) + \frac{1}{2}\omega CR \sin 2(\omega t_1 - \varphi) + 1]}}{\omega\sqrt{(4LC - C^2R^2)}} \cdot \varepsilon^{-(t - t_1)R/2L} \sin\left[\lambda(t - t_1) + \cot^{-1} - \left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega\sqrt{(4LC - C^2R^2)}} \right) \right] \quad (290)$$

This equation shows that the initial wave-forms of the current may be of a very complex nature, and examples are given in Fig. **326**.

Case II. $R = 2\sqrt{(L/C)}$. This corresponds to the critical case for the discharge of a condenser through an inductive resistance as the discharge is then just non-oscillatory.

The transient term of equation (287) now takes the form

$$A_3\varepsilon^{-(R/2L)t} + A_4\varepsilon^{-(R/2L)t}$$

$$\text{or} \qquad \varepsilon^{-(R/2L)t}(A_3 + A_4 t)$$

and accordingly equation (287) becomes

$$i = I_m \sin(\omega t - \varphi) + \varepsilon^{-(R/2L)t}(A_3 + A_4 t) \qquad . \qquad . \qquad (291)$$

The constants A_3, A_4, must be determined from the initial conditions.

By integrating equation (291) we obtain

$$\int i dt = q = -\frac{I_m}{\omega}\cos(\omega t - \varphi) - 2\frac{L}{R}\varepsilon^{-t(R/2L)}(A_3 + A_4 t + 2\frac{L}{R}A_4).$$

If initially the condenser is uncharged, then at the instant (t_1) of closing the circuit, we have

$$0 = I_m \sin(\omega t_1 - \varphi) + (A_3 + A_4 t_1)\varepsilon^{-(R/2L)t_1}$$

$$\text{and} \quad 0 = -\frac{I_m}{\omega}\cos(\omega t_1 - \varphi) - 2\frac{L}{R}\left[A_3 + A_4\left(t_1 + 2\frac{L}{R}\right)\right]\varepsilon^{-(R/2L)t}$$

$$\text{whence} \quad A_3 = -I_m\varepsilon^{(R/2L)t_1}\left[\left(1 + \frac{R}{2L}t_1\right)\sin(\omega t_1 - \varphi) - \frac{R^2 t_1}{4\omega L^2}\cos(\omega t_1 - \varphi)\right]$$

$$A_4 = \frac{R}{2L}I_m\varepsilon^{(R/2L)t_1}\left[\sin(\omega t_1 - \varphi) - \frac{R}{2\omega L}\cos(\omega t_1 - \varphi)\right]$$

EXAMPLES

THIS collection of examples includes questions set at the following examinations—

University of London, B.Sc. (Eng.)*—Reference *L.U.*

Institution of Electrical Engineers, A.M.I.E.E. Examination—Reference *I.E.E.*

City and Guilds of London Institute, Electrical Engineering Practice—Reference *C.G.*

The examples have been grouped according to subject matter and chapter.

I.—GENERAL PRINCIPLES AND FORMS OF REPRESENTATION (CHAPTERS I AND II)

1. Write down (*a*) the maximum value, (*b*) the root-mean-square value, (*c*) the frequency of the following—
(1) $100 \sin 500t$; (2) $(A + B) \sin (3\omega t - \theta + \varphi)$; (3) $P \cos \omega t - Q \sin \omega t$; (4) $A \sin (\omega t + \tan^{-1} b/a)$.
Write down also the phase difference with respect to $B \sin(\omega t - a)$ in cases (3) and (4).

2. Plot curves, in rectangular co-ordinates, for one cycle of the quantities $a = 10 \sin(\omega t + 10°)$, $b = 5 \sin(\omega t - 30°)$, and determine graphically the curve representing the sum of these quantities.

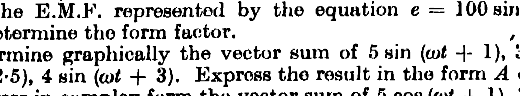
3. A sinusoidal current of 50 frequency has an R.M.S. value of 10 A. Write down the equation for the instantaneous value. If time is measured from a maximum positive value, what will be the instantaneous value after a lapse of (*a*) 0·0025 sec., (*b*) 0·0125 sec.? At what time, measured from a positive maximum value, will the instantaneous value be 7·07A.?

4. A circuit consists of two portions, *AB* and *BC*. The voltage across *AB* is 60 V., that across *BC* is 100 V., and their phase difference is 45°. What is the voltage between the terminals *A* and *C*, and what is its phase difference with respect to the voltage across *BC*.

5. Define "form factor" in connection with alternating E.M.F. wave forms. Calculate this factor in the case of (i) a V-shaped wave, (ii) a sine wave. For what purposes is a knowledge of form factor required? (*L.U.*)

6. An alternating current has the following values after equal time intervals beginning at zero: 0, 3, 4, 4·5, 5·5, 8, 10, 6, 0, – 3, – 4, –4·5, – 5·5, – 8, – 10. – 6, 0. Find the R.M.S. value and form factor.

7. Calculate the form factor of the wave-forms shown in the accompanying diagram.



(*a*) (*b*) (*c*) (*d*)

8. Plot the E.M.F. represented by the equation $e = 100 \sin \omega t + 13·4 \sin 3\omega t$. Determine (*a*) the R.M.S. value, (*b*) the mean value, (*c*) the form factor.

9. Plot the E.M.F. represented by the equation $e = 100 \sin \omega t - 13·4 \sin 3\omega t$, and determine the form factor.

10. Determine graphically the vector sum of $5 \sin (\omega t + 1)$, $3 \cos (\omega t - 1)$, $2 \sin (\omega t - 2·5)$, $4 \sin (\omega t + 3)$. Express the result in the form $A \cos (\omega t + \varphi)$.

11. Express in complex form the vector sum of $5 \cos (\omega t + 1)$, $3 \sin (\omega t - 1)$, $- 2 \cos (\omega t - 2·5)$.

12. Evaluate $(10 + j5)(5 - j3) / (4 - j3)$.

13. Show that if a conductor transmits simultaneously a sinusoidal

* The Engineering Examination Papers of the University of London are published regularly by the University of London Press.

alternating current of maximum value I_m, and a direct current, I_D, the I^2R loss in the conductor is equal to $\sqrt{(I_D^2 + \frac{1}{2}I_m^2)}/(I_D + I_m/\sqrt{2})$ of that which would be obtained if the two currents were transmitted over the same distance by separate conductors, the current densities being the same in the two cases.

II.—INDUCTANCE (CHAPTER III)

1. Prove the relationship between impressed E.M.F and current for an inductive circuit which is carrying a sinusoidal alternating current.

Such a circuit having a resistance of 15 Ω. and an inductance of 0·03 H is connected to a 200 V., 50-cycle supply system. Determine the current and the phase difference between current and impressed E.M.F.

2. An inductive circuit has a resistance of 60 Ω. and an inductance of 0·36 H., and is supplied at 220 V., 50 frequency. Calculate (*a*) the reactance of the circuit; (*b*) the impedance; (*c*) the current; (*d*) the phase difference between impressed E.M.F. and current.

3. An inductive coil when connected to a 100 V., 50-cycle supply takes a current of 1·3 A. When the coil is connected to a 24 V. D.C. supply the current is 1 A. Determine the inductance of the coil.

4. An inductive coil, having $R = 250$ Ω., $L = 1$ H, is connected to 230 V., 50-cycle mains. Calculate (1) the current; (2) the phase difference of this current with respect to the supply voltage; (3) the energy stored in the magnetic field of the coil at the instant when the supply voltage is (i) zero, (ii) a maximum.

5. A voltage which follows the law $e = 5t$ (where e is in volts and t in sec.) is applied to a reactor having an inductance of 0·5 H. and negligible resistance. The connection is made when $t = 0$ and is broken when $t = 10$. Find the R.M.S. value of the current during this period. Also find the energy stored in the reactor at the moment of switching off. Neglect all losses in the reactor. (*L.U.*)

6. A low-voltage release consists of a solenoid into which an iron plunger is drawn against a spring. The resistance of the solenoid is 28·5 Ω. When connected to 100 V. 50-cycle mains the current taken is at first 1·12 A., and falls to 0·43 A. when the plunger is drawn into the solenoid. Calculate the inductance of the solenoid and the maximum flux-turn linkages for both positions of the plunger. (*L.U.*)

7. The reluctance of the core of a certain transformer is 0·002. Find the coefficient of mutual inductance between the primary and secondary coils which have 1250 and 40 turns respectively, neglecting magnetic leakage.

8. A solenoid is wound with 1068 turns in a length of 60·4 cm. and is fitted with an internal search having 242 turns and a mean cross section of 16·2 sq. cm., the search coil occupying a symmetrical position at the centre of the solenoid. Find the E.M.F. induced in the search coil when an alternating current of 2 A. at 50 frequency is passed through the solenoid.

9. A solenoid, 30 cm. long, has 300 turns uniformly distributed and is supplied with an alternating current of 2 A. at a frequency of 1000 c.p.s. A small single-layer coil of 50 turns, with an end area of 10 cm. is placed inside the solenoid near the middle. Calculate the mutual inductance of the two coils and the E.M.F. induced in the smaller one when their axes are parallel. The permeability of the medium may be taken as unity. (*L.U.*)

10. The mutual inductive reactance between two similar circuits placed near together is 5 Ω. Each circuit alone has a resistance of 12 Ω. and a reactance of 9 Ω. When a current of 0·5 A. circulates in the closed secondary circuit, what is the current in the primary circuit? What voltage must be applied to the primary to establish this current? (*L.U.*)

11. Two coils A, B, of impedance $R_1 + j\omega L_1$ and $R_2 + j\omega L_2$ respectively are so wound on a wooden core that their mutual inductance is M. Find expressions for the currents in the coils when an alternating voltage V is established across coil A and coil B is closed by an impedance Z. How will these expressions be modified if the wooden core is replaced by iron? State any assumptions made. (*L.U.*)

12. Two wires are wound together on a core. At one end A the wires are connected together but the other ends B and C are kept separate. One end of a resistance R is connected to A, and between the other end, D, of R and B a voltage of 100 V. at 50 c.p.s. is applied. D is also connected to C.

If each coil has a resistance of 30 Ω. and an inductance of 0·5 H., and if the mutual inductance between the coils is 0·3 H., for what value of R will the currents in the coils be in exact phase opposition, and what will be the currents in the coils? (*L.U.*)

III.—CONDENSERS AND CAPACITANCE (CHAPTER IV)

1. Deduce an expression for the capacitance of a cylindrical condenser in terms of its dimensions and the specific inductive capacity of the dielectric.

A condenser consists of two co-axial cylinders, the outer diameter of the inner cylinder being 3 cm. and the inner diameter of the outer cylinder 3·5 cm. If the tubes are 50 cm. long and the space between them is filled with bakelite having a specific inductive capacity of 4, calculate the capacitance. State the units in which the result is given. (*L.U.*)

2. Calculate the capacitance of a 5-mile length of single-core cable for which the conductor diameter is 0·58 in., the thickness of insulation 0·18 in. and the dielectric constant 3.

3. Compare the properties of (*a*) air, (*b*) waxed paper, (*c*) mica when used as the dielectric of a condenser.

Calculate the capacitance of a condenser consisting of 49 metal sheets, each 5 cm. by 2 cm., the dielectric consisting of mica sheets (dielectric constant 6·5) each 0·003 cm. thick. Assume that the assembly is compressed until all air is expelled from between the plates. (*L.U.*)

4. A condenser is made up of two parallel circular metal discs separated by three layers of dielectric of equal thickness but having permittivities (dielectric constant) of 3, 4, and 5 respectively. The diameter of each disc is 25·4 cm. and the distance between them is 6 mm. Calculate the potential gradient in each dielectric and the energy stored in each when a steady P.D. of 1500 V. is applied between the discs. Neglect fringing. (*C.G.*)

(*Note.* Potential gradient = voltage/thickness of dielectric.)

5. A condenser of 20 μF. capacitance is connected in series with a 100 Ω. resistance to a 230 V., 50-cycle supply. Calculate the energy stored in the electric field of the condenser at an instant when the supply voltage is a maximum. (*L.U.*)

6. What is meant by the "charging current" of a condenser connected to an alternating current supply system? Draw a diagram showing relationship between charging current and impressed E.M.F.

Calculate the charging current of a 15 μF. condenser when supplied at 220 V., 50 frequency.

7. Two condensers having capacitances of 10 μF. and 20 μF. are connected (i) in parallel, (ii) in series, and the combination is supplied at 250 V., 50 frequency. Determine the line currents in each case.

8. Two condensers having capacitances of 0·05μF. and 0·12 μF. are connected in series across a 2500 V., 50-cycle circuit. Calculate the voltage across each condenser.

9. A condenser and a resistance are connected in series across 110 V., 50-cycle mains. Calculate the capacitance of the condenser in order that the resistance shall absorb 30 W. at 50 V.

10. A coil having R = 20 Ω. and L = 0·3 H. is connected in series with a 30 μF. condenser to a supply of 230 V. at 50 c.p.s.

Calculate (i) the current in the circuit, (ii) the voltage across the coil, (iii) the voltage across the condenser.

IV.—POWER AND POWER FACTOR (CHAPTER V)

1. What is meant by the "power factor" of a circuit?

Show how the power factor of a circuit may be determined experimentally, and draw a diagram of connections.

Calculate the power factor of an inductive resistance which takes a current of 5 A. and a power of 100 W. at 100 V.

2. What is the power factor of the circuit in Example 9 (III).

3. Calculate the current in, and the power factor of, a circuit of which the resistance is 5 Ω. and the inductance is 0·01 H. when the supply pressure is 200 V., 50 frequency.

4. Calculate the power supplied to, and the power factor of, the circuit in Example 10 (III).

5. A circuit contains three coils, A, B, C, in series. When 1 A. direct current is passed through them, the voltage drop across each, taken in order, is 5, 2, 10. When an alternating current of 1 A. and of frequency 50 is passed, the voltages are 5·6, 6, and 11. Find the power factor, the power factor of each coil and the power dissipated in each coil. (*L.U.*)

6. The voltage applied to a coil having $R = 200\,\Omega$., $L = 638$ mH., is represented by $e = 200 \sin 100\,\pi t$. Find a corresponding expression for the current and calculate the average value of the power taken by the coil. (*I.E.E.*)

7. The total load on a factory is 1000 kW. at a power factor of 0·95 lagging. If 800 kW. of the load has a power factor of 0·8 lagging, what is the power factor of the remaining portion of the load ?

8. A load of 2500 kW. at a power factor of 0·8 is supplied by two alternators operating in parallel. If the output of one machine is 1000 kW. at a power factor of 0·95 lagging, at what output is the other machine operating ?

9. In what way does power factor affect the cost of electricity supply and the heating of cables? A substation, which is connected to the generating station by a single cable, supplies the following loads: 1500 kW. at 0·85 power factor (lagging), 1000 kW. at 0·8 power factor (lagging), 1000 kW. at 0·75 power factor (lagging). Determine the power factor of the supply to the substation and the load that could be supplied through the cable at unity power factor and the same heating. (*I.E.E.*)

V.—SERIES CIRCUITS (CHAPTER VI)

1. Define the terms "admittance," "conductance," and "susceptance," with reference to alternating-current circuits. Calculate their respective values for a circuit consisting of a resistance of 20 Ω. in series with an inductance of 0·07 H. when the frequency is 50 cycles per second. (*C.G.*)

2. A non-inductive 10 Ω. resistance, A, and an inductive coil, B, are connected in series across a 230 V. 50-cycle supply. The voltages across A and B are 108 V. and 171 V. respectively. Find (*a*) the impedance, reactance, inductance and resistance of the coil, (*b*) the total power, and (*c*) the power factor. (*L.U.*)

3. Two inductive resistances A and B are connected in series. A has $R = 5\,\Omega$., $L = 0{\cdot}01$ H.; B has $R = 3\,\Omega$., $L = 0{\cdot}02$ H. If a sinusoidal voltage of 230 V. at 50 frequeney is applied to the whole, calculate (*a*) the current, (*b*) the power factor, (*c*) the voltage drops. Draw a complete vector diagram for the circuit. (*I.E.E.*)

4. A constant reactance is connected in series with a variable resistance across A.C. mains. Prove that the current locus is a circle. Taking a constant reactance of 10 Ω. and an applied P.D. of 100 V., demonstrate the above theorem by drawing current vectors for a number of values of the resistance between zero and infinity. What is the maximum power dissipated and at what value of the resistance does it occur? (*C.G.*)

5. A coil, for which $L = 0{\cdot}3$ H., $R = 10\,\Omega$., is connected in series with a variable resistance across a supply of 230 V., 50 frequency. Draw to scale the vector locus of the current when the resistance is varied from zero to infinity. Determine the value of the current when the variable resistance has values of (i) 15 Ω., (ii) 50 Ω., and (iii) 90 Ω. (*L.U.*)

6. A 5 μF. condenser is connected in series with a variable resistance across 230 V., 50-cycle mains. Draw the vector locus of the current as R varies from 0 to ∞. For what value of R will the power taken from the mains be a maximum? (*L.U.*)

7. A condenser of 10μ F. capacitance is connected in series with a coil having an inductance of 250 mH. and a resistance of 400 Ω. An alternating E.M.F. of 100 V., sine wave-form, and 50 frequency is applied to this circuit. What current will flow through it. (*L.U.*)

8. An alternating-current circuit includes two sections, AB and BC, in series. The section AB consists of two branches in parallel. The first of these is formed of a non-inductive resistance of 60 Ω. in series with a condenser of 50 μF.; while the second consists of a resistance of 60 Ω. having an inductance of 250 mH. The section BC consists of a resistance of 100 Ω. having an inductance of 300 mH. The frequency is 50 cycles per second and the voltage across section AB is 500 V. What is the voltage across BC? (*L.U.*)

9. A series circuit connected to a 230 V., 50-cycle supply consists of (1) a non-inductive resistance of 50 Ω., (2) an inductive coil for which $R = 15\ \Omega$., $L = 0{\cdot}1$ H., (3) a condenser of capacitance $C\ \mu$F. Determine the value of C for which the voltage across the condenser is equal to that across the inductive coil. For this condition calculate (1) the current, (2) the power, (3) the power factor, (4) the voltages across the separate parts of the circuit.

10. An inductive coil having a resistance of 35 Ω. and an inductance of 0·5 H. forms part of a series circuit, for which the resonance frequency is 55 cycles per second. If the supply frequency is 50 cycles and the voltage is 100 V., determine (1) the line current, (2) the power factor, and (3) the voltage across the inductive coil.

11. A condenser, the losses in which are negligible, is connected in series with an air choking coil having a self-inductance of 0·1 H. across the terminals of an alternator, the voltage of which is maintained constant at 100 V. The frequency is gradually increased until at 250 cycles per second the current reaches a maximum value of 50 A. Calculate the capacitance of the condenser and the ammeter reading at 200 cycles per second. (*C.G.*).

12. A resistance, a condenser and a variable inductance are connected in series across a 200 V., 50-cycle supply. The maximum current which can be obtained by varying the inductance is 314 mA., and the voltage across the condenser is then 300 V. Calculate the capacitance of the condenser and the values of the inductance and resistance. (*I.E.E.*)

13. A series circuit consists of a 20 μF. condenser and an inductive coil. When this circuit is supplied at a constant voltage of 10 V. and variable frequency the highest value of current which can be obtained on an ammeter connected in the circuit is 1 A., which occurs when the frequency is 50 c.p.s. Find the frequencies at which the current is (i) 0·5 A., (ii) 0·25A.

14. A closed circuit consists of a resistance of 5 Ω., an inductance of 0·2 mH. and a variable condenser connected in series. An E.M.F. of 1 V. at a frequency of $10^6/2\pi$ is induced in the circuit. Draw the current locus as the capacitance of the condensor is raised from 220 $\mu\mu$F. below to 200 $\mu\mu$F. above the value corresponding to resonance. (*L.U.*)

15. At a frequency for which $\omega = 796$ an E.M.F. of 6 V. sends a current of 100 mA. through a certain circuit. When the frequency is raised so that $\omega = 2866$, the same voltage sends only 50 mA. through the same circuit. Of what does the circuit consist? (*I.E.E.*)

16. A coil of inductance 20 mH. and resistance 2·2 Ω. has applied to it an alternating E.M.F. of 50 frequency. Near the coil there is an eddy-current path of inductance 0·5 μH. and resistance 50 $\mu\Omega$. The mutual inductance between this path and the coil is 10 μH.

Find the percentage change in the equivalent resistance and inductance of the coil caused by the eddy currents. (*L.U.*).

17. A coil A, having $R = 10\ \Omega$., $L = 0{\cdot}01$ H., is placed near a short-circuited coil, B, having $R = 20\ \Omega$., $L = 0{\cdot}02$ H. The mutual inductance between the coils is 0·006 H. If coil A is connected to a supply having a frequency of 1000 c.p.s., find its effective inductance and effective resistance. (*L.U.*)

18. A primary circuit, consisting of $C = 0{\cdot}0005\ \mu$F., $L = 1{\cdot}5$m H., $R = 45\Omega$. is coupled magnetically with a closed secondary circuit, consisting of $C = 0{\cdot}002\ \mu$F., $L = 0{\cdot}5$ mH., $R = 15\ \Omega$. The mutual inductance is 1 mH. What is the effective series impedance of the primary circuit at a frequency of 100 kc. p.s.? (*I.E.E.*)

19. Two circuits, each of $L = 2$ mH., $R = 200\ \Omega$., $C = 0{\cdot}002\ \mu$F., are coupled by a mutual inductance of $10\ \mu$H. At what frequency will an E.M.F. of 1 V. in one of the circuits produce maximum current in the other? What will be the values of the current in each circuit? (*I.E.E.*)

VI.—PARALLEL CIRCUITS (CHAPTER VI)

1. Two inductive resistances, having reactances of $5\ \Omega$. and $2\ \Omega$., and resistances of $3\ \Omega$. and $4\ \Omega$., respectively, are connected in parallel and supplied at 100 V. Calculate the line current.

2. Two circuits, the impedances of which are given by $8 - j7$, $5 + j6$, are connected in parallel across a 100 V. supply. Calculate the current passing through each circuit, and the total, or line, current. Find also the phase difference between the supply voltage and the currents.

3. A 200 V., 50 cycle alternator is loaded with two choking coils which may be connected either in series or parallel. One coil has a resistance of $1{\cdot}5\ \Omega$. and an inductance of 60 mH.; the other coil has a resistance of $2{\cdot}5\ \Omega$. and an inductance of 30 mH. Determine the current in the circuits and the power factor of the load (i) when the coils are connected in series, (ii) when they are in parallel.

4. A resistance, R, and a condenser, C, are connected in parallel across A.C. supply mains, the frequency being $\omega/2\pi$. Obtain expressions for (1) the joint impedance, giving both the trigonometric and symbolic forms, (2) the cosine of the angle of phase difference between the line voltage and the line current. Find the values of R and C which will give a line current of 5 A. and a power factor of 0·8 when connected to a 230 V. 50-cycle system.

5. A capacitance, C, and a resistance, R, are in series across A.C. mains of frequency $\omega/2\pi$. Find expressions for the values of an equivalent circuit consisting of a capacitance K shunted by a resistance S. Taking $C = 2\ \mu$F., $R = 800\ \Omega$., and $\omega = 500$, draw vector diagrams in each case when the supply voltage $= 200$ V. (*L.U.*)

6. A supply at 200 V., 50 cycles is applied to two circuits in parallel. One of these takes a current expressed by $10\ \underline{/-30^\circ}$ A. The other circuit consists of a condenser of $10\ \mu$F.

Express as complex quantities (*a*) the current from the supply; (*b*) the combined impedance of the two circuits. (*L.U.*)

7. A voltage, having a frequency of 50 c.p.s. and expressed by $V = 200 + j100$, is applied to a circuit consisting of an impedance of $50\ \underline{/30^\circ}\ \Omega$. in parallel with a capacitance of $10\ \mu$F. Find (*a*) the reading on an ammeter connected in the supply circuit, (*b*) the phase difference between the current and the voltage. (*L.U.*)

8. A voltage of $200\ \underline{/30^\circ}$ V, is applied to two circuits A and B connected in parallel. The current in A is $20\ \underline{/60^\circ}$ A, and that in B is $40\ \underline{/-30^\circ}$ A. Find the kVA. and kW. in each branch circuit and the main circuit. Express the current in the main circuit in the form $A + jB$. (*C.G.*)

9. The two branches of a parallel circuit consist of (i) an inductive resistance, and (ii) a shunted condenser connected in series with an inductionless resistance. The inductive resistance has an impedance of $60\ \Omega$. and the ratio of reactance to resistance is 3. The condenser has a capacitance of $10\ \mu$F., and is shunted by a resistance of 0·1 megohm. The value of the inductionless resistance connected in series with the shunted condenser is $100\ \Omega$. The supply pressure is 200 V., and the frequency is 50 cycles per second.

Determine (*a*) the current in the two branches, (*b*) the line current, (*c*) the power supplied.

10. A circuit, having $R = 10\ \Omega$. and $L = 0{\cdot}04$ H., is connected in parallel with a $100\ \mu$F. condenser across 230 V., 50-cycle mains. Calculate (1) the current in each branch circuit, (2) the phase difference between these currents, (3) the line current, (4) the power taken from the supply, (5) the power factor.

11. A condenser is placed in parallel with two inductive loads, one of 20 A. at 30° lag, and one of 40 A. at 60° lag. What must be the current in the condenser so that the current from the external circuit shall be at unity power factor? (*C.G.*)

12. A resistance of $23\ \Omega$., an inductive coil ($L = 0{\cdot}4$ H, $R = 13\ \Omega$.), and a $90\ \mu$F. condenser are connected in parallel across a 230 V., 50-cycle supply. Find the values of (*a*) the total current, (*b*) the current in the condenser, (*c*) the current in the inductive coil at the instants when the current in the resistance is 8A. (*L.U.*)

13. A coil tuned by a condenser C is loosely coupled to a high-frequency generator. The values of C to produce resonance at frequencies of 1 and 1·4 megacycles per sec. are 800 and 400 $\mu\mu$F. respectively. Find the self-capacitance of the coil. (*I.E.E.*)

14. An air-cored choking coil is subjected to an alternating voltage of 100 V. The current taken is 0·1 A. and the power factor 0·2 when the frequency is 50. Find the capacitance which, if placed in parallel with the coil, will cause the main current to be a minimum. What will be the impedance of this parallel combination (*a*) for currents of frequency 50, (*b*) for currents of frequency 40? (*L.U.*)

15. Show how condensers are connected for improving power factor. It is desired to install a condenser to obtain 200 kVA. (leading) on a 600 V. system at a frequency of 50 cycles per second. If each element has a capacitance of 1 μF., how many elements will be needed? (*C.G.*)

16. Two impedances, $\boldsymbol{4 + j5}$, and $\boldsymbol{8 + j10}$, are connected in parallel across 200 V., 50-frequency mains. Find (*a*) the admittance, conductance and susceptance of each branch, and of the entire circuit; (*b*) the total current and its power factor.

What value of capacitance must be connected in parallel with the combination to raise the resultant power factor to unity? (*C.G.*)

17 A single-phase motor takes a current of 80 A. on full-load when its power factor is 0·83. What capacitance must be placed in parallel with it to make the power factor unity upon a 500 V., 50 frequency circuit? (*L.U.*)

18. The load on a single-phase alternating-current supply system is 100 kW. at a power factor of 0·71 (lagging). If phase advancing apparatus is available for parallel connection, taking leading current at a power factor of 0·1, what must be its load in kVA. if the power factor of the whole system is to be raised to (*a*) 0·8, (*b*) 0·9, and (*c*) 0·95.

19. A 60 h.p., single-phase, 440 V., 50-cycle induction motor has a full-load efficiency of 90 per cent, and a full-load power factor of 0·89 (lagging). This machine is shunted with a condenser in order to bring up the resultant power factor to 0·95 (lagging). Assuming zero power factor in the condenser, calculate its capacitance. (*L.U.*)

20. A coil, with $\underset{\cdot}{Z} = \boldsymbol{20 + j50}\ \Omega$., has induced in it an E.M.F. of 10 V. Across the coil are connected two circuits in parallel, the impedances of which are $\boldsymbol{100 + j0}$ and $\boldsymbol{60 - j40}$, respectively. Find the current which flows in the coil. (*I.E.E.*)

VII.—SERIES-PARALLEL CIRCUITS (CHAPTER VI)

1. An alternating E.M.F. of 230 V. at 1000 c.p.s. is applied to a circuit consisting of a non-inductive resistance in series with a 0·05 μF. condenser. When an electrostatic voltmeter is connected across the condenser a reading of 100 V. is obtained. If the capacitance, in μF., of the voltmeter is given by

$$C = 10^{-3} + 5 \times 10^{-6}\ V^2,$$

where V is the voltage at the terminals of the instrument, find the current in the circuit when the voltmeter is disconnected. (*L.U.*)

2. Two reactive circuits when carrying 50-cycle currents have the same impedance and also the same power factor ($= 1/\sqrt{2}$). The first circuit consists of a condenser shunted by a resistance of 200 Ω. The second circuit consists of a choking coil. Find (1) the capacitance of the condenser, and (2) the inductance and resistance of the choking coil. If these circuits are placed in series across 400 V., 50-cycle mains, draw a vector diagram for the voltages and currents. (*I.E.E.*)

3. A circuit, consisting of a condenser in series with a resistance of 10 Ω., is connected in parallel with a coil, having $L = 55{\cdot}2$ mH. and $R = 10$ Ω., to a 100 V. 50-cycle supply. Calculate the value of the capacitance for which the current taken from the supply is in phase with the voltage. Show that for the particular values given the supply current is independent of the frequency. (*L.U.*)

4. A coil, having $R = 100$ Ω., $L = 0{\cdot}04$ H., is connected in series with a 1000 Ω. resistance which is shunted by a condenser. Determine the capacitance of the condenser if the current in the coil is to be closely independent of frequency between 0 and 100 c.p.s., and state the alteration in the current when the frequency is 2000 c.p.s. (*I.E.E.*)

5. Three impedances A, B, C, have constants as follows: A – resistance = 50 Ω., series inductance = 0·1 H.; B – resistance = 20 Ω., series inductance = 0·4 H.; C – Resistance = 20 Ω., series capacitance = 200 μF.

A and B are connected in parallel and C is connected in series with the combination. Find the current taken from a 100 V. 50-cycle supply system. (*L.U.*)

6. A series-parallel circuit consists of two parallel branches A, B, and a series branch C.

The branch impedances are: $Z_A = 10 + j8$, $Z_B = 9 - j6$, $Z_C = 3 + j2$.

Determine the current in A and B when the voltage across C is $100\,(1 + j0)$. Determine also the phase difference between these currents. (*L.U.*)

7. A circuit includes two sections AB and BC in series. The section AB consists of two branches in parallel, one having $R = 60$ Ω. and $C = 50\mu$F. in series, and the other having $R = 60$ Ω. and $L = 250$ mH. in series. The section BC has $R = 100$ Ω. and $L = 300$ mH. in series. If the voltage across AB is 500 V. at 50 c.p.s., what is the voltage across BC? (*I.E.E.*)

8. A series-parallel circuit, consisting of a series impedance Z_3 and two parallel-connected impedances Z_1, Z_2, is supplied at 100 V., 50 c.p.s. If $Z_1 = x + j4$, $Z_2 = 8 + j0$, $Z_3 = 3 + j5$, find the locus of the line current as x varies from 0 to ∞. From the locus read off the value of the current when $x = 4$ Ω. (*L.U.*)

9. A resistance of 3 Ω. and a capacitive reactance of 5 Ω. are in parallel and this combination is in series with an inductive reactance of 7 Ω. The whole is shunted by a resistance of 4 Ω. Find in symbolic notation the impedance of the whole circuit. (*L.U.*)

10. The arms of a T-network $ABC - BD$ are as follow: AB and BC each consist of a coil for which $R = 40$ Ω., $L = 4$ mH.; BD consists of a 10 μF. condenser. Across A and D is established a P.D. of 6 V. at a frequency of 796 c.p.s. Find the current in a resistance of 40 Ω. connected across C and D, and its phase angle with regard to the applied voltage. (*I.E.E.*)

11. An inductive coil, for which $R = 300$ Ω., $L = 0{\cdot}7$ H., is shunted by a resistance R_1 and the combination is connected in series with a 15 μF. condenser across a 230 V, 50-cycle supply. Find the value of R_1 which gives zero phase difference between line voltage and line current. Find also the line current under these conditions and the voltage at the terminals of the condenser.

12. A circuit, A, consisting of a condenser C shunted by a resistance R, is connected in series with another circuit, B, consisting of an inductive coil ($L = 0{\cdot}05$ H., $R = 5$ Ω.) shunted by a 100 Ω. resistance, and the combination is supplied at 230 V., 50 c.p.s. Find the values of R and C in order that the

voltages across A and B shall have a phase difference of 90° and that the voltage across B shall be four times that across A.

13. For a certain A.C. experiment it is necessary to have a circuit through which the current is maintained constant while the power factor is varied. For this purpose a coil, having $L = 7$ mH. and R negligible, is connected in parallel with a 0 – 20 Ω. variable resistance R_1, and a second variable resistance, R_2, is connected in series with the combination. Give a vector diagram and determine the range required for R_2 to maintain a constant line current of 40 A. when R_1 is varied from 0 to 20 Ω., the voltage across the whole circuit being maintained constant at 100 V., 50 c.p.s. What will be the maximum and minimum power factor under these conditions? (*L.U.*)

14. A coil ($L = 2$ mH., $R = 8$ Ω.) is connected in series with a circuit consisting of a resistance of 12 Ω. in parallel with a variable condenser. The combination is supplied at a constant voltage of 40 V. and a frequency for which $\omega = 5000$.

Draw the vector current-locus and explain its theory. From the locus diagram read the currents when the capacitance of the condenser has values of 5 and 25 μF. (*L.U.*)

VIII.—SINGLE-PHASE TRANSFORMERS (CHAPTER VIII)

1. A transformer has its primary winding connected to mains whose voltage varies according to the sine law at a frequency of 50. The secondary coil has 50 turns and gives 100 V. when on open circuit. The section of the transformer core is 20 sq. in. Determine the maximum value of the flux density in the core. Prove the formula used. (*L.U.*)

2. Draw a vector diagram for a transformer, showing the phase relations between the primary and secondary voltages and currents. A single-phase transformer with a ratio 6600/400 takes a no-load current of 0·7 A. at 0·24 power factor. If the secondary supplies a current of 120 A. at 0·8 power factor (lagging) estimate the current taken by the primary. (*I.E.E.*)

3. Explain what is meant by (*a*) equivalent resistance, (*b*) leakage reactance of a transformer. Calculate these values from the following test figures obtained on the primary side of a step-down transformer, the secondary being short-circuited. Applied P.D. 40 V., current 60 A., power input 800 W. (*C.G.*)

4. A 50-cycle single-phase transformer has a ratio of transformation of 5 to 1 and a full-load secondary current of 200 A. The resistance and reactance of the primary are 0·8 Ω. and 2·5 Ω. respectively, and the corresponding values for the secondary are 0·04 Ω. and 0·1 Ω. If the low-voltage winding is short-circuited, what voltage must be applied to the other winding so that full-load current may be obtained in the former? Neglect the no-load current. (*L.U.*)

5. A 10 kVA., 2000/200 V. transformer was tested (i) with the secondary winding open circuited, (ii) with the secondary winding short circuited, and the results were as follow, all readings being taken on the primary (2000 V. side)—

(i) *Open circuit test*—2000 V., 0·2 A., 280 W.; (ii) *short circuit test*—150 V., 5·0 A., 225 W.

Explain the meaning of these results and the information which can be deduced from them.

How would the above readings be modified if the tests had been carried out by applying suitable voltages to the secondary winding, the primary winding being short circuited in the second test? (*L.U.*)

6. A 1000 kVA., single-phase, 50-cycle transformer has a full-load secondary current of 2500 A., and the ratio of transformation is 5/1. The resistances of the primary and secondary are 0·04 Ω. and 0·0015 Ω. respectively, and the corresponding reactances are 0·25 Ω. and 0·08 Ω. What voltage applied to the secondary terminals will cause full-load current to circulate in the short-circuited primary windings. (*L.U.*)

7. Develop a simple expression for the pressure-drop in a transformer

If the copper loss in a transformer is 1 per cent of the full load output at unity power factor, and the inductive pressure is 3 per cent of the normal pressure, find the regulation at full-load when the power factor is 0·8. (*C.G.*)

8. Draw and explain the complete vector diagram for a transformer supplying a lagging load, and give a graphical construction for obtaining the voltage regulation for full-load current and various lagging and leading power factors.

A transformer has a reactive drop of 4 per cent and a resistance drop of 2 per cent. Find the lagging power factor at which the voltage change is a maximum and the value of this regulation. (*C.G.*)

9. Show that an A.C. transformer has a maximum efficiency when the copper loss is equal to the iron loss. A 100 kVA. transformer has a full-load copper loss of 2500 W. and an iron loss of 1600 W. If the transformer supplies a load of 0·8 power factor calculate (i) the efficiency at full load, (ii) the maximum efficiency and the corresponding load. (*L.U.*)

10. A 20 kVA. single-phase transformer, ratio 1100/220 V., has an iron loss of 200 W. The resistance of the primary winding is 0·25 Ω. and that of the secondary is 0·012 Ω.: the corresponding leakage reactances are 1·1 Ω. and 0·055 Ω. respectively. Calculate the percentage resistance and reactance drops at full load 0·8 power factor. At what percentage of full load will the efficiency be a maximum? (*I.E.E.*)

11. Deduce the conditions under which (i) the efficiency of a transformer is a maximum, (ii) the power factor at which the voltage drop is a maximum. A transformer has voltage drops of 1·5 per cent and 4 per cent of its rated voltage due to resistance and leakage reactance respectively. Calculate (*a*) the maximum percentage voltage drop on full load, (*b*) the efficiency on half load at unity power factor if the iron losses are equal to the copper losses at full load. (*C.G.*)

12. Give the rules which govern the parallel operation of transformers. Two transformers *A* and *B* give the following results: with the l.-v. side short-circuited *A* takes a current of 10 A. when 200 V. is applied to the h.-v. side, the power input being 1000 W. Similarly, *B* takes 30 A. at 200 V., the power input being 1500 W. On open circuit both transformers gave a voltage of 2 200 V. on the l.-v. side when 11,000 V. was applied to the h.-v. side. These transformers are connected in parallel on both sides, the h.-v. side being supplied at 11,000 V. Calculate the current and power supplied by each transformer to a load requiring 200 A. at 0·8 power factor (lagging). Neglect the no-load currents of the transformers. (*L.U.*)

13. Two transformers, *A* and *B*, of equal rating share a load of 180 kW. at 0·9 power factor (lagging). At full-load the voltage drop due to resistance in transformer *A* is 1 per cent of the normal terminal voltage, and that due to reactance 6 per cent. The corresponding figures for transformer *B* are 2 per cent and 5 per cent respectively. Find the load in kW. on each transformer. (*L.U.*)

14. Two transformers, *A* and *B*, have a ratio of transformation of 3300 to 220. When run on short circuit, *A* took 600 W. at 100 V. on the primary to make 230 A. flow in the secondary; *B* took 1100 W. at 80 V. to make the same current (230 A.) flow in its secondary. The two transformers are run in parallel on the same primary and secondary bus-bars, with a total load of 100 kW. at 0·8 power factor. Find approximately how the load will be divided between the transformers. (*L.U.*)

15. Three transformers of equal ratings share a load of 200 kVA. at 0·8 power factor. Calculate the loadings if the voltage drops due to resistance and leakage reactance at full load are: 2 per cent (*RI*) and 4 per cent (*XI*) for transformer *A*, 2 per cent (*RI*) and 6 per cent (*XI*) for transformer *B*, 2·5 per cent (*RI*) and 8 per cent (*XI*) for transformer *C*.

IX.—POLYPHASE CIRCUITS (CHAPTER IX)

1. A balanced Δ-connected load, connected to a 400 V., three-phase system, takes 10 kVA. at 0·8 power factor. Calculate the resistance of each branch

of a balanced Y-connected non-inductive load which, when connected to the same supply, will take the same power.

2. Three resistances each of $500\,\Omega$. are connected in star to a 400 V., 50-cycle, three-phase supply. If three condensers, when connected in Δ to the same supply, take the same line currents, calculate the capacitance of each condenser and the line currents. (*L.U.*)

3. A three-phase induction motor is supplied from mains at 440 volts. It is coupled to a pump taking 50 B.H.P. The efficiency of the motor is 92 per cent and its power factor is 90 per cent. What is the current in each phase? (*C.G.*)

4. Explain, with the aid of vector and connection diagrams, how you would measure the power input to a three-phase motor. The readings of two wattmeters properly connected for this test are respectively 2·5 kW. and 0·25 kW., the latter reading being obtained after reversing the connections of the current coil. Find the power and power factor. (*C.G.*)

5. Two three-phase alternators are running in parallel and supplying power to a transmission line symmetrically loaded. Two wattmeters properly connected give readings of 300 and 900 kW. respectively, and both wattmeter deflections follow the variations of the load. The readings of ammeters connected in the line and in the circuits of the two alternators are in the ratio 5 : 3 : 4. Determine how much load each alternator is taking, and the readings that would be indicated by each of four wattmeters properly connected in pairs in the circuit of each alternator. (*L.U.*)

6. Plot a graph showing the variation in power factor with the ratio of the wattmeters readings in three-phase power measurements by two wattmeters when the load is balanced.

The power input to a 2000 V., 50-cycle three-phase motor running on full load at an efficiency of 90 per cent is measured by two wattmeters which indicate 300 kW. and 100 kW. respectively. Calculate (*a*) the input, (*b*) the power factor, (*c*) the line current, (*d*) the horse-power output. (*I.E.E.*)

7. A 20-h.p., 500 V., three-phase induction motor is working at full load from supply mains at normal voltage and frequency. Under these conditions the power factor of the motor is 0·87 and the efficiency is 89 per cent. Two wattmeters are connected in the circuit as follows. Both current coils are connected in series with each other in supply main *R*. One end of each pressure coil is connected to supply main *R*, the other ends being connected to supply mains *B* and *Y* respectively. Find the readings of each instrument. It can be assumed that the motor currents are balanced. (*L.U.*)

8. Two three-phase generators operating in parallel supply a non inductive load of 500 kW. at 3,300 volts. If the output from one machine is 200 kW. at 0·75 power factor, lagging, what is the output from the other machine. (*L.U.*)

9. A 440 V., 50-cycle induction motor takes a line current of 45 A. at a power factor of 0·8 (lagging). Three Δ-connected condensers are installed to improve the power factor to 0·95 (lagging). Calculate the kVA. of the condenser bank, and the capacitance of each condenser. State the conditions which determine the most economical power factor at which to operate a motor. (*I.E.E.*)

X.—TRANSMISSION LINES AND CABLES (CHAPTER X)

1. A system supplies a load which varies in a period of 24 hours as follows: 6 a.m. to 12 noon, 1000 kW.; 12 noon to 1 p.m. 100 kW; 1 p.m. to 6 p.m. 1000 kW.; 6 p.m. to 6 a.m., 300 kW.

The energy is transmitted over a line in which the losses at full load (i.e. 1000 kW.) are 5 kW., and is transformed by a transformer having a no-load loss of 6 kW. and a copper loss (at an output of 1000 kW.) of 8 kW. Calculate the total energy losses in transmission and conversion during the period of 24 hours. (*I.E.E.*)

2. An alternator supplies current to two single-phase motors located some

distance apart, each motor taking 7·5 kW. at a power factor of 0·8. The resistance of the line wires between the alternator and one motor is 0·2 Ω. and that of the line wires between the motors is also 0·2 Ω. Determine the voltage at the alternator in order that the voltage at the further motor may be 100 V.

3. A single-phase load of 200 kVA. is delivered at 2500 V. over a transmission line having $R = 1·4$ Ω., $X = 0·8$ Ω. Calculate the current, voltage, and power-factor at the sending end when the power factor of the load is (*a*) unity, (*b*) 0·8, lagging, (*c*) 0·8 leading. (*I.E.E.*)

4. Explain, with the aid of a vector diagram the effect of power factor upon the voltage drop in a transmission line. A line, having $R = 0·05$ Ω., $X = 0·07$ Ω., delivers 10 kW. at 230 V., 0·8 power factor (lagging). Calculate the voltage regulation and the efficiency of the transmission. (*L.U.*)

5. Determine the necessary cross-section of each wire of a three-phase transmission line, 2 miles in length, designed to deliver 500 kW. at 6000 V., unity power factor and 95 per cent efficiency. The specific resistance of the material used is 0·7 microhm per inch cube. (*L.U.*)

6. A three-phase supply at 400 V., 50 frequency, and 0·9 power factor is required for a factory 1·3 miles from a generating station. The total power at the factory is 500 kW. If the voltage lost in transmission is 10 per cent of the received voltage, calculate the necessary cross-section of each conductor. Assume the resistance of a cable 1 mile long and 1 sq. in. in cross-section is 0·43 Ω. The effects of inductance and capacitance may be neglected (*L.U.*)

7. Show that when the fall of voltage due to resistance and reactance is small compared with the line voltage, the fall of voltage along a three-phase transmission line per ampere per mile is given by the expression $\sqrt{3}(R \cos \varphi + X \sin \varphi)$, where R is the resistance per mile of conductor, X the reactance per mile of conductor, and $\cos \varphi$ the power factor of the load.

Find the fall of voltage along a three-phase transmission line, the line pressure at the load being 30,000 V. and the length of the line being 30 miles, when 5000 kVA. are delivered at a power factor of 0·8, the current lagging. The resistance and reactance per mile are 0·72 Ω. and 0·6 Ω. respectively. (*L.U.*)

8. Prove that in a symmetrically arranged three-phase transmission line the inductive drop and resistance drop between any two wires is the same as that which would occur if half the total power were transmitted at the same voltage and frequency along two of the wires.

An overhead three-phase line consists of three wires, each 0·8 in. in diameter, spaced 4 ft. apart. The current flowing per wire is 300 A. at 50 cycles. Calculate the resistance and inductive drop per mile of line; the specific resistance of copper is 0·67 microhm per inch cube. (*L.U.*)

9. What load at 0·8 power factor, lagging, can be delivered by a three-phase line 5 miles long with a pressure drop of 10 per cent. The station voltage is 11,000 V., and the resistance and reactance per mile of line are 0·09 Ω. and 0·08 Ω. respectively. (*C.G.*)

10. 15,000 kVA. is received at 33 kV., 0·85 power factor (lagging) over an 8-mile three-phase overhead line. Each line has $R = 0·29$ Ω. per mile and $X = 0·65$ Ω. per mile. Calculate (*a*) the voltage at the sending end, (*b*) the power lost in the line, (*c*) the percentage regulation. (*I.E.E.*)

Calculate also (*d*) the efficiency of transmission, (*e*) the power factor at the sending end of the line, (*f*) the power input to the sending end of the line.

11. A three-phase, 50-cycle, generating station supplies an inductive load of 5000 kW. at a power factor of 0·7 by means of an overhead transmission line 5 miles long, with conductors symmetrically arranged. The resistance per mile of each wire is 0·61 Ω., and the self-inductance per mile of the loop formed by any two conductors taken together is 0·0035 H. The pressure at the receiving end is maintained constant at 10,000 V. If a condenser is connected across the load to increase the power factor at the receiving end from 0·7 to 0·9, calculate (*a*) the value of capacitance required per phase,

(*b*) the station voltage when the condenser is in use, (*c*) the station voltage when the condenser is disconnected. (*C.G.*)

12. A transmission line worked at 11 kV. between phases has a resistance per phase of 8 Ω. and a reactance per phase to neutral of 11 Ω. It supplies a load having a power factor of 0·8 (lagging). The frequency is 50 c.p.s. A 25 μF. condenser is connected at the receiving end of the line between each phase and the neutral. At what kW. load will the regulation be zero? (*L.U.*)

13. Draw and explain the vector diagram for a transmission line assuming that half the line capacitance is concentrated at each end of the line.

A 50-frequency, three-phase line, 100 km. long, delivers a load of 40,000 kVA. at 110 kV. and a lagging power factor of 0·7. The line constants (line to neutral values) are: resistance, 11 Ω.; inductive reactance, 38 Ω.; capacitive susceptance, 3×10^{-4} mho. Find the sending-end voltage, current, power factor and power input. (*C.G.*)

14. A three-phase, 50-cycle transmission line is 50 miles long and delivers 2500 kW. at 30 kV., 0·8 power factor (lagging). Calculate the voltage at the generator end of the line if each conductor has $R = 0{\cdot}4\ \Omega$. and $X = 0{\cdot}5\ \Omega$. per mile. If an extra load consisting of condensers having $C = 1{\cdot}5\ \mu$F. to neutral is connected at the middle of the line, calculate the voltage at the generator end. (*L.U.*)

15. A three-phase load of 9000 kVA. at 0·9 power factor (lagging) is received at 60 kV. and 50 c.p.s. from an overhead transmission feeder 50 miles long, for which $R = 0{\cdot}67\ \Omega$., $X = 0{\cdot}67\ \Omega$. per conductor per mile, and the capacitance to neutral is 0·014 μF. per mile. Calculate the voltage and the power factor at the sending end of the line. (*L.U.*)

16. A three-phase load of 500 kW., at 11 kV., 0·8 power factor (lagging), is supplied by two overhead lines *A* and *B* operating in parallel. The resistance and reactance per conductor are: 4 Ω. and 6 Ω. respectively for line *A*, and 4 Ω. and 2 Ω. respectively for line *B*. Calculate accurately the current and the power supplied to the load by each overhead line. (*L.U.*)

17. A substation receives 10,000 kW. from a central distributing station by means of duplicate overhead three-phase lines operating in parallel. The lines follow different routes. For line *A*, $R = 15\ \Omega$., $X = 16\ \Omega$. per conductor; and for line *B*, $R = 13\ \Omega$., $X = 15\ \Omega$. per conductor. The voltage at the *distributing* station is 66 kV. and the power factor at the substation is 0·8 (lag). Calculate the current in each transmission line. (*L.U.*)

18. Two parallel three-phase feeders, *A* and *B*, supply 500 kW. at unity power factor, and 160 kW. at 0·8 power factor (lagging) respectively, a voltage being injected into each line of feeder *B*. The constants per line are: *A*, $R = 1{\cdot}1\ \Omega$., $X = 2\ \Omega$.; *B*, $R = 0{\cdot}9\ \Omega$., $X = 1{\cdot}8\ \Omega$. Find the injected voltage in terms of the receiving-end line voltage, and its phase difference with respect to the current in *B*. (*L.U.*)

19. A three-phase load at 0·8 power factor (lagging) is supplied at 10 kV. through a transmission line *A* fed by two other transmission lines *B* and *C* which are connected in parallel. Calculate the voltage at the generator end of *B* and *C* when the current in *B* is 100 A. Calculate also its phase relationship to the voltage at the load. The resistances per conductor are: *A*, 1 Ω.; *B*, 2 Ω.; *C*, 2 Ω.; and the reactances per conductor are: *A*, 2 Ω.; *B*, 4 Ω.; *C*, 6 Ω. (*L.U.*)

20. A three-phase transmission line 20 miles long delivers 1000 kW. at 30 kV., 0·8 power factor (lagging). Calculate the voltage at the generator end if the resistance and reactance per mile of conductor are 1·25 Ω. and 0·6 Ω. respectively.

If a 30/10 kV. transformer is connected at the end of this line, calculate the voltage at the generator end of the line if the load of 1000 kW. at 0·8 power factor is delivered at the secondary side of the transformer at 10 kV. Referred to the neutral on the secondary side the equivalent resistance and reactance of the transformer are 0·8 Ω. and 2·5 Ω. respectively. (*L.U.*)

21. A three-phase transmission line is 40 miles long. Each conductor has

a resistance of 0·4 Ω. per mile and a reactance of 0·6 Ω. per mile. It is fed at one end through a step-up transformer for which the ratio of transformation is 1 : 3, the equivalent resistance per phase referred to the secondary is 2 Ω., and the equivalent reactance per phase referred to the secondary is 8 Ω.

The voltage and current at the mid-point of the line are 33 kV. (between lines) and 100 A. respectively, the power factor being 0·8 (lagging). Calculate (*a*) the voltage at the low-voltage terminals of the transformer, (*b*) the equivalent resistance and reactance "to neutral" of the load at the end of the line. (*L.U.*)

22. Compare the weights of copper required in a cable transmission system utilizing (*a*) constant direct current, (*b*) single-phase, (*c*) three-phase three-wires. Assume the same transmitted power and maximum voltage between conductors, and the same percentage loss in each case. Assume also unity power factor and balanced load. (*I.E.E.*)

23. The measured capacitance between any two cores of a three-phase cable is 5μ F. Calculate how many kVA. are needed to keep the cable charged when connected to 10,000 V., 50 cycle, three-phase bus-bars. (*C.G.*)

24. In a symmetrical three-core, three-phase cable with earthed sheath, the capacitances per mile are 0·1 μF. between sheath and bunched conductors; 0·12 μF. between one conductor and the other two joined to the sheath. If the length of the cable is 10 miles, the line voltage 33,000, and the frequency 50, calculate the charging current in each core. (*I.E.E.*)

25. An overhead three-phase transmission line is 105 miles long and has a characteristic impedance of $450\underline{/-15^\circ}$ Ω. at 50 c.p.s. and a propagation constant of $1{\cdot}9 \times 10^{-3}\underline{/75^\circ}$ per mile. What must be the voltage at the generating end if the voltage at the receiving end is 132 kV. with a load of 20,000 kW. at unity power factor? (*L.U.*)

26. Explain how the primary constants of a long transmission line may be found from measurements of the impedance of the line. Calculate the resistance, inductance and capacitance per mile of a 50-cycle overhead line 175 miles long, the measured impedances of which are $1500\,\underline{/-75^\circ}$ Ω. on open circuit, and $150\,\underline{/45^\circ}$ on short circuit. (*L.U.*)

XI. ALTERNATORS (CHAPTER XI)

1. Prove that the frequency, f, of the E.M.F. generated by an alternator having p pole-pairs and running at n r.p.s. is given by $f = pn$. The field system of a 50-cycle alternator has a sinusoidal flux per pole of 10^7 lines. Calculate the E.M.F. generated in one turn which spans $\frac{2}{3}$ of a pole pitch. (*L.U.*)

2. A single-phase alternator has 20 poles, 6 stator slots per pole, and the stator winding consists of full-pitch coils. With all slots wound, 10 conductors in each slot, the alternator is run at 300 r.p.m. with its fields excited to give a flux of 10^7 lines per pole. Calculate the generated voltage in the stator winding.

If this voltage is applied to the primary winding of a transformer which has 1000 turns, calculate the maximum value of the flux in the transformer core. Prove any formula used. (*L.U.*)

3. Explain the effect on the R.M.S. value of the E.M.F. of an alternator of distributing the armature coils and of skewing the armature slots. Define distribution factor (or breadth coefficient) and calculate its value for a three-phase, single-layer winding with 3 slots per pole per phase. Assume a sinusoidal flux distribution. (*C.G.*)

4. Calculate the "winding distribution factor" (or "breadth coefficient") for the winding of a single-phase alternator having four full-pitch coils per pair of poles located in slots 15° apart, the flux distribution in the air gap being sinusoidal.

5. Deduce an expression for the distribution factor for the nth harmonic of the induced E.M.F., in a winding occupying g' slots per pole per phase, there being g slots per pole with full-pitch coils. Evaluate the expression for

the fundamental and third harmonic is an eight-pole alternator having 72 slots, the phase spread being 3 slots per pole. (*I.E.E.*)

6. The stator of a two-pole 3000 r.p.m., three-phase turbo-alternator is wound with a double-layer winding in 54 slots, there being four conductors per slot. The pitch of the coils is two slots less than the pole-pitch. The flux per pole is 25 megalines and is distributed sinusoidally over the pole-pitch. Calculate the no-load terminal voltage if the windings are star connected. (*L.U.*)

7. A three-phase, 50-cycle, Y-connected alternator has open-circuit characteristic as follows—

Open-circuit terminal volts .	0	4250	6 370	8 500	10 300	12 200	13 300
Ampere-turns per pair of poles	0	8000	12 000	16 000	20 000	28 000	36 000

The stator winding has a leakage reactance of 4 Ω. per phase and has negligible resistance. Calculate the generated stator voltage, at an output of 2000 kVA., 11,000 V., 0·8 power-factor (lag). If at this output the stator reaction field ampere-turns are 10,000 per pair of poles, calculate the number of ampere-turns required on each pair of main poles and the voltage regulation of the alternator.

8. Tests on a 15,000 kVA., 11,000-V., three-phase, 50-cycle, Y-connected turbo-alternator gave the following results—

Field amp.-turns per pole, thousands . .	10	18	24	30	40	45	50
Open-circuit line E.M.F. (kV.) . . .	4 9	8 4	10·1	11·5	12 8	13·3	13·65
Line voltage (kV.) with full-load current, zero power factor		0				10·2	

Find the armature-reaction amp.-turns, the leakage reactance, and the regulation for full load at 0·8 power factor (lagging). Neglect resistance. (*I.E.E.*)

XII.—INDUCTION MOTORS (CHAPTER XII)

1. Prove the relation between the slip and the rotor output of a three-phase induction motor working at constant voltage and frequency.

A three-phase induction motor has a synchronous speed of 500 r.p.m. and runs at 486 r.p.m. when giving a torque of 488 lb.-ft. Find the rotor efficiency and the rotor copper loss. (*L.U.*)

2. Show that full-load power is required to obtain full-load torque in a polyphase induction motor. Neglect stator losses.

A 500 V., 50 frequency, six-pole, three-phase motor develops 20 h.p. inclusive of mechanical losses when running at 975 r.p.m., the power factor being 0·87. Calculate (*a*) the slip, (*b*) rotor copper losses, (*c*) total input if the stator losses are 1500 W., (*d*) line current, (*e*) number of cycles per minute of the rotor E.M.F. (*C.G.*)

3. A 50 h.p., 500 V., three-phase induction motor has an efficiency of 85 per cent and a power factor of 0·8 at full load. What is the voltage across each phase of the motor and the current in each phase when the motor is on full load and is (*a*) star-connected, (*b*) delta-connected? (*L.U.*)

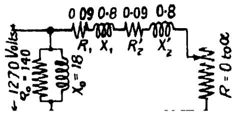
4. A three-phase induction motor with squirrel-cage rotor is rated as follows: 25 h.p., 400 V., 960 r.p.m., 50 c.p.s. The power factor at full load is 0·9 and the efficiency is 90 per cent. When switched on to the line, the motor takes 5 times full-load current and gives twice full-load torque. Find the line current and the motor torque, if an auto-transformer with a 50 per cent tapping, is used for starting purposes. Neglect the magnetizing current of the transformer. Draw a simplified diagram of connections for the starting apparatus. (*L.U.*)

5. A $7\frac{1}{2}$ h.p., 400 V., three-phase induction motor with squirrel-cage rotor, has its stator windings arranged for star-delta starting. The equivalent impedence per phase of the motor with rotor locked is 12·5 Ω., and the rotor I^2R loss at full load is 225 W. Determine the approximate line current at starting and the starting torque (as a percentage of the full-load torque) when the motor is started by a star-delta starter.

. Assume the power factor at full load to be 0·87 and the efficiency at full load to be 0·88. (*L.U.*)

6. A three-phase, 100 h.p., 550 V., 50-cycle induction motor has both stator and rotor windings Y-connected, the ratio (rotor turns/stator turns) being 0·7. The rotor resistance per phase is 0·2 Ω. and the leakage inductance 0·003 H. Determine (*a*) the rotor starting current on normal voltage with the slip rings short-circuited, (*b*) the rotor power factor at starting, (*c*) the full-load rotor current (slip = 3 per cent), (*d*) the rotor power factor at full load, (*e*) the external resistance per phase to obtain a starting current of approximately 60 A. in the stator. The effect of stator impedance may be neglected. What generally would be the effect on the results if the stator impedance were taken into account? (*L.U.*)

7. Show that the performance of a polyphase induction motor operating with constant applied voltage and frequency can be approximately represented by the equivalent circuit shown in the accompanying diagram.



Point out the assumptions made and develop a vector diagram for the circuit which enables the line and load currents to be determined directly. With the values given—which relate to one winding of a star-connected three-phase motor—determine (*a*) the maximum power input, (*b*) the maximum power expended in the resistance *R*, (*c*) the value of *R* which gives the maximum power factor. (*L.U.*)

8. Draw to scale the circle diagram of a 25 h.p., 400 V., 50-cycle, six-pole, three-phase induction motor from the following test figures—*No-load test*, 400 V., 12 A., power factor 0·15; *Short-circuit test*, 200 V., 85 A., power factor 0·3; *Ratio of rotor to stator losses on short circuit*, 0·9. Determine from the diagram the full-load current and power factor, and the percentage pull-out torque. (*I.E.E.*)

9. The following data refer to a 75 h.p., 500 V., 16-pole, 50-cycle, three-phase, slip-ring induction motor—*Running light*, 500 V., 35 A., 2235 W.; *Standstill, rotor locked and slip-rings short circuited*, 137 V., 87·5 A., 5720 W.; *Standstill, slip-rings open circuited*, 500 V. (stator), 196 V. (rotor); *Stator resistance* (measured between line terminals), 0·16 Ω.; *Connections of stator and rotor windings, Y*. Draw the circle diagram and deduce therefrom (i) the full-load current and power factor, (ii) the external resistance per phase to give 1·5 times full-load torque at starting. (*L.U.*)

10. The following particulars apply to a three-phase, 200 h.p., 5000 V., 25-cycle, eight-pole induction motor with star-connected stator and rotor.—Turns per phase: 576 (stator), 56 (rotor); resistance per phase, 3·69 Ω (stator), 0·032 Ω. (rotor); reactance per phase, 16·1 Ω. (stator), 0·074 Ω. rotor; magnetizing current, 5·5 A.; iron loss, 2·1 kW.; friction and windage loss, 1·3 kW. Draw the circle diagram and determine two values of rotor starting resistance which will give twice full-load torque at starting. Determine also the starting currents. (*L.U.*)

XIII.—THREE-PHASE TRANSFORMERS (CHAPTER XIII)

1. A 500 kVA., three-phase, 50-cycle transformer has a voltage ratio (line voltages) of 33,000/11,000 and is Δ/Y connected. The resistances per phase are: h.v. winding, 35Ω.; l.v. winding, 0·876 Ω. The iron loss is 3050 W. Calculate the efficiency at full load and half load at (*a*) unity power factor, (*b*) 0·8 power factor. (*I.E.E.*)

2. Determine the currents in the various branches of a three-phase, *Y*-connected auto-transformer loaded with 400 kW. at 0·8 power factor (lagging), and having a ratio 440/550 V. Neglect voltage drops, magnetizing current and all losses. (*I.E.E.*)

3. A 50-cycle, three-phase, 500 kVA., 6600/440 V., Δ/Y transformer is to work with $B_{max} = 12{,}500$ lines per cm.² The net area of iron in each core is 400 cm.² Determine the number of turns in each phase of both windings and the cross-sectional areas of the conductors. A suitable current density is 250 A. per cm.² If the iron loss at normal voltage and frequency is 9 kW.,

and the resistances per phase are 4 Ω. (primary) and 0·0052 Ω. (secondary), calculate the load at which maximum efficiency is obtained. (*L.U.*)

4. A three-phase transformer rated at 300 kVA., 11,000/415 V., 50 cycles is to be designed. The primary windings are to be Δ connected and the secondary windings *Y* connected. The approximate value of the voltage per turn is to be 10·5. Calculate (i) the number of primary and secondary turns, and the cross-sectional area of their conductors, if the current density is to be 2 A. per mm.^{2}; (ii) the cross-sectional area of each core if the flux density is not to exceed 10 kilo lines per sq. cm. Draw up a specification for the secondary winding. (*L.U.*)

5. A three-phase transformer delivers 100 kVA. on its h.v. side to a capacitive load. The ratio of transformation is 440/3300. The equivalent resistance and reactance per phase, referred to the secondary, are 3 Ω. and 9 Ω. respectively. Calculate the primary voltage if the secondary voltage is 3300.

If the exciting current (assumed constant) is 10 A. at 0·1 power factor, calculate the total input current and its power factor. (*L.U.*)

6. A three-phase, 50-cycle transformer has a ratio of transformation of 30/10 kV. The secondary voltage is 10 kV., and it supplies a load consisting of two circuits *A* and *B* in parallel. The phases of circuit *A* are star-connected and each takes a current of 100 A. at 0·8 power factor (lag). Circuit *B* consists of a star-connected condenser bank, each phase having a capacitance of 100 μF. Calculate the voltage regulation of the transformer and the power factor on its primary side. Transformer particulars (referred to neutral) are—*Primary*, $R = 2$ Ω., $X = 6$ Ω.; *Secondary*, $R = 0{\cdot}2$ Ω., $X = 0{\cdot}6$ Ω. (*L.U.*)

7. Two single-phase transformers are used to convert a balanced three-phase into a balanced two-phase pressure. The pressure on the three-phase side is 6600 V. and that on the two-phase side is 2200 V. Find the number of turns in each transformer when the induced pressure per turn is 10 V. (*L.U.*)

8. Explain, with connection and vector diagrams, the transformer connections to obtain a two-phase supply from a three-phase system. Two 100 V. single-phase furnaces take loads of 600 kW. and 900 kW. respectively at a power factor of 0·71, and are supplied from 6·6 kV., three-phase mains through Scott-connected transformers. Calculate the currents in the three-phase lines, neglecting transformer losses. (*C.G.*)

9. Two electric furnaces are supplied with single-phase current at 80 V. from a three-phase 11,000 V. system by means of two single-phase Scott-connected transformers with similar secondary windings. When the load on one transformer is 500 kW. and the load on the other is 800 kW., what current will flow in each of the three-phase lines (i) at unity power factor, (ii) at 0·5 power factor? Neglect the phase displacement and losses in the transformers. (*C.G.*)

10. State the conditions to be fulfilled for the satisfactory parallel operation of two three-phase transformers. Two *Y*/*Y* three-phase transformers *A* and *B*, of equal rating, share a balanced load of 400 kVA. at a power factor of 0·8 (lag). Calculate the load on each transformer. The voltage drops per phase corresponding to the full-load currents, and expressed as percentages of the normal secondary voltages, are: 1 per cent (*RI*) and 5 per cent (*XI*) for transformer *A*, and 2 per cent (*RI*) and 6 per cent (*XI*) for transformer *B*. (*L.U.*)

11. Calculate the currents supplied by each of two three-phase transformers, working in parallel and delivering a total load current of 200 A. The transformers have equal ratios of transformation, and their equivalent impedances, referred to the secondary windings, are $2 + j5$ ohms per phase for transformer *A* and $2 + j3$ ohms per phase for transformer *B*. Calculate also the angle of phase displacement between the load current and the currents in the two transformers. (*L.U.*)

12. Determine the relative voltages of the secondary windings of the three-phase/twelve-phase transformer connections shown in Fig. 189.

13. Two similar three-phase transformers, one core-type and the other

shell-type, both Y-connected on the primary and secondary sides, are connected to the same 11 kV. supply mains. The voltage between the primary star points (which are not connected together or to any other point in the system) is found to be 3000 V. Explain clearly the reason for this voltage, and describe methods by which it could be reduced. (*L.U.*)

XIV.—NON-SINUSOIDAL WAVE FORMS (CHAPTER XIV)

1. An E.M.F. is represented by $v = 1000 \sin \omega t + 250 \sin 3\omega t + 200 \sin 5\omega t$ Calculate the reading that will be given by an electrostatic voltmeter connected to the circuit.

2. An E.M.F. $e = 100 \sin \omega t + 8 \sin 3\,\omega t$ is applied to a circuit which has a resistance of 1 Ω., an inductance of 0·02 H, and a capacitance of 60 μF. A hot-wire ammeter is connected in series with the circuit, and a hot-wire voltmeter is connected to the terminals. Calculate the ammeter and voltmeter reading and the power supplied to the circuit. Given $\omega = 300$. (*L.U.*)

3. An alternating current represented by $i = 10 \sin \omega t$ is superimposed upon a direct current of 80 A. What is the R.M.S. value of the resultant current ?

4. An E.M.F. represented by the equation $e = 150 \sin 314t + 50 \sin 942t$ is applied to a condenser having a capacitance of 20 μF. What is the R.M.S. value of the charging current ? (*C.G.*)

5. An alternating voltage $e = 100 \sin 100\pi t - 30 \sin 300\pi t + 10 \sin 500\pi t$ is applied to a coil having $R = 10$ Ω. and $L = 0{\cdot}0318$ H. Determine an expression for the wave-form of the current flowing and calculate the total power consumed in the circuit. (*L.U.*)

6. A coil, having $R = 2$ Ω. and $L = 0{\cdot}01$ H., carries a current which obeys the law—$i = 50 + 20 \sin 300t$—where i is in amp. and t in sec. A moving-iron ammeter, a moving-coil voltmeter and a dynamometer wattmeter are used to indicate the current, voltage and power respectively. Determine the readings of the instruments and the law governing the p.d. Neglect all losses in the instruments. (*L.U.*)

7. Two circuits, having impedances at 50 c.p.s. of $(10 + j6)$ Ω. and $(10 - j6)$ Ω. respectively, are connected in parallel across the terminals of an A.C. system, the wave-form of which is represented by $e = 100 \sin \omega t + 35 \sin 3\omega t + 10 \sin 5\omega t$, the fundamental frequency being 50 c.p.s. Determine the ratio of the readings of two ammeters, of negligible resistance, connected one in each circuit. (*I.E.E.*)

8. The capacitance of a 20-μF. condenser is checked by direct connection to an alternating voltage, which is supposed to be sinusoidal, an electrostatic voltmeter and a dynamometer ammeter being used for the measurement. If the voltage actually follows the law $e = 100 \sin 250t + 20 \sin (500t - \varphi) + 10 \sin (750t - \alpha)$, calculate the value of the capacitance as obtained from the direct ratio of the instrument readings. (*L.U.*)

9. A coil having $L = 0{\cdot}1$ H. and $R = 100$ Ω. is connected in series with a condenser across a supply, the voltage of which is given by $e = 200 \sin 314t + 5 \sin 3454t$. What capacitance will be required to produce resonance with the 11th harmonic? Find (*a*) the equation of the current, and (*b*) the R.M.S. value of the current, if this capacitance is in circuit. (*L.U.*)

10. A voltmeter, designed for the measurement of third harmonic components in a 50-cycle system, consists of a moving-iron milliammeter connected in series with an inductance and a condenser ($C = 0{\cdot}25\mu$F.), the whole circuit having $R = 250$ Ω. and being tuned to 150 c.p.s. Calculate (*a*) the total inductance in the circuit, (*b*) the error in the reading if the frequency falls to 49·5 cycles per sec., (*c*) the error if the frequency is correct, but the voltage wave contains a fifth harmonic having an amplitude equal to 40 per cent of the third. (*I.E.E.*)

11. A star-connected, three-phase alternator, the phase E.M.Fs. of which are symmetrical but non-sinusoidal, supplies a balanced star-connected load., Show that if the E.M.F. wave-form contains 3rd, 9th, 15th, etc., harmonics, a difference of potential will exist between the neutral points of generator

and load, and that if these neutral points are connected by a fourth line wire the current in this wire is made up of components having frequencies 3, 9, 15, etc., times the fundamental frequency.

Calculate the R.M.S. value of this current if the alternator E.M.F. follows the law $e = 850 \sin 314t + 120 \sin 942t + 50 \sin 2826t$. The impedance per phase of the alternator is $(0{\cdot}02 + j0{\cdot}12)\ \Omega$. at 50 c.p.s. and the resistance per phase of the load is $10\ \Omega$.

XV.—MAGNETIC CIRCUITS (CHAPTER XV)

1. The following data refer to a 50-cycle transformer. Calculate and plot the B–H loop. Applied primary volts, 110 (R.M.S.); number of primary turns, 90; length of mean magnetic path, 125 cm.; cross-section of magnetic path, 45·9 cm.^{2} The applied voltage is sinusoidal and voltage drops may be neglected. The wave-form of the no-load current is as follows, the angles being measured from the start of the positive half-wave of the applied voltage—

Angle (°)	0	20	40	60	80	100	120	140	160
Current (A.)	− 4 97	− 3	− 1	+ 0 25	1	1·69	2·39	3·18	4 18

2. The connection between the magnetizing current and flux for a particular alternating-current electromagnet is shown by the following table, which gives values for one-half of the magnetization loop, the complete loop showing the usual symmetry with respect to the axes—

Flux (kilolines)	0	30	75	120	150	179	176	165	138	102	60	0
Magnetizing current (A.)	5·5	6·25	8	10·5	14·25	20·5	13·5	6·8	0	− 3	− 4·5	− 5·5

Plot the loop on squared paper, and by its aid deduce and plot the wave-form of the magnetizing current when the flux follows a sine law, the amplitude of the flux being equal to the maximum value of the flux in the above magnetization loop. (*L.U.*)

3. Deduce an expression for the watts per cm.^{3} wasted in eddy currents in a thin core plate of a transformer, in terms of the frequency, the specific resistance of the material, the maximum flux density and the thickness of the plate.

In a 440 V., 50-frequency transformer, the total iron loss is 2500 W. When the applied voltage is 220 V. at 25 c.p.s., the corresponding loss is 850 W. Calculate the eddy-current loss at normal voltage and frequency. (*C.G.*)

4. Determine the magnitude of the hysteresis and of the eddy current losses at 25, 50 and 60 cycles per second for a transformer which gave the following open-circuit data.

Applied volts	100	75	50	25
Cycles per second	50	37 5	25	12 5
Watts input	85	58·6	35 6	16

Give a full explanation of the principles underlying the method adopted. (*L.U.*)

5. An iron-core choking coil has a magnetic length of 120 cm. and a cross-sectional area of 25 cm.^{2} There are 100 turns of wire of negligible resistance. The maximum flux density is 11,000 lines per cm.^{2} for which 5 amp.-turns per cm. are required. Calculate for a frequency of 50 c.p.s. (*a*) the VA. supplied, (*b*) the inductance of the coil. Find also these quantities when an air gap of 2 mm. is introduced in the iron core. (*C.G.*)

6. A single-phase core-type transformer has the following dimensions: cross-section of each core 515 sq. cm., distance between axes of cores 53 cm., cross-section of each yoke 650 sq. cm., distance between axes of yokes 96 cm. Calculate the flux produced by 1330 ampere-turns having given the following relationship between B and H for the magnetic circuit—

B	6000	9000	10500	13750
H	1·3	2·6	5·9	18·6

(*L.U.*)

·7. Calculate the no-load current for a single-phase transformer having given the following data: Primary voltage 2200, supply frequency 50, no-load loss 500 W., number of turns in primary winding 1200, magnetic

cross-section of core 35 sq. in., magnetic length of core 80 in., permeability (at flux density corresponding to normal primary voltage) 1200.

8. A transformer for 10 kVA., 2200 V. (primary), 50 cycles, takes 100 W. at rated voltage and frequency when its secondary is open. If the magnetizing current is 90 per cent of the exciting current, what is the no-load power factor of the transformer ? What is the exciting current in amperes and as a percentage of the rated full-load current ? (*L.U.*)

9. A single-phase core-type transformer has a core cross-section (net iron) of 60 cm.^{2} and a yoke cross-section (net iron) of 65 cm.^{2} The primary winding has 1350 turns and the secondary winding has 148 turns. Calculate the no-load current and the voltage at the secondary terminals when a voltage of 2200 V. at 50 frequency is applied to the primary. The mean magnetic length of each core is 30 cm., and that of each yoke is 23·5 cm. The overall length of each yoke is 27 cm.

Magnetic data of the sheet-steel laminations are—

Flux density (lines per cm.^{2})	8000	10,000	12,000	13,000
Ampere turns per cm.	1·6	2·55	5·9	9·35
Core loss at 50 cycles (W. per lb.)	0 53	0·8	1·1	1 3

Assume the weight of one cm.^{3} of the steel laminations to be 1/58 lb. (*L.U.*)

10. The net cross-section of iron in each core and yoke of a single-phase core-type transformer is 50 cm.^{2} and 55 cm.^{2} respectively. The magnetic length of each core is 35 cm. and that of each yoke is 30 cm. Calculate the voltage per turn at 50 c.p.s. to give an iron loss of 70 W. The density of the laminations is 0·0165 lb per cm.^{3}, and the iron loss (P_i) at 50 c.p.s. is—

B_m (lines per cm.^{2})	6000	8000	10,000	12,000
P_i (W per lb.)	0·3	0·48	0 72	1·01

11. A choking coil consists of a ring-shaped core of 1·5 sq. in. section and 8 in. mean diameter. The ring is wound with 1500 turns having a resistance of 400 Ω. Calculate the current in the winding when it is connected to a 220 V., 50-cycle supply of sinusoidal wave-form. The permeability of the iron may be taken as constant at 10,000, the iron losses at 0·5 W. per lb., and the density of iron 7·8 gm. per cm.^{3}. (*I.E.E.*)

12. An air-gap choking coil has a back-E.M.F. of 200 V. when taking 10 A. from a 50-cycle supply. Calculate the effective length of the air-gap. Assume that (*a*) the number of turns is 400, (*b*) the maximum flux density in the air-gap is 8000 lines per cm.^{2}, (*c*) the magnetic reluctance of the iron core is 6 per cent of that of the air-gap, (*d*) the magnetic leakage and losses are negligible. (*I.E.E.*)

13. The nearly closed iron core of a choking coil has a net iron cross-section of 20 cm.^{2} and the mean length of the iron path is 50 cm. This core is to be worked at a flux density of 10,000 lines per cm.^{2} and at this density it requires 3·5 amp.-turns per cm. of magnetic length, and the iron loss is 0·025 W. per cm.^{3} at 50 cycles. If the coil is to absorb 50 V. at a current of 10 A. on a 50-cycle circuit, calculate (*a*) the number of turns, (*b*) the length of the air gap, and (*c*) the phase difference between the terminal voltage and current. The resistance of the coil may be neglected. (*L.U.*)

14. A choking coil, normally used in series with a mercury-vapour lamp on a 50-cycle supply, has a net core cross-section of 15 cm.^{2}, an air-gap of 2 mm., and a magnetic length of 25 cm. The coil is wound with 500 turns, and the resistance of the winding is 1·6 Ω. The normal current is 2·28 A. at 50 c.p.s., and the core loss is 17 W. Calculate the voltage at the terminals of the coil. Magnetic data of the laminations are

Flux density (lines per cm.^{2})	4000	6000	8000	10,000
Ampere turns per cm.	0·6	1·05	1·6	2·55

XVI.—MEASURING INSTRUMENTS AND MEASUREMENTS (CHAPTERS XII, XIII, XV)

1. A 150 V. moving-iron voltmeter takes a current of 0·06 A. and has an inductance of 1·105 H. The total resistance is 2490 Ω., of which 695 Ω. is

in the magnetizing coil. Calculate the frequency error when used on a 50-cycle circuit, assuming the d.c. calibration to be correct. Show how the frequency error may be compensated by means of a condenser shunting the series resistance, and calculate the capacitance required in this case. (*I.E.E.*)

2. The relationship between the inductance, current, and position of the moving system of a 2 A. moving iron ammeter is—

Scale reading (amp.) .	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Deflection (Degrees) .	16	26	36·5	49·5	61·5	74·5	86·5
Inductance (μH.) .	573·2	574·2	575·2	576·6	577·8	578·8	579·5

Deduce an expression for the deflecting torque in terms of the rate of change of inductance with position of moving system, and calculate the deflecting torque at scale readings of 1 A. and 2 A. (*L.U.*)

3. A moving-iron voltmeter—range 0–125 V.—has its scale between 0–25 V. divided according to a square law (i.e. the deflection is proportional to the square of the voltage at the terminals of the instrument) and the remainder of the scale (between 25–125 V.) is divided according to a straight-line law (i.e. equal increments of deflection correspond to equal increments of voltage at the terminals of the instrument). The scale divisions immediately adjacent to, and on each side of, the 25-V. mark are of equal lengths. The full-scale deflection is 120°, the torque for full-scale deflection is 0·2 gm.-cm., and the resistance of the instrument is 1500 Ω. Calculate the rate of change of the inductance of the instrument with deflection for the square-law portion of the scale. (*L.U.*)

4. What difficulties are encountered in designing a shunt for a moving-iron ammeter? It is proposed to use a non-inductive shunt to increase the range of a 10 A. moving-iron ammeter to 100 A. The resistance of the instrument, including the leads to the shunt, is 0·06 Ω., and the inductance is 15 μH. at full scale. If the combination is correct on a d.c. circuit, find the error at full scale on a 50-cycle a.c. circuit. (*L.U.*)

5. The coil dimensions of a dynamometer voltmeter giving 90° deflection for 50 V. are so arranged that the inductance of the instrument increases uniformly as the pointer is deflected from zero to the top of the scale. The initial inductance is 0·25 H., and the torque for full-scale deflection is 0·4 gm.-cm., the current being 0·05 A. Determine the difference between a.c. (50-cycle) and d.c. readings at (*a*) 50 V., (*b*) 25 V. (*I.E.E.*)

6. A rectifier has the following characteristics—

Volts at terminals .	+ 1·2	+ 1·0	+ 0·8	+ 0·6	+ 0·4	0	− 0·6	− 1·2
Current in mA. .	+ 20	+ 12	+ 6·6	+ 2	+ 0·8	0	− 1·2	− 2·4

The rectifier is connected in series with a permanent-magnet moving-coil instrument having a total resistance of 120 Ω., and the circuit is connected to a sinusoidal supply of 2·5 V. (R.M.S.). Calculate the ammeter reading and draw the wave-form of the current passing through the instrument. (*L.U.*)

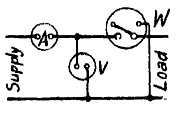
7. A single-element rectifying device is connected in series with a moving-coil ammeter and a thermal ammeter, and it is found that the circuit has a resistance of 100 Ω. to a current in one direction, and an infinitely great resistance to a current in the opposite direction. If a sinusoidal E.M.F. of 10 V. (R.M.S.) is applied to this circuit, what will be the reading of each instrument, and what will be the power taken from the supply and the power factor of the system? (*I.E.E.*)

8. An electrostatic voltmeter has the fixed inductors in the form of cylindrical segments. The inner radius of the outer segments is 2·575 cm. and the outer radius of the inner segments is 2·425 cm. Between these inductors a needle, 0·5 mm. thick and cylindrical in form, can rotate. If the angular breadth of the needle segments is 115°, the mean radius is 2·5 cm. and the active length is 8 cm., calculate the torque in gm.-cm. when 200 V. is applied between the needle and the fixed inductors. (*L.U.*)

9. An electrostatic voltmeter consists of two attracted plates, one movable

each surrounded by a guard ring so as to make the "edge-effect" negligible. It is observed that the application of 10 kV. between the plates results in a pull of 500 dynes on the movable plate. Determine the alteration of the capacitance of the system resulting from a change in position of the movable plate of 1 mm. Diameter of movable plate, 10 cm. (*I.E.E.*)

10. A wattmeter is connected up as in the accompanying diagram to measure the power supplied to an alternating-current circuit.



The following readings were taken: $A = 2{\cdot}4$, $V = 200$, $W = 22$. The resistance of the pressure circuit of the wattmeter is 8000 Ω., that of the current coil 0·46 Ω., and that of the voltmeter 2000 Ω. What are the true watts supplied to the load, and what is its power factor? Neglect the inductances of the instruments. (*L.U.*)

11. The following data refer to an electro-dynamic wattmeter having a current range of 3 A. and a pressure range of 2500 V.

Resistance of pressure-coil circuit, 25,000 Ω.; inductance of pressure coil circuit, 0·532 μH.; resistance of fixed coils, 5·5 Ω.

Determine the error when this instrument is used on a 100 cycle circuit at power factors of (*a*) 0·9, and (*b*) 0·15 (lagging), the current being 3 A.

12. An electro-dynamic wattmeter has a shunt coil with a resistance of 750 Ω. and a series resistance of 2250 Ω. A condenser of 1 μF. capacitance is arranged so that it can be shunted across the series resistance. If two readings of the wattmeter are taken, W_1 without the condenser shunt and W_2 with the condenser shunt connected, determine a formula by which the power factor of the circuit in which the power is being measured can be found in terms of these readings. Frequency 50 c.p.s. (*L.U.*)

13. A wattmeter having a moving coil with $R = 100$ Ω., $L = 20$ mH., is used for measuring both power and reactive VA. in a single-phase 300-cycle circuit. For power measurements a 3000 Ω. resistance is connected in series with the moving coil, and for reactive VA. measurements a good condenser is substituted for this resistance. Calculate (1) the capacitance of this condenser, assuming the calibration of the instrument to remain unchanged, (2) the error in the "reactive" reading when the power factor of the load is 0·95 (lagging). (*I.E.E.*)

14. Explain clearly how phase compensation in the voltage-coil circuit of an induction-pattern wattmeter may be obtained by shunting the pressure coil of the instrument with a non-inductive resistance. What are the principal objections to this method of phase compensation?

Work out the value of shunt resistance required for an instrument whose voltage coil ($L = 4{\cdot}78$ H., $R = 298$ Ω.) is joined in series with an external inductance of 3 H. and an external resistance of 190 Ω. The supply voltage is 200 V. at 50 c.p.s. What current will flow in the voltage coil? (*L.U.*)

15. The current and flux are in phase in the series system of an induction pattern watt-hour meter, but there is an angular departure of 3° from quadrature between the voltage and flux of the pressure system. The speed of the rotor on full load at unity power factor is 40 r.p.m. Assuming the meter to register correctly under this condition, calculate its percentage error on quarter load at 0·5 power factor. (*L.U.*)

16. A 50 A., 230 V. meter is put under test, and it is found that the disc makes 61 revolutions in 37 sec. when full load is being carried. If the normal disc speed is 520 revolutions per kWh., what is the meter error, and how could the error be corrected? (*I.E.E.*)

17. In a deflectional frequency meter working on the principle of electrical resonance, there are two parallel circuits each consisting of an inductance and a capacitance in series. One circuit has $C = 1$ μF. and is tuned to a frequency of 60 c.p.s. The other has $C = 1{\cdot}5$ μF. and is tuned to a frequency below 50 c.p.s. The resistance of each circuit is 100 Ω. What must be the inductance of the second circuit, and to what frequency must it be tuned,

in order that the current in both circuits shall be the same at a frequency of 50 c.p.s.? (*L.U.*)

18. The current taken by a small iron-cored choking coil is measured by means of an a.c. potentiometer. A 1 Ω. shunt is inserted in series with the winding and the voltage across the latter is measured directly on the potentiometer. The readings are—Voltage across shunt, + 0·8 V. in-phase, − 0·75 V. quadrature; voltage across winding, + 1·3 V. in-phase, + 0·3 V. quadrature. Assuming sinusoidal voltage and current, determine (1) the core loss, (2) the magnetizing current. (*I.E.E.*)

19. When using an electrostatic wattmeter to measure the dielectric loss in a leaky condenser at high voltage, the following readings were obtained: Voltage applied to the condenser, 30,000 V.; voltage between the moving vanes of the wattmeter and the earthed end of the high voltage supply, 250 V.; voltage across the 130 Ω. resistance connected between the quadrants, 26·8 V.; deflection of the wattmeter (scale divisions), − 37; constant of the wattmeter, 3·52.

A step-up transformer was used to supply the 30,000 V. and the vanes of the wattmeter were connected to a tapping brought out from the secondary winding of this transformer near the earthed end.

Calculate the power factor and the dielectric loss of the condenser at 30,000 V.

Prove any formulæ used, and indicate the circumstances under which the wattmeter deflection is likely to be negative. (*L.U.*)

20. An alternating-current bridge is arranged as follows: The arms AB and BC consist of non-inductive resistances of 100 Ω., the arms BE and CD of non-inductive variable resistances, the arm EC of a condenser of capacitance 1 μF., the arm DA of an inductive resistance. The alternating-current source is connected to A and C and the telephones to E and D. A balance is obtained when the resistance of the arm CD is 50 Ω., and the arm BE 2500 Ω.

Calculate the resistance and inductance of the arm DA.

If there are harmonics in the wave-form of the source of alternating current, what will be the effect? (*L.U.*)

21. The four arms of a bridge network are made up as follows: AB, a resistance of 50 Ω. in parallel with an inductance of 0·1 H.; BC, a resistance of 100 Ω.; CD, an unknown resistance in parallel with a condenser of unknown capacitance; DA, a resistance of a 1000 Ω. A 50-cycle supply voltage is maintained between A and C, and a vibration galvanometer is connected to B and D. Find the values of the unknown resistance and capacitance for which the vibration galvanometer will be undeflected. (*L.U.*)

22. A small single-phase motor, in series with a 10 Ω. resistance, is connected to a 220 V. supply. The voltages across the motor and resistance are 200 V. and 40 V. respectively. Determine the power consumption and the power factor of the motor. (*I.E.E.*)

23. Describe briefly methods which may be used to determine the phase rotation of a three-wire, three-phase system. In a phase-rotation test on a 200 V., three-phase system, a 20 Ω. non-inductive resistance in series with a choking coil (X = 15·6 Ω., R = 2·2 Ω.) was connected to the R and Y lines. A very high resistance voltmeter was connected between the B line and the junction of the resistance and choking coil. Find the two possible readings of the voltmeter, stating the phase rotation in each case. (*L.U.*),

24. In a test to determine the phase sequence of a 415 V., 50-cycle, three-phase supply, a 430 Ω. resistance was connected in series with a 5-μF. condenser between two lines A and B. A voltmeter connected between the junction of the resistance and the condenser, and the third line, C, indicated 182 V. Draw the vector diagram and determine the phase sequence. If the phase sequence is reversed, what will be the reading on the voltmeter? (*L.U.*)

25. Give an account of the precautions to be observed in the design and

construction of precision current transformers for use with a standard wattmeter.

Deduce an expression for the correction factor of a wattmeter (which has a negligible inherent error) when used with a current transformer having ratio and phase-angle errors of p per cent and $q°$ respectively, and a potential transformer which has a ratio error of x per cent and a phase-angle error of $y°$. (*C.G.*)

26. Two single-phase watt-hour meters are connected through current and voltage transformers to a three-phase, three-wire system. The ratios of the transformers may be considered to be correct, but the phase displacement of the voltage transformers is 15 minutes (leading) and that of the current transformers is 3 minutes (leading). The power factor of the system is 0·75 (lagging). Determine the relative speeds of the meter discs. (*I.E.E.*)

27. A voltage transformer, ratio 3300/110, has the following measured characteristics—primary resistance, 4000 Ω.; secondary resistance, 3 Ω.; exciting current, 0·0038 A. at 0·3 power factor. The phase displacement between the primary and secondary (reversed) voltages is 10 minutes (leading) when the burden is 15 VA. at unity power factor. Determine (*a*) the true turn ratio, (*b*) the primary reactance (assuming this to be equal to the equivalent reactance). (*I.E.E.*)

28. A 5 VA., 400/100 V., 50-cycle potential transformer was tested, at 50 c.p.s. with its normal burden, according to the method shown in Fig. 279 (p. 439), balance being obtained with—R_1 = 7086 Ω., R_2 = 8000 Ω., R_3 = 5000 Ω., C = 0·00021 μF., switch S in position 2. Calculate (1) the ratio error, (2) the phase-angle error.

29. A current-transformer has a bar primary and 400 secondary turns. The secondary burden is a circuit making, with the secondary winding, a total resistance of 1·25 Ω. and a reactance of 0·75 Ω. The core requires 100 amp.-turns for magnetization and 60 amp.-turns for losses. Find (1) the primary current, (2) the ratio error, and (3) the phase-angle error when the secondary current is 5A. How could the ratio error be corrected for this current? (*I.E.E.*)

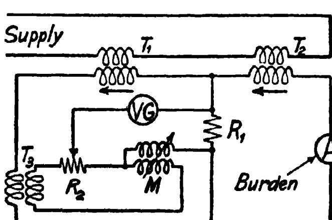
30. The mumetal core of a current transformer has a cross-sectional area of 15 cm.² and a magnetic length of 28 cm. The 5-A. secondary winding has 120 turns and a resistance of 0·15 Ω. Calculate the ratio and phase-angle errors of this transformer when it is connected to a 50-cycle supply system, and the secondary is loaded with a burden of 15 VA. at a power factor of 0·6 (lagging), the secondary current being 5 A.

Data of the magnetic properties of the core for a frequency of 50 c.p.s. are as follows—

Induced E.M.F. (mV.) per turn per cm.² of core cross section	0·6	1·0	1·5	2·0
Magnetizing reactive mVA. per cm.³ of core	0·0045	0·0105	0·0205	0·0335
Iron loss (mW.) per cm.³ of core	0·0023	0·0066	0·016	0·029

(*L.U.*)

31. The accompanying diagram shows the connections of a method of determining the ratio and phase angle errors of a current transformer. T_1 is a standard transformer and T_2 is the transformer under test, both of which have the same nominal ratio. T_3 is another standard transformer of 1 : 1 nominal ratio; M, a standard variable mutual inductance; R_1, R_2, non-inductive resistances.



Balance is obtained at a supply frequency of 50 c.p.s. with the following results: R_1 = 10 Ω., R_2 = 0·08 Ω., M = 556 μH.

The errors of T_1 and T_3 are—*Ratio error*: T_1, 0·2 per cent; T_3, 0·5 per cent; *phase-angle error*: T_1, 20′ leading; T_3, 50′ leading.

Calculate the ratio and phase-angle errors of T_2. Give briefly the theory of the method. (*L.U.*)

XVII.—CALCULATION OF THREE-PHASE CIRCUITS (CHAPTERS XIX, XX)

1. On a symmetrical three-phase system with a phase-sequence A, B, C, a capacitance reactance of 8 Ω. is connected between lines B and C. A resistance of R ohms and an inductive reactance of X ohms are connected in series between phases B and A. Find the respective values of R and X which will cause the current in phase B to become zero. (*L.U.*)

2. If, to the system of Ex. 1, a resistance R_1 in series with a capacitive reactance X_C is connected across lines C and A, determine the values of R_1 and X_C which will cause the currents in these lines to be twice the former value with zero current in line B.

If the line voltage is 200 V., calculate the power and volt-amperes in the whole system.

3. A three-phase supply, giving sinusoidal voltages of 400 V. at 50 c.p.s., is connected to three terminals marked R, Y, B.

Between R and Y is connected a resistance of 100 Ω., between Y and B an inductance of 318 mH. and negligible resistance, and between B and R a condenser of 31·8 μF.

Determine (1) the current flowing in each line and the total power supplied; (2) the resistance of each phase of a balanced star-connected, non-reactive, load, which will take the same total power when connected across the same supply. (*L.U.*)

4. A three-phase, 11 kV. alternator supplies a load of 15,000 kVA. at 0·8 power factor and also feeds a 132 kV. transmission line through a step-up transformer. The no-load current of this transformer is 60 A. at 11 kV., 0·25 power factor. If the transmission line is unloaded, determine the kW. and kVA. output from the alternator and the power factor at which it is operating. The open-circuit impedance of the line is $1250\angle -70°$ Ω. per phase, and resistance and reactance voltage drops in the transformer may be ignored.

5. An unbalanced Δ-connected load is supplied from a 400 V. symmetrical, sine-wave three-phase system having a phase sequence RYB. The impedances (in ohms) of the load are: $Z_{RY} = \mathbf{10 + j5}$; $Z_{YB} = \mathbf{15 + j15}$; $Z_{BR} = \mathbf{5 + j8}$. Calculate (1) the line currents, (2) the power supplied to the load, (3) the resistance of each phase of a balanced Y-connected load which will take the same power, but at unity power factor. Determine also (4) the impedances of a Y-connected load which is equivalent to the Δ-connected, (5) the voltage between the neutral point of this load and the neutral point of three Y-connected resistances of equal values connected to the system.

6. An unbalanced star-connected load is supplied from a three-phase, three-wire system at a line voltage of 100 V. The current taken by one branch (A) is 20 A. at a power factor of 0·8 lagging, and that taken by a second branch (B) is 10 A. at a power factor of 0·75 (lagging). Determine the current in, and the power factor of, the third branch, C. Also determine the total power supplied to the load and the readings of two wattmeters connected, according to the two-wattmeter method, with the current coils in the branches A, B.

7. Three impedances having values of $\mathbf{3 + j6}$, $\mathbf{4 + j3}$, and $\mathbf{5 + j4}$ ohms respectively are connected in star across three-phase supply mains. Calculate, proving any formulae used, the values of the three impedances which, when connected in delta across the same supply mains, will behave in exactly the same way as the above star-connected system. (*L.U.*)

8. A symmetrical three-phase, 440 V. system supplies a Y-connected load, of which the branch resistances are: A = 10 Ω., B = 13 Ω., C = 15 Ω.

Calculate the voltage to earth of the load star-point, assuming the neutral of the supply to be earthed. Phase sequence A, B, C. (*I.E.E.*)

9. A three-phase star-connected load is connected between three line terminals R, Y, B, the impedances (in ohms) of the load being: $1 + j2$ between R and the star point, $2 + j3$ between Y and the star point, and $3 + j4$ between B and the star point. The phase sequence is RYB. If the line voltage is 400 V., calculate the voltage between R and the star point. (*L.U.*)

10. A three-phase, four-wire, 400/230 V. system supplies (1) a balanced load which is connected between the lines R, Y, B, and the neutral, each phase taking a current of 20 A. at 0·8 power factor, (2) a single-phase load, taking 15 A. at a power factor of 0·9, connected between line R and the neutral. Determine the current in each of the lines R, Y, B, and the current in the neutral.

11. In a four-wire, three-phase distribution system, with 240 V. between lines and neutral, there is a balanced motor load of 500 kW. at 0·8 power factor. Lamp loads connected between the lines and the neutral absorb 50 kW., 150 kW., and 200 kW. respectively. Calculate the current in each line and the neutral. (*I.E.E.*)

12. A combined power and lighting load is supplied by a three-phase, four-wire distribution system. The three-phase motor load absorbs 1000 kW. at a power factor of 0·8, while the lamps between the outers and the neutral take 200, 300, and 400 kW. respectively. Calculate the current in each of the four wires when the supply pressure between outers is 400 volts. (*C.G.*)

13. What are the advantages of the three-phase, four-wire system of distribution? Compare the weight of copper required for the distributor cables in such a system with that required in a three-wire, d.c. system. State exactly the bases of comparison.

A 440/254 V., three-phase, four-core cable supplies an unbalanced load represented by the following impedances, in ohms, connected between the R, Y, and B line wires respectively and the neutral: $16 + j12$, $14 - j21$, and $25 + j0$. The phase sequence is R, Y, B. Calculate the current in each conductor of the cable and the readings on each of three wattmeters connected in each line to neutral. (*C.G.*)

14. The following apparatus is connected to a three-phase, four-wire system: A resistance of 10 Ω. between line A and neutral; a 400 μF. condenser between line B and neutral; a resistance of 2 Ω. in series with an inductance of 0·01 H. between line C and neutral. The frequency of the supply is 50 cycles per second, and the voltage between each line and neutral is 230 V. Find the current in the neutral and its phase angle with the current in line A. Phase sequence A, B, C. (*L.U.*)

15. Load resistances R_a, R_b, R_c are connected in delta. If R_1, R_2, R_3 are the equivalent star values, calculate these values when $R_a = 5$ Ω., $R_b = 10$ Ω., and $R_c = 15$ Ω., proving any formula used. If this star-connected system is supplied from a four-wire, three-phase system with 230 V. between each line and neutral, calculate the current in the neutral and the power supplied to the load. (*L.U.*)

16. Three substations A, B, C, forming part of a three-phase, 11 kV. distributing system are connected to form a ring-main by three feeders having impedances in ohms per phase as follows: AB, $2 + j4$; BC, $3 + j5$; CA, $2 + j3$. Power is supplied at A, and loads are taken at B and C; the load at B being 50 A. at 0·8 power factor (lagging), and that at C being 40 A. at 0·9 power factor (lagging). If the voltage at A is maintained at 11 kV., calculate the voltage at B and the current in feeder BC. The power factors of the loads may be assumed to be relative to the voltage at A. (*L.U.*)

17. A ring-main supplies three substations B, C, D from a distributing centre A, the system being three-phase with 6600 V. between lines at the distributing centre. The loads at the substations are: 40 A. at B, 20 A. at C, 30 A. at D, all at unity power factor. The four sections of the ring-main have

resistances and reactances as follows: *AB*, $R = 1{\cdot}2\ \Omega.$, $X = 0{\cdot}8\ \Omega.$; *BC*, $R = 0{\cdot}9\ \Omega.$, $X = 0{\cdot}6\ \Omega.$; *CD*, $R = 0{\cdot}6\ \Omega.$, $X = 0{\cdot}4\ \Omega.$; *DA*, $R = 1{\cdot}5\ \Omega.$, $X = 1{\cdot}0\ \Omega.$ Find (1) the voltage at each substation, and (2) the current in each section of the ring-main.

If an induction regulator is installed in the section *AB* at the distributing centre, *A*, and gives a boost of 50 V. in the three lines, in phase with the line-to-neutral voltage in each case, what will be the new distribution of currents in the sections of the ring-main and the voltages at the substations?

XVIII.—SYMMETRICAL COMPONENTS (CHAPTER XXI)

1. The readings of the three line ammeters on a three-phase, three-wire system are 10·5, 13, and 8 A. Determine the symmetrical components.

2. The three current vectors of a three-phase, four-wire system have the following values: $I_A = 7 + j0$, $I_B = -12 - j13$, $I_C = -2 + j3$. Find the symmetrical components. The phase sequence is *A*, *B*, *C*. (*L.U.*)

3. Derive an expression for the symmetrical components of an unbalanced three-phase system of vectors. Explain, with the aid of circuit diagrams, how the voltage and current components may be measured, and show how the circuits must be modified when zero-sequence components are present in the currents of a four-wire system. If in a three-wire, three-phase system the line voltmeters indicate 100, 60, and 90 V., what are the symmetrical components of the voltage. (*L.U.*)

4. A star-connected, three-phase generator having an E.M.F. per phase of 7000 V. is supplying bus-bars through a star/star step-up transformer, all the star points being earthed. An earth fault occurs on the red phase on the high-voltage side of the transformer. Determine the current in each phase of the generator. The reactance of the alternator to positive, negative, and zero sequence currents is 4 Ω., 3 Ω., and 1 Ω. respectively, and the reactance of the transformer to positive sequence currents is 3 Ω. (referred to the low-voltage side).

State briefly what would have been the general effect on the fault currents in the generator if (*a*) the star point on the low-voltage side of the transformer had not been earthed, (*b*) the low-voltage winding of the transformer had been connected in delta, the other two star points remaining earthed in each case. (*L.U.*)

XIX.—TRANSIENTS (CHAPTER XXIII)

1. A condenser of 10 μF. capacitance charged to 20,000 V. is suddenly discharged through an inductance of 0·001 H. Find the maximum current and the frequency of the resultant oscillation. (*L.U.*)

2. A condenser, $C = 0{\cdot}1\ \mu$F., is charged to a voltage V and then connected in series with a circuit having $L = 0{\cdot}2$ H., $R = 40\ \Omega$. Calculate the frequency of the oscillating current.